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Proving the Pythagorean Theorem via Infinite Dissections

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Abstract. Novel proofs of the Pythagorean Theorem are obtained by dissecting the squares on the sides of the abc triangle into a series of infinitely many similar triangles.

The diagrams shown in Figure 1 are the bases of two of the classical proofs of the Pythagorean Theorem. One, according to tradition, is by Pythagoras himself, and the other is by Bhaskara. Common to both diagrams is the fact that a smaller square appears inside a larger square.

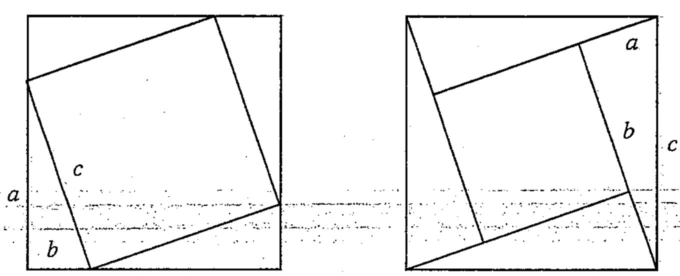

Figure 1. Classical proofs by Pythagoras and Baskara

What if we place a scaled-down copy of these diagrams over the smaller squares? What happens if the process is repeated ad infinitum? As shown in Figure 2, following a spiral or a zigzag pattern, we obtain dissections of the original square into infinitely many scaled copies of the abc-triangle. These dissections, in turn, give rise to several novel proofs of the Pythagorean Theorem, some of which are a mix of geometry, algebra, and infinite (geometric) series; the others are purely geometrical.

Consider the dissection of the $(a+b) \times (a+b)$ square in either of the two diagrams on the left in Figure 2. The corresponding sides of the similar right triangles in consecutive layers have a ratio $c/(a+b)$; hence, the ratio of their areas is $c^2/(a+b)^2$.

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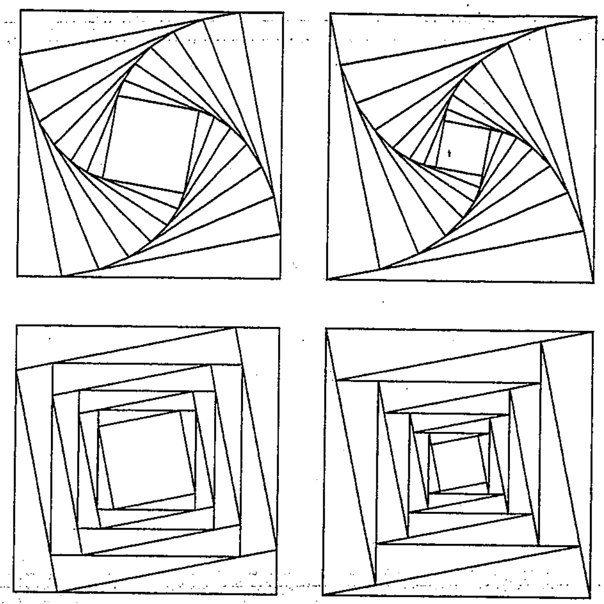

Figure 2. Pythagorean iterations

Therefore, the area of the $(a+b) \times (a+b)$ square can be expressed as the sum of an infinite geometric series,

$$(a+b)^2 = 2\sum_{n=0}^{\infty}(ab)\left(\frac{c^2}{(a+b)^2}\right)^n = 2ab\frac{1}{1-\frac{c^2}{(a+b)^2}} = 2ab\frac{(a+b)^2}{(a+b)^2-c^2},$$

which yields

$$(a+b)^2 - c^2 = 2ab,$$

and $a^2+b^2=c^2$ follows.

A similar argument can be made using the $c \times c$ square in the diagrams on the right in Figure 2. In this case, we have

$$c^2 = \sum_{n=0}^{\infty}(ab)\left(\frac{(a-b)^2}{c^2}\right)^n = 2ab\frac{1}{1-\frac{(a-b)^2}{c^2}} = 2ab\frac{c^2}{c^2-(a-b)^2},$$

and again, $a^2+b^2=c^2$ is obtained.

