Take Home Exam - Math 215 - Spring 2013

Write all your proofs with care, using full sentences and correct reasoning.

1. Prove by induction on n that

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6,$$

for all positive integers n.

2. (a) Prove that there is a $1-1$ correspondence between the set $Z \times Z$ of pairs (n, m) where n and m are integers and the set N of all natural numbers.

(b) Let

$$P = \{2, 3, 5, 7, 11, 13, \cdots\}$$

be the set of prime numbers. Show that there is a $1-1$ correspondence between P and the set N of all natural numbers. You can use the fact that there are infinitely many prime numbers. Note that it is not enough to just say that the set of prime numbers is infinite and so is the set of all natural numbers. You need to explain how (in principle) to make a list of all the prime numbers. *For extra credit, give a proof that there are infinitely many prime numbers.*

3. Prove that the single statement B is equivalent to:

$$(A \vee B) \wedge (A \Rightarrow B).$$

In your proof, do *not* use truth tables. Use the fact that

$$X \Rightarrow Y \ = (\sim X) \vee Y,$$

and give a completely algebraic proof. You should also use other logical identities if they are relevant. For example, you may want to use the distributive law which states that

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$$

and that

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z).$$

4. (a) Define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $g \circ f : X \longrightarrow Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that if $g \circ f$ is surjective, then g is surjective.
(b) Prove that in any list of eight distinct positive integers at least two of them will have the same remainder on division by seven.
(c) Prove the following statement: Given n people who can shake hands with one another (where $n > 1$), then there is always a pair of people who will shake hands with the same number of people.

5. Given sets A and B, consider the following statements about a function $f : A \longrightarrow B$.

(i) $\forall b \in B, \exists a \in A$ such that $f(a) = b$.
(ii) $\forall a \in A, \exists b \in B$ such that $f(a) = b$.
(iii) $\exists b \in B, \forall a \in A, f(a) = b$.
(iv) $\exists a \in A, \forall b \in B, f(a) = b$.

Please note that it is best to read
"$\forall x \in X$" as "for every $x \in X$".
One of these statements is the definition for f to be a surjective mapping from A to B. Which one is it? For the remaining statements, explain what each one says and give an example of a mapping on specific sets for which it is satisfied. Can you give an example of a mapping for which it is not satisfied?

6. (a) Recall that we associate a subset of the natural numbers to a sequence $s = (s_1, s_2, \cdots)$ of 0's and 1's by the assignment

$$Set[s] = \{n \in N | s_n = 1\}.$$

For example

$$Set[(1, 0, 1, 0, 1, 0, \cdots)] = \{1, 3, 5, 7, \cdots\}.$$

Make your best guess about the set associated with the sequence

$$s = (1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1 \cdots).$$

(b) Let X and Y and Z be three sets. Suppose that X is in 1-1 correspondence with Y and that Y is in 1-1 correspondence with Z. Show that it follows that X is in 1-1 correspondence with Z.

(c) Recall that a set S is said to be *countable* if it can be put into $1-1$ correspondence with the natural numbers N. Let $P(N)$ denote the set of all subsets of the natural numbers N. Prove that $P(N)$is an uncountable set. Give a proof of this statement from first principles.

(d) Apply the Cantor diagonal process to the following list of sequences to produce a sequence that is not on the list. Give the first six terms of the sequence that you get from the diagonal process.

$$s_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \cdots)$$
$$s_2 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 \cdots)$$
$$s_3 = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0 \cdots)$$
$$s_4 = (0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0 \cdots)$$
$$s_5 = (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 \cdots)$$
$$s_6 = (0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1 \cdots)$$
$$\cdots$$

7. Let SF denote all infinite sequences of zeroes and ones that have only *finitely* many 1's in the sequence. Prove that SF is a countable set. (Note that such sequences are eventually an infinite string of zeroes.) If you take a complete list of all sequences in SF and apply the Cantor diagonal process, what will happen? You will get an infinite sequence that is not on the list. Why does this not imply that the list is incomplete for SF?

8. Let C^n_r denote the binomial choice coefficient. Thus C^n_r is equal to the number of r-element subsets of a set with n-elements. This is sometimes phrased as the number of ways to choose r things from n things.

(a) State the binomial theorem for $(x + y)^n$ in terms of the coefficients C^n_r.

(b) Prove that

$$2^N = \sum_{r=0}^{N} C^N_r$$

for all natural numbers N. Explain how your proof is related to the binomial theorem.