free generators of $\pi_1(\mathscr{F}_n)$ are also free generators for $\pi_1(\mathscr{F}_{n+1})$, for in each case there is a deformation retraction of $\mathscr{F}_{n+1}$ onto $\mathscr{F}_n$.

In Cases (4), (5), (6) the old boundary curve is replaced by two or three new ones, some of which may be spanned by discs in $\mathscr{F} - (\mathscr{F}_n \cup \Delta'_{n+1})$. When these discs are added the result is replacement of the old boundary curve by (a) one, (b) two, or (c) three new ones. In Case (a) the free generators of $\pi_1(\mathscr{F}_n)$ are also free generators for $\pi_1(\mathscr{F}_{n+1})$ by the same argument as above.

In Case (b) we get one new generator. Recall that free generators for $\pi_1(\mathscr{F}_n)$ are found by collapsing $\mathscr{F}_n$ onto a bouquet of circles $\mathscr{B}_n$. It is clear that $\mathscr{F}_{n+1}$ can be collapsed onto $\mathscr{B}_n$ plus one new edge (corresponding to Δ'_{n+1}), hence the free generators of $\pi_1(\mathscr{F}_n)$ will serve as free generators for $\pi_1(\mathscr{F}_{n+1})$ in conjunction with one new generator which passes through Δ'_{n+1}. In Case (c), which can only arise from (6), we get two new generators passing through Δ'_{n+1}.

This completes the proof of the claim.

It follows that there are nested *free* presentations of $\pi_1(\mathscr{F}_1)$, $\pi_1(\mathscr{F}_2)$, $\pi_1(\mathscr{F}_3), \ldots$, and hence a free presentation of $\pi_1(\mathscr{F})$, by the concluding argument of 4.1.7. □

EXERCISE 4.2.2.1. Construct surfaces to realize the free groups on $1, 2, 3, \ldots$ and a countable infinity of generators.

4.2.3 Wirtinger Presentation of Knot Groups

A *knot* $\mathscr{K}$ is a simple closed polygonal curve in R^3. $\mathscr{K}$ of course need not be an actual polygon, but only the image of one under a homeomorphism of all R^3 (this is to exclude wild embeddings, see 4.2.6). During the nineteenth century the study of knots and their classification was pursued on an experimental basis, but with the advent of the fundamental group decisive results could be obtained for the first time. The key observation is that when $\mathscr{K}$ is a *trivial knot* (isotopic to the circle in R^3) the group of its complement is infinite cyclic. Thus if we can show that the *knot group* $\pi_1(\mathsf{R}^3 - \mathscr{K})$ is not infinite cyclic for a particular knot $\mathscr{K}$ we have a topologically sound proof that $\mathscr{K}$ is not trivial.

The first method for computing knot groups was introduced by Wirtinger around 1904 in his lectures in Vienna, but not given wide circulation until its publication in Tietze 1908. We begin with intuitive explanation of the method.

Any knot $\mathscr{K}$ can be given by a projection on the plane with no multiple points which are more than double, and with indication being given, at each double point, which branch of $\mathscr{K}$ is uppermost. Figure 156 shows a projection of the trefoil knot. The double points are called *crossings*. If we break the lower branch at each crossing we obtain a finite set of arcs α_i,

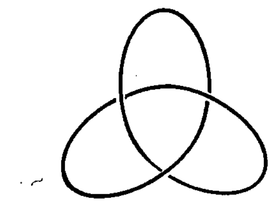

Figure 156

and it is intuitively clear that $\pi_1(\mathsf{R}^3 - \mathscr{K})$ is generated by loops a_i which pass around these arcs. Thus we have as many generators as there are crossings. We choose an orientation for the knot $\mathscr{K}$, and then orient the generators a_i around the arcs α_i by the right-hand screw convention (Figure 157). It is also convenient to order the subscripts to follow a circuit round $\mathscr{K}$, so that the lower arc α_i into a crossing is followed by the arc α_{i+1} out of the crossing. Referring to Figure 158, we see that for the crossing of type (1) the curve $a_i a_j^{-1} a_{i+1}^{-1} a_j$ contracts to a point, hence we have the relation

$$a_i a_j^{-1} a_{i+1}^{-1} a_j = 1 \quad \text{or} \quad a_j a_i = a_{i+1} a_j.$$

For the crossing of type (2) we have

$$a_i a_j a_{i+1}^{-1} a_j^{-1} = 1 \quad \text{or} \quad a_i a_j = a_j a_{i+1}.$$

(The other two possibilities correspond to the opposite orientation of $\mathscr{K}$, and hence give relations of one of these two forms.)

