Linking Numbers in Branched Covers

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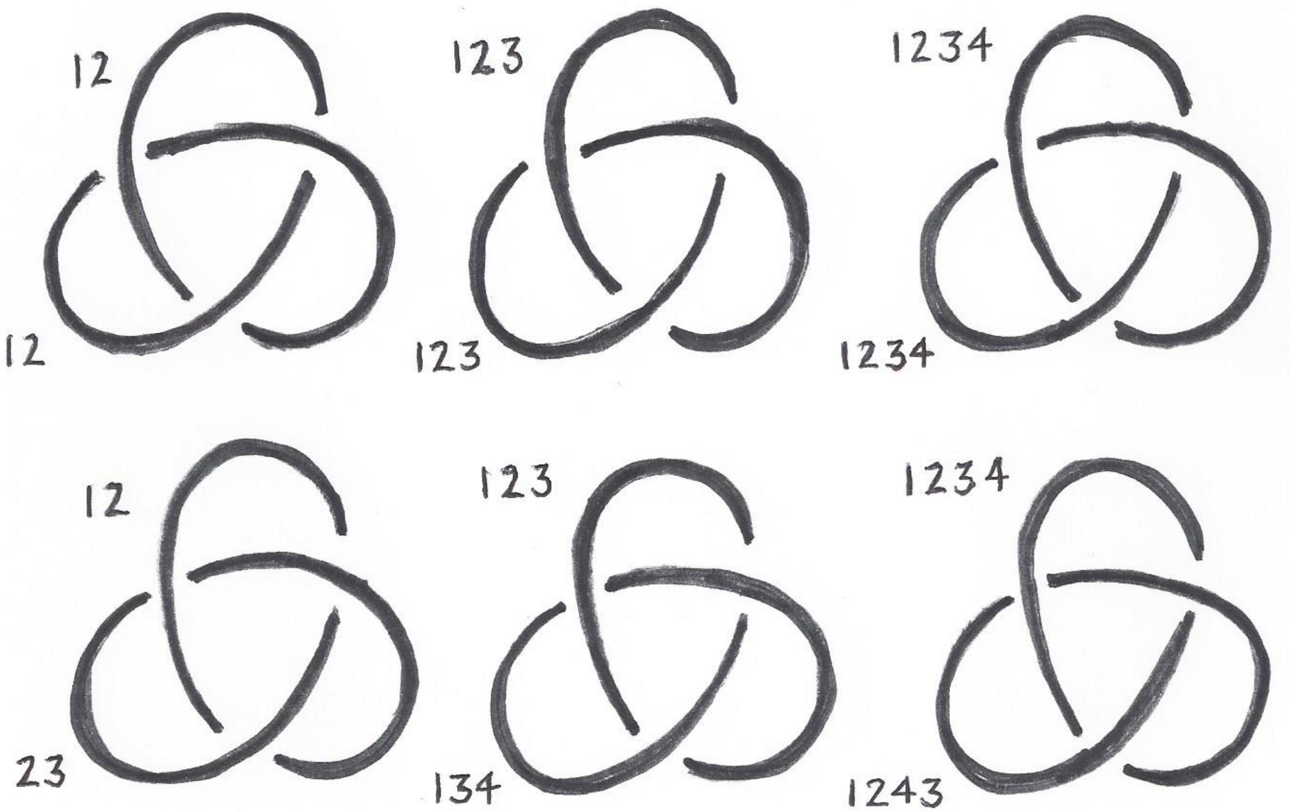
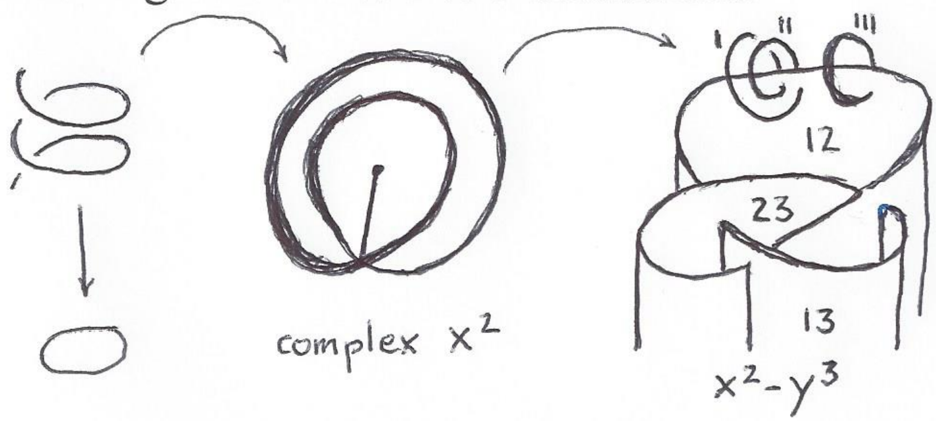
Seifert and Threlfall's "Lehrbuch der Topologie" has a beautiful chapter on covering spaces, at the end of which they give Kneser's examples of the 3- and 4-sheeted covers of the trefoil knot. Adding the double cover, we get:

