Philip of Spain, promoting astrology, and recommending salt as a fertilizer. Napier's book *Mirifici logarithmorum canonis descriptio* (A Description of the Wonderful Rule of Logarithms) was published earlier (1614) than Bürgi's (1620), and he is generally regarded as the discoverer of logarithms.

These developments, naturally, facilitated the calculation of π by the Archimedean method or its modifications, and accuracies far beyond any possible practical use were obtained. We shall come back to this point in the next chapter, but first we shall examine the theoretical progress made during this period. The main achievement was that of another amateur mathematician, François Viète, Seigneur de la Bigotière (1540-1603). He was a lawyer by profession and rose to the position of councillor of the *Parlement* of Brittany, until forced to flee during the persecution of the Huguenots. The next six years or so during which he was out of favor, he spent largely on mathematics. With the accession of Henry IV, a former Huguenot, Viète was restored to office, becoming Master of Requests (1580) and a Royal Privy Councillor (1589). He endeared himself to the king by breaking the Spanish code made up of some 500 cyphers, thus enabling the French to read all secret enemy dispatches. Thereupon the Spanish, with singular one-track-mindedness, accused him of being in league with the devil.

Viète made important contributions to arithmetic, algebra, trigonometry and geometry. He also introduced a number of new words into mathematical terminology, some of which, such as *negative* and *coefficient*, have survived. His attack on π, though still proceeding along general Archimedean lines, started with a square rather than a hexagon, and resulted in the first analytical expression giving π as an infinite sequence of algebraic operations.

His procedure consisted (essentially) of relating the area of an n-sided polygon to that of a $2n$-sided polygon (see figure on opposite page). The area of an n-sided polygon is

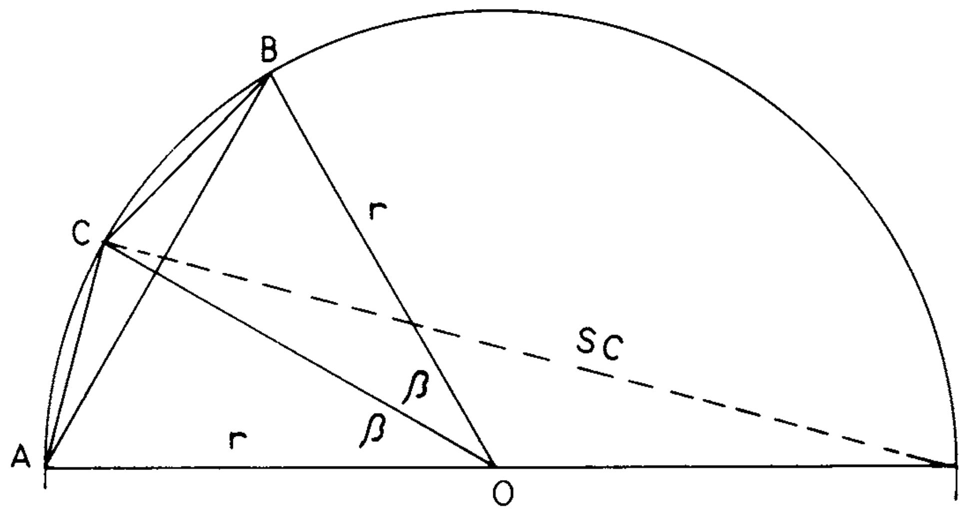
$$\begin{aligned} A(n) &= n \text{ times area of triangle } OAB \\ &= \tfrac{1}{2} n r^2 \sin 2\beta \qquad (1) \\ &= n r^2 \cos\beta \sin\beta \qquad (2) \end{aligned}$$

Similarly,

$$A(2n) = n r^2 \sin\beta \qquad (3)$$

so that from (2) and (3),

$$A(n)/A(2n) = \cos\beta \qquad (4)$$



Vièete's method of finding π.

and if we double the sides of the polygon again, we have

$$\frac{A(n)}{A(4n)} = \frac{A(n)}{A(2n)} \times \frac{A(2n)}{A(2^2 n)} = \cos\beta \cos(\beta/2) \tag{5}$$

Continuing like this k times, we obtain

$$\frac{A(n)}{A(2^k n)} = \frac{A(n)}{A(2n)} \times \frac{A(2n)}{A(4n)} \times \ldots \times \frac{A(2^{k-1} n)}{A(2^k n)}$$

$$= \cos\beta \cos(\beta/2) \ldots \cos(\beta/2^k) \tag{6}$$

But if k tends to infinity, the area of a regular polygon with 2^k sides is indistinguishable from that of a circle, so that

$$\lim_{k \to \infty} A(2^k n) = \pi r^2. \tag{7}$$

Substituting (7) and (2) in (6), we have

$$\pi = \frac{\tfrac{1}{2} n \sin 2\beta}{\cos\beta \cos(\beta/2) \cos(\beta/2^2) \cos(\beta/2^3)\ldots} \tag{8}$$

Viète chose a square to start with, so that $n = 4$, $\beta = 45°$, $\cos\beta = \sin\beta = \sqrt{\tfrac{1}{2}}$.

Also, each of the cosine factors in (8) is expressible in terms of the preceding factor through the half-angle formula

$$\cos(\theta/2) = \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\cos\theta)} \tag{9}$$

so that we finally have

$$\pi = \frac{2}{\sqrt{\tfrac{1}{2}} \times \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})} \times \sqrt{[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})}]} \times \ldots} \tag{10}$$

and this is Viète's expression, published in 1593 in his *Variorum de rebus mathematicis responsorum liber VIII* (Various mathematical problems, vol. 8). Actually, his derivation used the supplementary chord of a polygon, that is, the chord joining a point of a polygon on a circle to the other end of the diameter, marked *SC* in the figure on p. 93. (Note that this is the supplementary chord of a $2n$-sided, not of an n-sided polygon.) Viète showed that the supplementary chord of n-sided polygon is to the diameter as the area of the polygon with n sides is to that of the polygon with $2n$ sides. But since that ratio is simply $\cos\beta$, the derivation given above conveys the essence of Viète's procedure.