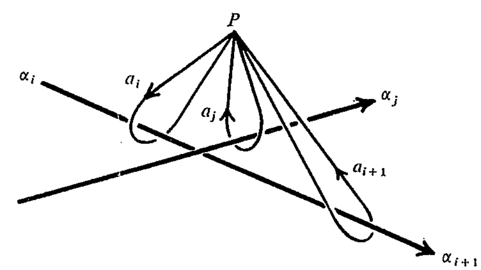

Figure 157

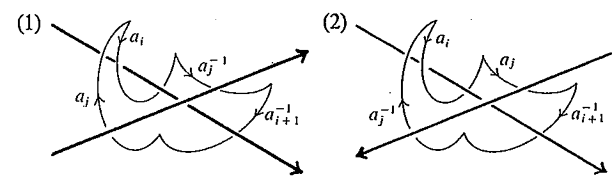

Figure 158

4.2.4 Proof of the Wirtinger Presentation

We now give a rigorous derivation of the presentation using the Seifert–Van Kampen theorem.

We can assume that each arc α_i of the knot $\mathscr{K}$ lies in the plane $z = 1$ except for a vertical segment at each end which goes down to $z = 0$. The final point of α_i can then be joined to the initial point of α_{i+1} by a segment β_j in the plane $z = 0$ passing under the upper arc of the crossing α_j, to complete the knot $\mathscr{K}$. We now remove from R^3 the "tunnel" neighbourhood $\mathscr{N}$ of $\mathscr{K}$ swept out by a cube of side ε which travels with its midpoint on $\mathscr{K}$ and faces parallel to the axes, where ε is small enough to ensure that $\mathsf{R}^3 - \mathscr{N}$ is a deformation retract of $\mathsf{R}^3 - \mathscr{K}$ (Figure 159). $\mathsf{R}^3 - \mathscr{N}$ can now be expressed as the union of open sets $\mathscr{A}$ and $\mathscr{B}$ which reflect the generators and relations, respectively, of $\pi_1(\mathsf{R}^3 - \mathscr{N})$.

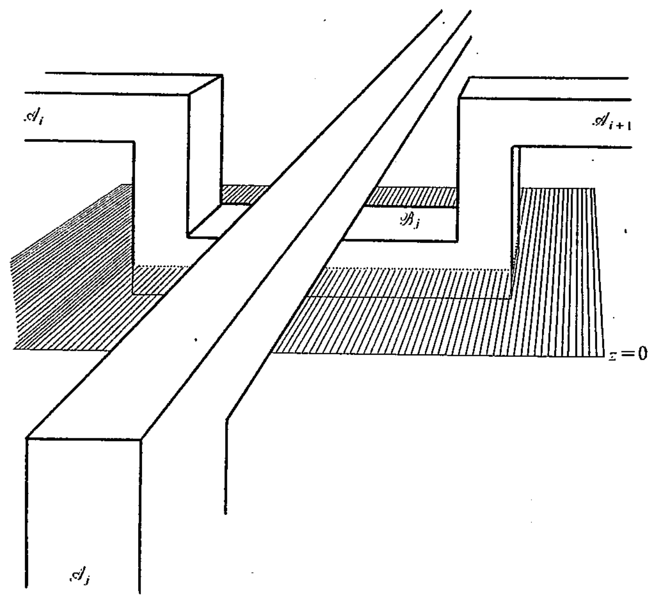

Figure 159

$\mathscr{A} = \{z > 0\} - \mathscr{N}$ has a deformation retraction onto a bouquet of circles $a_1, \ldots, a_n$, where a_i is a loop passing under the tunnel $\mathscr{A}_i$ containing α_i. The reader may become more convinced of this by first deforming $\mathscr{A}$ so that the "hollows" $\mathscr{B}_j$ containing the β_j are pressed down to $z = 0$, then pulling the tunnels $\mathscr{A}_i$ into parallel with each other, as in Figure 160 (cf. the cube with holes in 3.3.2).

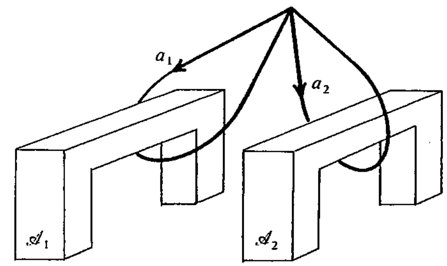

Figure 160

Hence $\pi_1(\mathscr{A}) = \langle a_1, \ldots, a_n; - \rangle$.