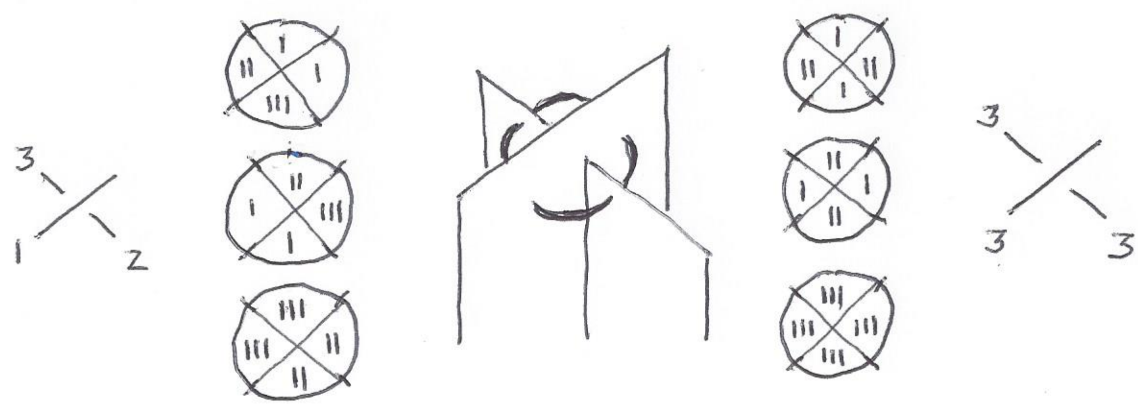


The top three are cyclic and the last is locally cyclic (and octahedral) but the other two (dihedral and tetrahedral) have multiple branch curves and thus, in most cases, linking numbers. We shall show how to calculate them – algebraically, in the general case, and sometimes just by looking at pictures. Relying on the time-honored techniques of legal argument (finger-pointing and arm-waving) ours is not the sort of proof usually offered by mere non-amateur mathematicians.

Branched covering spaces were discovered by Heegaard. His 1898 thesis looked at zeros of the function x^2-y^3 in two complex variables, i.e., $x^2=y^3$. Today's topologists will recognize this as the fundamental group of the complement of the trefoil knot, and that is what Heegaard got by extending Riemann's surfaces to 3-manifolds and projecting sterographically onto S^3 the self-intereections of a 4-manifold immersed in real 6-space. You can read about it (in English) in Moritz Epple's paper in Historia Mathematica 22 (1995), 371-401. The following pictures of covering spaces may help explain how to get from 1 to 2 to 3 dimensions.



The last one is Heegaard's thesis example, as illustrated by Brauner in 1928. The connections between 2-cells in various sheets of this covering are not all that easy to see, so we show below how a little circle under a crossing lifts to three circles in the cover, depending on whether the crossing has just one or all three Fox colors.