Let us now place scaled copies of the diagrams in Figure 2 over the three squares on the two legs and the hypotenuse of a right triangle. Two examples are shown in Figure 3. For each square, we are looking at a dissection into an infinite number of scaled copies of the original right triangle. Pairs of triangles in the squares on the legs are matched with triangles in the square on the hypotenuse, so that the sums of the areas of the former equal the areas of the latter triangles. With Bhaskara we may say: Behold!

The proofs presented here have a bit of a calculus flavor, in that each relies explicitly or implicitly on infinite series. Other unconventional proofs of the Pythagorean Theorem that borrow from calculus include B. Mazur [**4**], where limits are used, and M. Hardy [**3**] and M. Staring [**5**], where differentials are used.

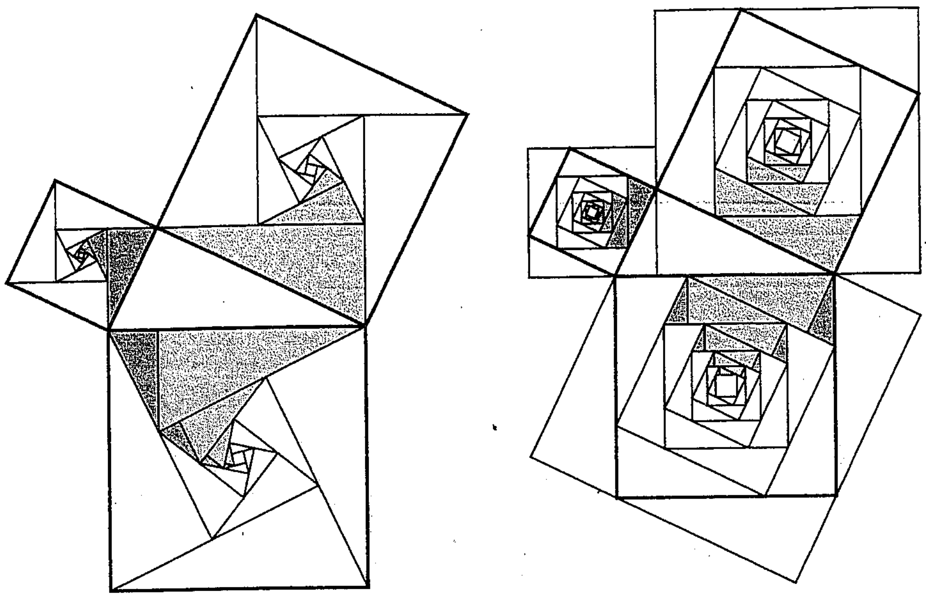

Figure 3. Proofs without words

Finally, we mention that diagrams similar to those in Figure 2 have appeared before, although not in the context of proving the Pythagorean Theorem. Two examples are: Chapter 17 of C. Alsina and R. B. Nelsen [**1**], and the logo of the Bridges Pécs 2010 conference [**2**].

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Stirling's Formula and Its Extension for the Gamma Function

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Abstract. We present new short proofs for both Stirling's formula and Stirling's formula for the Gamma function. Our approach in the first case relies upon analysis of Wallis' formula, while the second result follows from the log-convexity property of the Gamma function.

The well-known formula of Stirling asserts that

$$n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n} \quad \text{as } n \to \infty, \tag{1}$$

in the sense that the ratio of the two sides tends to 1. This provides an efficient estimation to the factorial, used widely in probability theory and in statistical physics.

Articles treating Stirling's formula account for hundreds of items in JSTOR. A few of the most relevant references may be found in [2], [3], [5], and [7].

As was noticed by Stirling himself, the presence of π in the formula (1) is motivated by the Wallis formula,

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdots(2n)\cdot(2n)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdots(2n-1)\cdot(2n-1)\cdot(2n+1)},$$

that is,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n+1}}\cdot\frac{(2n)!!}{(2n-1)!!} = \sqrt{\frac{\pi}{2}},$$

where $n!! = n\cdot(n-2)\cdots 4\cdot 2$ if n is even, and $n\cdot(n-2)\cdots 3\cdot 1$ if n is odd.

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