$\mathscr{B} = \{z < \varepsilon/2\} - \mathscr{N}$ is an open half-space with "trenches" containing the segments $\mathscr{B}_j$ dug out of it. It is clearly simply connected, so $\pi_1(\mathscr{B}) = \{1\}$.

$\mathscr{A} \cap \mathscr{B} = \{0 < z < \varepsilon/2\} - \mathscr{N}$ is an infinite plate with n holes in it (the upper halves of the trenches), hence

$$\pi_1(\mathscr{A} \cap \mathscr{B}) = \text{free group of rank } n.$$

The typical generator of $\pi_1(\mathscr{A} \cap \mathscr{B})$, a circuit round a trench (Figure 161), has the form $a_i a_j^{-1} a_{i+1}^{-1} a_j$ in $\pi_1(\mathscr{A})$ (or $a_i a_j a_{i+1}^{-1} a_j^{-1}$ for the second type of crossing) and 1 in $\pi_1(\mathscr{B})$. Thus the Seifert–Van Kampen theorem gives precisely the Wirtinger relations for

$$\pi_1(\mathscr{A} \cup \mathscr{B}) = \pi_1(\mathsf{R}^3 - \mathscr{N}) = \pi_1(\mathsf{R}^3 - \mathscr{K}). \qquad \square$$

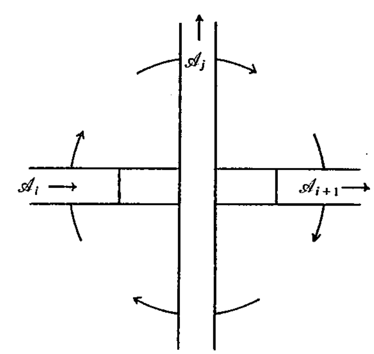

Figure 161

EXERCISE 4.2.4.1. Show that any one Wirtinger relation is a consequence of the remainder. (*Suggestion*: Instead of using $\mathscr{B}$ to seal the tunnels $\mathscr{B}_j$ at the bottom of $\mathscr{A}$, use a separate open set $\mathscr{C}_j$ to seal each $\mathscr{B}_j$, where $\mathscr{C}_j$ is an open cube with $\mathscr{B}_j$ removed from its top. Then show

$$\begin{aligned}\pi_1(\mathsf{R}^3 - \mathscr{N}) &= \pi_1(\mathscr{A} \cup \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_n)\\ &= \pi_1(\mathscr{A} \cup \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_{i-1} \cup \mathscr{C}_{i+1} \cup \cdots \cup \mathscr{C}_n).)\end{aligned}$$

:XERCISE 4.2.4.2. Generalize the Wirtinger method to compute $\pi_1(\mathbb{R}^3 - \mathscr{G})$, where $\mathscr{G}$ is ny graph embedded in $\mathbb{R}^3$. Show in particular that the relation at a vertex of degree has the form

$$a_{i_1} a_{i_2} \cdots a_{i_n} = 1$$

/hen $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}$ is the clockwise sequence of edges into the vertex.

·.2.5 The Simplest Knot and Link

Ve now compute the groups for the trefoil knot and the two-crossing link.

(i) The two-crossing link (Figure 162). At X we read off the relation

$$a_2 a_1 a_2^{-1} a_1^{-1} = 1 \quad \text{or} \quad a_1 a_2 = a_2 a_1$$

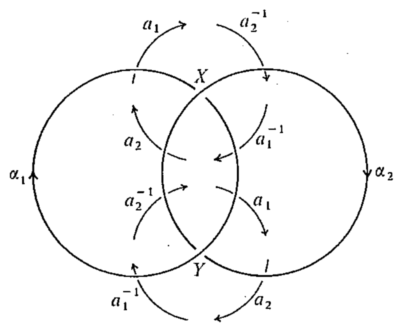

Figure 162

nd at Y we find the same relation. Thus the group of the two-crossing nk is

$$\langle a_1, a_2; a_1 a_2 = a_2 a_1 \rangle$$

r the free abelian group of rank 2. The group of ($\mathbb{R}^3$-two unlinked circles) ; the free group of rank 2 (why?) and hence we have a proof of the non-·iviality of the link.