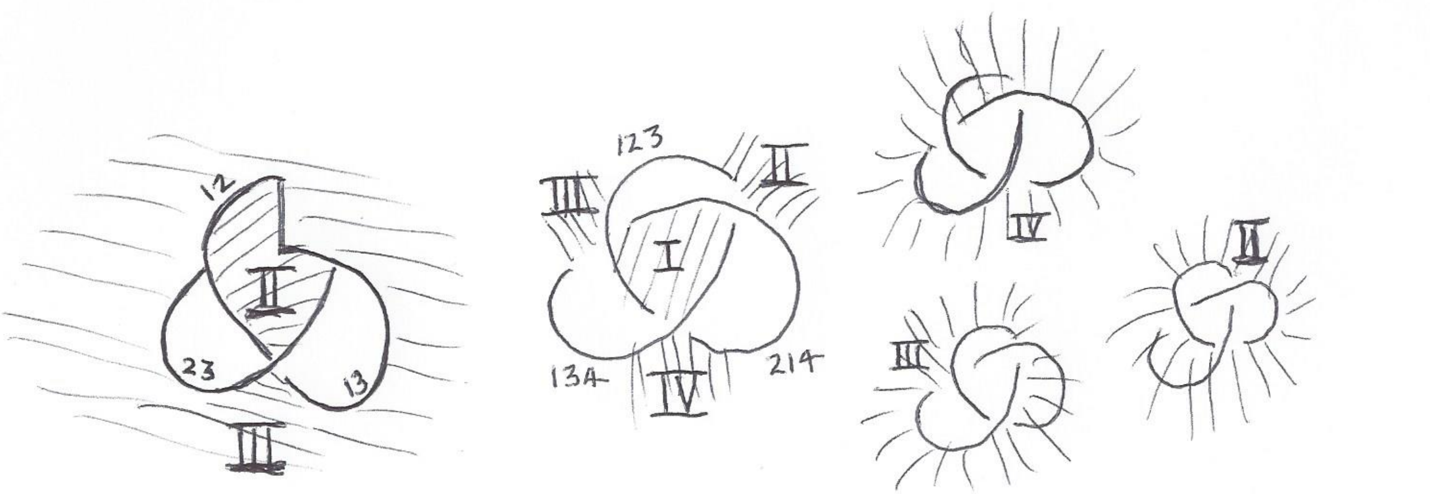


Note that in most cases a lift of this circle passes through four (not just three) different 2-cells of the natural lift of the cellular decomposition that Wirtinger spoke about in 1905. This is because most of those 2-cells connect different sheets of the covering space. Don't be surprised if it takes a while to sort this out in your mind. The three 3-D pictures that comprise the covering space, although superimposed on each other, are nicely ordered (like ducks in a row).

For the general case of an N-sheeted cover, arising from a homomorphism of the knot group onto a transitive permutation group on N letters, it's easy to keep track of what's connected to what by writing on one side of each segment of the knot diagram the permutation that corresponds to the associated meridian. Each letter of each such permutation may them be associated biuniquely with a 2-cell in the cover that sits below that segment, namely the one that can be “seen” from the side of each segment on which permutations are written and from the copy of S^3 that that letter represents. Note that this cellular decomposition of the cover has N 3-cells, CxN 2-cells, CxN vertical 1-cells and another Cx(N-X) 1-cells, running horozontally over the knot, where C is the number of segments in the diagram and X reflects the loss resulting from branching along the knot. These 2-cells, related to each other by the 1-cells that join them, may be used to construct a surface that cobounds any branch curve (or a multiple thereof) where possible. Assigning dummy coefficients to the 2-cells and reading relations off the 1-cells yields a set of linear equations that's either inconsistent or solvable in rational numbers. Tossing out N-1 of them, to create a maximal cave, and working relations through crossings in the style of Wirtiger's knot group algorithm, simplifies this straightforward algebraic task.

To count the intersections of a cobounding surface with some other branch curve, just push the latter a little bit off of the 1-cells it sits in, and off of the 2-cell grid, into the 3-cells that form the various sheets of the cover. Any such resulting curve, kept epsilon-close to the branch, will intersect 2-cells of the cobounding surface precisely in accordance with the linking number. Note that there are countless different ways to do this. Isn't it nice that they all yield the same numerical result.

Now we get to the easy part – "the royal road" to calculating linking numbers. Forget about the cellular decomposition and all that nasty algebra and just look around in the cover for a free-floating surface that cobounds a branch of multiple branching index. This is, of course, a feckless endeavor where homology decrees that no such surface exists, and difficult where only a multiple of such a branch can be cobounded. Still, it's a very pleasant surprise when it works, and it works for both of the examples we started out with, even for the tetrahedral trefoil cover, where the linking number is 3/2.



Attached are 100 3-colored examples and a handful of 5-fold ones. An unexpected payoff from all this fooling around with pictures was a couple of papers relating covering spaces to each other in obvious ways, but yielding some surprising theorems.