Viète's result represents one of the milestones in the history of π, and it is also the high point of renaissance mathematics connected with π; therefore it deserves a few comments.

First, we note that Viète was still imprisoned by the idea of the Archimedean polygon; he is, in fact, one of the last men to use a polygon in furthering the theory of calculating π (the others were Descartes, Snellius and Huygens, as we shall see in Chapter 11), though he was far from last in using it for numerical evaluation.

Second, Viète was the first in history to represent π by an analytical expression of an infinite sequence of algebraic operations. (As a matter of fact, he was also the first to use the term "analytical" in mathematical terminology, and the term has survived.) The idea of continuing certain operations *ad infinitum* was of course much older. Archimedes had used it, and before him Antiphon expressed the principle of exhaustion, as we have seen on p. 37. Viète was familiar with the Greek classics, and he refers to Antiphon in his treatment; his approach executed Antiphon's idea mathematically. However, there is a vast difference between expressing an idea qualitatively and giving quantitative instructions how to execute it, and Viète was the first to achieve this. Viète's expression, in fact, is the first known use of an infinite product, whether connected with π or not.

Third, Viète was a typical child of the Renaissance in that he freeley mixed the methods of classical Greek geometry with the new Arabic art of algebra and trigonometry. The idea of substitution is algebraic, and the square roots in his expression come from the trigonometric half-angle formula for the cosine, but otherwise his treatment is entirely Greek, based on considerations of area ratios involving the supplementary chord. Had he tried to express the mouthful *ratio of the supplementary chord to the diameter* trigonometrically, he would have found that it equals $\cos\beta$, a much more easily manipulated quantity. He would then have obtained our formula (8) above, and a man of Viète's stature could hardly have overlooked that by expressing β in radian measure and setting $\theta = 2\pi/n$, he would have obtained the formula

$$\theta = \frac{\sin\theta}{\cos(\theta/2)\cos(\theta/2^2)\cos(\theta/2^3)\ldots}, \qquad (\theta < \pi) \tag{11}$$

which Leonard Euler obtained in a quite different way almost 200 years later. Viète's result (10) is a special case of (11) for $\theta = \pi/2$.

Fourth, Viète did not yet know the concept of convergence and did not worry whether his infinite sequence of operations would "blow up" or not. We need not worry either, because as far as the formula for π is concerned, it is sufficient to take k arbitrarily large, but finite. However, if you are a friend of mathematical rigor and this kind of "sloppy engineering mathematics" disgusts you, rest assured: The convergence of Viète's formula was proved by F. Rudio in 1891.[53]

Fifth, it should be noted that Viète's formula is of almost no use for numerical calculations of π; the square roots are much too cumbersome, and the convergence is slow. Viète himself did not use it for his calculation correct to 9 decimal places; he used the Archimedean method without substantial modification by taking a polygon of 393,216 sides (the number is obtained by 16 successive doublings of the original hexagon). This enabled him to reduce the Archimedean bounds to

$$3.1415926535 < \pi < 3.1415926537 \tag{12}$$

but this is only a minor success compared with the formula (10) which he derived. In the vast majority of practical applications, $\pi = 22/7$ is good enough, and to obtain better and better approximations is only a matter of drudgery. In contrast, Viète's formula (10) is an entirely new formulation, one that can be manipulated and investigated. Indeed, if no better expressions had been discovered since then, this one could have served for investigating some of the properties of π that no number of decimal places can reveal.

Finally, the fact that Viète introduced a number of mathematical terms that have survived to the present is a point of interest. Terminology and symbolism cannot, by themselves, solve anything; but if not conveniently chosen, they can spoil a lot. The long sentence for the circle ratio (p. 76) could not be squared or subjected to other mathematical operations as the symbol π can, and Viète's example shows that one can miss an important turning to a wide new field by using Greek geometry instead of the concise symbolism of trigonometry. In Viète's time, mathematical notation was still a long way from what it is now. The operator symbols $+$, $-$, $=$ had only very recently been introduced, and algebra often described an equation in words rather than symbols. The unknown quantity (our "x") had a strange name. The Italians, through whom algebra mostly came to Europe from the Arabs, called it *cosa*, the "thing" (the same word as the Mafia uses in *cosa nostra*), and the word went into other languages, e.g., in German it became *die Coss*, and in Latin it became *numerus cossicus*; in English, algebra was known as *the Cossike arte*. Robert Recorde (1510-1558) tells us that *in nombers Cossike, all nombers haue not rootes; but soche only emongest simple cossike nombers are rooted, whose nomber hath a roote, agreable to the figure of his denomination.*

There are other quaint little tid-bits in mathematical terminology. For example, the origin of the words *tangent* and *secant* are clear enough, but where does *sine* come from? It comes from a translator's error in the Toledo translation center (p. 83). Arabic script, like Hebrew script, consists of consonants, with the vowels punctuated underneath, and the latter are often omitted. The sine, which one would expect to be called "half-chord" in analogy with the secant and tangent, was given a name by the Hindus, which the Arabs took over, and which they spelled by the consonants *jb*. When Robert of Chester, one of the Toledo translators, translated al-Khowarizmi's *Algebra* from Arabic into Latin in 1145, he encountered this word without knowing its Hindu origin; supplying the missing vowels, he found the Arabic word for bay or inlet, and the Latin for bay, inlet or cavity is *sinus*.