(ii) The trefoil knot (Figure 163). At X we read off

$$a_1 a_3^{-1} a_2^{-1} a_3 = 1. \tag{1}$$

t Y

$$a_3 a_2^{-1} a_1^{-1} a_2 = 1. \tag{2}$$

t Z

$$a_2 a_1^{-1} a_3^{-1} a_1 = 1. \tag{3}$$

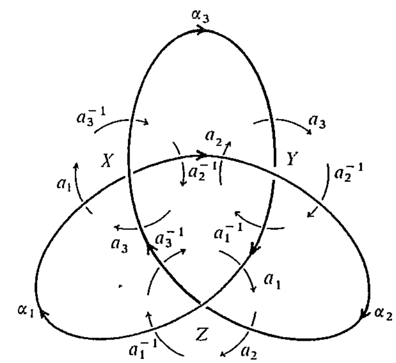

Figure 163

Solving (1) for a_1 and substituting the result in (2) and (3) we find that each yields the equation

$$a_3 a_2 a_3 = a_2 a_3 a_2. \tag{4}$$

It follows that

$$(a_2 a_3 a_2)^2 = (a_3 a_2 a_3)(a_2 a_3 a_2) = (a_3 a_2)^3$$

which we write

$$a^2 = b^3 \tag{5}$$

by setting $a = a_2 a_3 a_2$, $b = a_3 a_2$. But $a_2 = ab^{-1}$, $a_3 = b^2 a^{-1}$ so a, b are in fact generators and (4) is a consequence of (5). Thus we have the presentation

$$G_{2,3} = \langle a, b; a^2 = b^3 \rangle$$

for the group of the trefoil knot.

We now investigate whether $G_{2,3}$ is infinite cyclic. Notice that the group S_3 of permutations on three symbols is also a model of the relation $a^2 = b^3$, namely, take

$$a = (1\ 2), \qquad b = (1\ 2\ 3).$$

Hence any relation in $G_{2,3}$ is also valid in S_3 under this interpretation of a, b. It follows that $ab = ba$ is not a relation of $G_{2,3}$, since $ab \neq ba$ in S_3, and therefore $G_{2,3}$ is not infinite cyclic because all elements commute in the infinite cyclic group. (The representation of $G_{2,3}$ by S_3 is due to Wirtinger, who used it to construct a covering of $\mathbb{S}^3$ branched over the trefoil knot, see 1.1.4. The same covering had already been considered by Heegaard 1898, who made the surprising discovery that the covering manifold is also $\mathbb{S}^3$.)

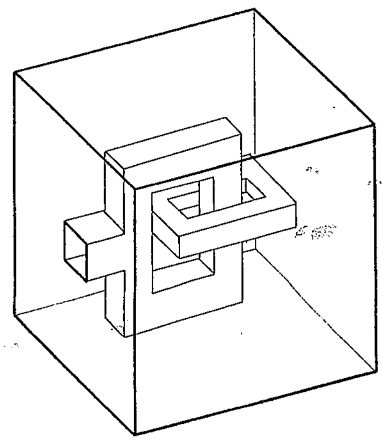

Figure 164

EXERCISE 4.2.5.1. Show that the cube with holes shown in Figure 164 has a deformation retraction onto the torus, and hence gives an alternative derivation of the group of the two-crossing link.

4.2.6 The Fox–Artin Wild Arc

A simple polygonal arc $\mathscr{A}$ in R^3 has the property that $\pi_1(\mathsf{R}^3 - \mathscr{A}) = \{1\}$. Fox and Artin 1948 call a simple arc $\mathscr{A}$ in R^3 *wild* if there is no homeomorphism of R^3 which maps $\mathscr{A}$ onto a polygon, in particular if $\pi_1(\mathsf{R}^3 - \mathscr{A}) \neq \{1\}$. Figure 165 shows an example of a wild arc (the limit points P and Q are included in the arc).

The generators we shall use for $\pi_1(\mathsf{R}^3 - \mathscr{A})$ are loops a_n, b_n, c_n for all integers n, placed as shown in Figure 166. $\mathsf{R}^3 - \mathscr{A}$ is the union of sets $\mathscr{C}_n$ obtained by removing cubes centred on P, Q at the positions shown in

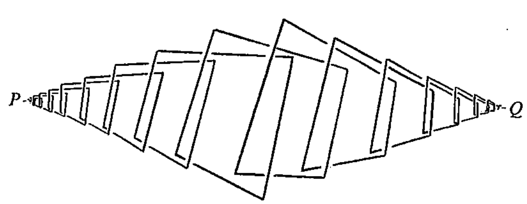

Figure 165

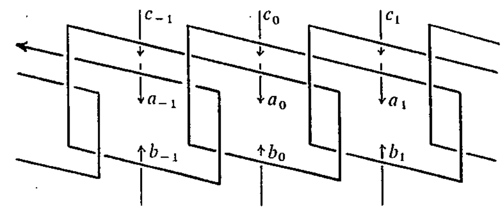

Figure 166

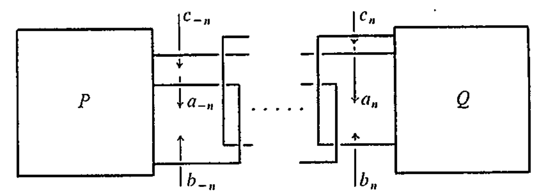

Figure 167

Figure 167. The generators of $\pi_1(\mathscr{C}_n)$ are a_m, b_m, c_m for $-n \leq m \leq n$ and the relations are

$$\left.\begin{aligned} a_{m+1} &= c_{m+1}^{-1} c_m c_{m+1} \\ b_m &= c_{m+1}^{-1} a_m c_{m+1} \\ c_{m+1} &= b_m^{-1} b_{m+1} b_m \end{aligned}\right\} \quad \begin{array}{c} \text{Wirtinger relations at the crossings;} \\ \\ -n \leq m \leq n \end{array}$$

together with relations

$$c_{-n} a_{-n} = b_{-n}, \qquad c_n a_n = b_n$$

at the ends (shrinking the cubes to points and using Exercise 4.2.4.2).

By removing a small tunnel neighbourhood of the arc (of diameter which $\to 0$ as $n \to \infty$) we can replace $\mathscr{C}_n$ by a finite simplicial complex, so it follows from 4.1.7 that the generators of $\pi_1(\mathsf{R}^3 - \mathscr{A})$ are a_n, b_n, c_n as claimed, and the relations are (for all integers n)

$$c_n a_n = b_n \tag{1}$$

$$a_{n+1} = c_{n+1}^{-1} c_n c_{n+1} \tag{2}$$

$$b_n = c_{n+1}^{-1} a_n c_{n+1} \tag{3}$$

$$c_{n+1} = b_n^{-1} b_{n+1} b_n. \tag{4}$$

Substituting (2) in (1) and (3) gives

$$c_n c_n^{-1} c_{n-1} c_n = b_n, \quad \text{that is,} \quad b_n = c_{n-1} c_n \tag{5}$$

and

$$b_n = c_{n+1}^{-1} c_n^{-1} c_{n-1} c_n c_{n+1}. \tag{6}$$

Substituting (5) in (4) gives

$$c_{n+1} = c_n^{-1}c_{n-1}^{-1}c_n c_{n+1}c_{n-1}c_n$$

or

$$c_{n-1}c_n c_{n+1} = c_n c_{n+1}c_{n-1}c_n \tag{7}$$

which is the same as the result of eliminating b_n between (5) and (6). Thus we can use the c's as generators, with the defining relations (7).

It can then be verified that these relations hold in the nontrivial group generated by the permutations (1 2 3 4 5) and (1 4 2 3 5) when c_n is interpreted as (1 2 3 4 5) for n odd and (1 4 2 3 5) for n even, hence $\pi_1(\mathsf{R}^3 - \mathscr{A}) \neq \{1\}$ □

In constructing the wild arc $\mathscr{A}$ we have also constructed a *wild ball* (the tunnel neighbourhood of $\mathscr{A}$) and *wild sphere* (the boundary of the wild ball). The first examples of such objects were given by Antoine 1921, based on an even more paradoxical object, a *wild Cantor set* in R^3. The ordinary Cantor set, obtained by the "middle-third" construction on the unit interval, has a simply connected complement in R^3. However, a wide variety of descending sequence constructions lead to homeomorphic images of the Cantor set; the one used by Antoine iterates the construction of linked solid tori inside a solid torus (Figure 168). Four linked tori are constructed again within each inner torus, and so on. The intersection of all these tori is a Cantor set in R^3 called *Antoine's necklace.* Antoine showed geometrically that its complement is not simply connected, and this was confirmed by calculation of the fundamental group by Blankenship and Fox 1950.

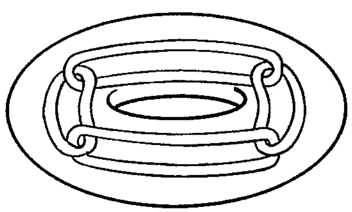

Figure 168

Fox and Artin 1948 also showed the wildness of the arc $\mathscr{A}'$ obtained by altering $\mathscr{A}$ so that the crossings are alternately over and under. $\mathscr{A}'$ is in fact the "chain stitch" of knitting, infinitely extended in both directions. Its group is calculated similarly, but turns out to be slightly more complicated than that of $\mathscr{A}$. It is interesting to note that the infinite chain stitch was pictured in the first ever paper on knot theory, Vandermonde 1771.

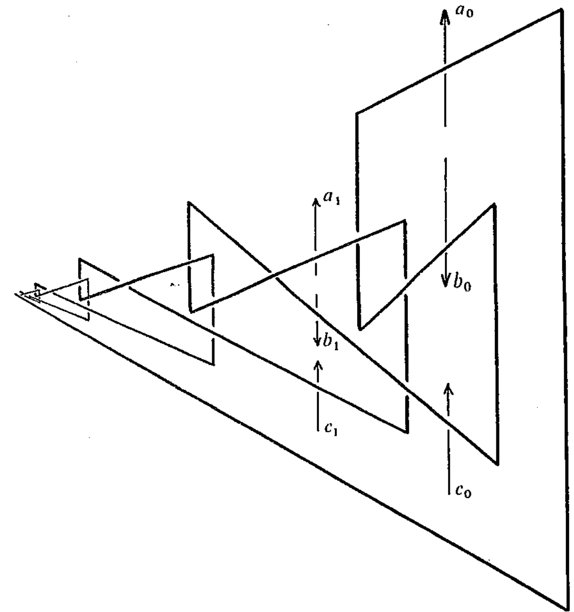

Figure 169

EXERCISE 4.2.6.1 (Fox 1949). Show that the group of the simple closed curve in Figure 169 is generated by $b_0, b_1, b_2, \ldots$ subject to the relations

$$b_1b_0b_1^{-1} = b_2b_1b_2^{-1} = b_3b_2b_3^{-1} = \cdots$$

and find a permutation representation which shows it is nonabelian.

4.2.7 Torus Knots

Consider a solid cylinder $\mathscr{C}$ with m line segments on its curved face, equally spaced and parallel to the axis. If the ends of $\mathscr{C}$ are identified after a twist of $2\pi(n/m)$, where n is an integer relatively prime to m, we obtain a single curve $\mathscr{K}_{m,n}$ on the surface of a solid torus $\mathscr{T}$ (Figure 170). Assuming that $\mathscr{T}$ lies

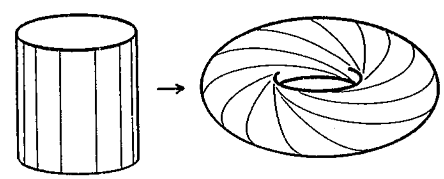

Figure 170

in R^3 in the "standard" way, which means among other things that $\pi_1(\mathrm{R}^3 - \mathscr{T})$ is infinite cyclic, the curve $\mathscr{K}_{m,n}$ is called the (m, n) *torus knot*. We now compute $\pi_1(\mathrm{R}^3 - \mathscr{K}_{m,n})$, following the method of Seifert and Threlfall 1934.

If we drill out a thin tubular neighbourhood $\mathscr{N}$ of $\mathscr{K}_{m,n}$ from R^3, the effect on $\mathscr{T}$ is to gouge out a narrow channel from its surface, and similarly on the surface of $\mathrm{R}^3 - \mathscr{T}$. $\mathscr{T} - \mathscr{N}$ and $(\mathrm{R}^3 - \mathscr{T}) - \mathscr{N}$ then meet along an annulus $\mathscr{L}_{m,n}$ which, like $\mathscr{K}_{m,n}$, results from m parallel strips on the cylinder being joined up after a twist of $2\pi(n/m)$ (Figure 171). $\pi_1(\mathscr{L}_{m,n})$ is infinite cyclic and generated by the centre line $l_{m,n}$ of $\mathscr{L}_{m,n}$. $\pi_1(\mathscr{T} - \mathscr{N})$ is also infinite cyclic and generated by the axis a of $\mathscr{T}$. Since $l_{m,n}$ results from m circuits of $\mathscr{T}$ we have

$$l_{m,n} = a^m$$

in $\pi_1(\mathscr{T} - \mathscr{N})$. Similarly, $\pi_1((\mathrm{R}^3 - \mathscr{T}) - \mathscr{N})$ is infinite cyclic, generated by a loop b through the "hole" in $\mathscr{T}$, and

$$l_{m,n} = b^n$$

in $\pi_1((\mathrm{R}^3 - \mathscr{T}) - \mathscr{N})$.

Then if we expand $\mathscr{T} - \mathscr{N}$ and $(\mathrm{R}^3 - \mathscr{T}) - \mathscr{N}$ slightly across $\mathscr{L}_{m,n}$ to open sets $\mathscr{A}, \mathscr{B}$ which intersect in a neighbourhood of $\mathscr{L}_{m,n}$, the Seifert–Van Kampen theorem becomes applicable, and we obtain the presentation

$$G_{m,n} = \langle a, b;\, a^m = b^n \rangle$$

for the group of the (m, n) torus knot. □

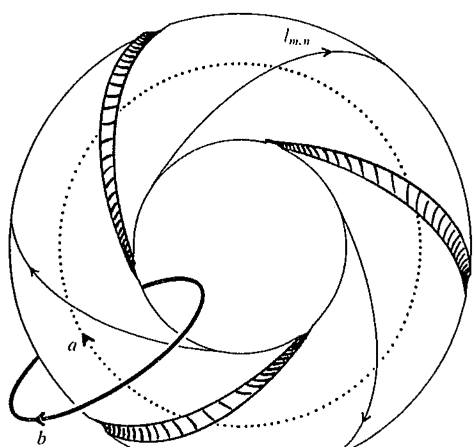

Figure 171

EXERCISE 4.2.7.1. Show that the (m, n) torus knot is the same as the (n, m) torus knot.

EXERCISE 4.2.7.2. Let $\mathscr{H}_n$ be the solid body (*handlebody*) bounded by an orientable surface of genus n which is standardly embedded in R^3, so that $\pi_1(\mathrm{R}^3 - \mathscr{H}_n)$ is the free group of rank n. Show that if $\mathscr{K}$ is a simple curve on the surface of $\mathscr{H}_n$ then $\pi_1(\mathrm{R}^3 - \mathscr{K})$ has a presentation with $n + 1$ generators and n relations. (*Hint*: Attach a thin handle $\mathscr{H}$ to $\mathscr{H}_n$ which follows $\mathscr{K}$ just above the surface of $\mathscr{H}_n$ and has its ends at neighbouring points on $\mathscr{K}$. Show that $\mathscr{H}_n \cup \mathscr{H}$ is a standardly embedded handlebody $\mathscr{H}_{n+1}$, then cut $\mathscr{H}_{n+1}$ so as to obtain a body whose complement is homeomorphic to $\mathrm{R}^3 - \mathscr{K}$.)

4.2.8 Lens Spaces

The (m, n) lens space is a 3-dimensional manifold introduced by Tietze 1908, by means of the following construction. On the surface of a solid ball B^3 one draws an equatorial circle and m equally spaced meridians, dividing the upper hemisphere into triangles $\Delta_1, \ldots, \Delta_m$ and the lower hemisphere into triangles $\Delta'_1, \ldots, \Delta'_m$, where Δ'_i is below Δ_i (Figure 172). The upper hemisphere is then identified with the lower after twisting it through $2\pi(n/m)$, that is, a point P with latitude and longitude (θ, ϕ) is identified with the point P' with latitude and longitude $(-\theta, \phi + 2\pi(n/m))$, where $\phi + 2\pi(n/m)$ is reduced mod 2π. Thus Δ_i is identified with Δ'_{i+n} after inversion ($i + n$ reduced mod m).

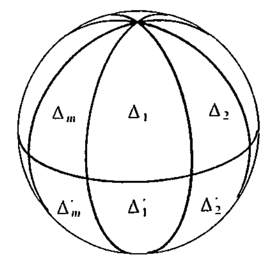

Figure 172

It is evident that if m, n have a common divisor d then the result is expressible more simply as the $(m/n, n/d)$ lens space, so we may as well assume that m, n are relatively prime. Likewise, there is no point in taking $n \geq m$.

Many properties of the (m, n) lens space will come to light in Chapter 8, in particular the fact that it is a manifold and the reason for the name "lens space." For the moment we wish only to compute its fundamental group.

By virtue of 4.1.5, we can forget about the interior of B^3, and just compute π_1 of the surface complex which results from identification of the two hemispheres. Since each point below the equator is identified with a point

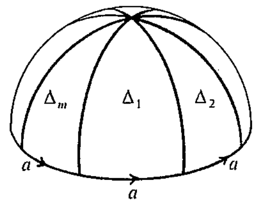

Figure 173

above, it suffices in turn to find out what becomes of the upper hemisphere when the identifications on its boundary, the equator, are carried out.

Since m, n are relatively prime, the numbers

$$1,\ 1+n,\ 1+2n,\ \ldots$$

run through all values $1, 2, \ldots, m$ when reduced mod m. This means that corresponding points on the bases of any two triangles are identified (Figure 173) or that the equator is wrapped m times round a circle corresponding to the base of a triangle. Thus our surface complex is a disc with boundary identified with the path a^m round a circle a. Its fundamental group is therefore $\langle a; a^m\rangle$, the cyclic group of order m, by 3.4.4. □

4.3 Surface Complexes and Subgroup Theorems

4.3.1 Surface Complexes and Groups

A *surface complex* $\mathscr{F}$ consists of three sets $\{P_i\}$, $\{e_j\}$, and $\{\Delta_k\}$ of elements called *vertices*, *edges*, and *faces* respectively, subject to certain incidence relations. The vertices and edges constitute a graph $\mathscr{G}$ (see 2.1.2) called the 1-*skeleton* of $\mathscr{F}$, and each face Δ_k is incident with a certain closed path b_k in $\mathscr{G}$ called its *boundary path*.

There is no harm in thinking of $\mathscr{F}$ being realized by actual points, line segments, and discs embedded in some euclidean space, where Δ_k is a disc with its boundary identified with a closed path b_k and different edges and discs are disjoint except where identifications force boundary points into coincidence. Comparison with 4.1.3 will then show that the group we are about to define combinatorially is the familiar fundamental group of $\mathscr{F}$. However, the purely combinatorial approach will suffice for the results we wish to derive, and no appeal will be made to general continuity considerations. Thus the situation is comparable with Chapter 2; the geometric language could in principle be dispensed with, but it seems to convey the most natural explanation of certain group-theoretic results. Indeed, it could be said that those results follow from viewing groups themselves as surface complexes.

The (combinatorial) *fundamental group of* $\mathscr{F}$, $\pi_1(\mathscr{F})$ is the group defined by extending the path product operation to equivalence classes of closed edge paths from some vertex P. Paths p, p' are equivalent if one can be converted to the other by a finite sequence of operations of the following types:

(i) insertion or removal of spurs;
(ii) insertion or removal of boundary paths of faces.

Paths which are equivalent by operations (i) above will be called *freely* equivalent. The equivalence class of p will be denoted $[p]$.

We know from 2.1.7 that generators for all edge paths based at P can be found by constructing a spanning tree $\mathscr{T}$ of $\mathscr{G}$, and for each edge $e_i = P_m P_n$ taking the closed path

$$a_i = w_m e_i w_n^{-1},$$

where w_r denotes the unique reduced path in $\mathscr{T}$ from P to P_r. In fact, any closed path $p(e_i)$ from P is freely equivalent to the corresponding product of the a_i's, $p(a_i)$.

For each boundary path

$$b_k(e_i) = e_{i_1}^{\varepsilon_1} e_{i_2}^{\varepsilon_2} \cdots e_{i_n}^{\varepsilon_n} \quad (\text{where } e_j = \pm 1)$$

of a face Δ_k we have a relation

$$b_k(a_i) = a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \cdots a_{i_n}^{\varepsilon_n} = 1 \tag{k}$$

because $b_k(a_i)$ is freely equivalent to the path $wb_k(e_i)w^{-1}$, where w is the unique reduced path from P to the initial point of $b_k(e_i)$. Conversely, any relation in $\pi_1(\mathscr{F})$ is a consequence of the relations (k), because the result of insertion (deletion) of $b_k(e_i)$ in a closed path $p(e_i)$ from P is freely equivalent to the result of insertion (deletion) of $b_k(a_i)$ in the path $p(a_i)$.

It follows immediately that any group $\mathscr{G}$ can be realized as the combinatorial fundamental group of a surface complex $\mathscr{F}$, by taking a bouquet of circles $a_1, a_2, \ldots$ and attaching a face Δ_k with boundary b_k for each relation $b_k(a_i) = 1$ of G. Furthermore, some of the useful topological properties of the (topological) fundamental group have combinatorial counterparts. We now establish some for use in later sections.

(a) *$\pi_1(\mathscr{F})$ does not change under elementary subdivisions of $\mathscr{F}$ or their inverses* (1.3.8).

Subdivision of an edge means replacing some e_i by $e_i'e_i''$ and a_i by $a_i'a_i''$ accordingly. But if we extend the spanning tree $\mathscr{T}$ to reach the new vertex it must include exactly one of e_i', e_i'', say e_i''. Then $a_i'' = 1$ and all we have done to $\pi_1(\mathscr{F})$ is to change the presentation by replacing a_i by a_i'.

When a face is subdivided we can assume that the new edge e_l begins at the initial vertex of the boundary path r_k (since a defining relator can be replaced by a cyclic permutation of itself). Then if $r_k = r_k'r_k''$ is the subdivision