THE NEW ELEMENTS OF MATHEMATICS

by

CHARLES S. PEIRCE

Edited by

CAROLYN EISELE

VOLUME IV

MATHEMATICAL PHILOSOPHY

1976

MOUTON PUBLISHERS
THE HAGUE – PARIS

HUMANITIES PRESS
ATLANTIC HIGHLANDS N.J.

5

QUALITATIVE LOGIC (736)

PREFACE

This book is not intended to guide children in their first attempts to use their minds, nor does it address itself particularly to persons of great experience, who, while they may still correct their tendencies to be a little too credulous or too sceptical concerning any kind of evidence, have long since passed the time when any consideration of the theory of drawing inferences could influence their practice,—although such persons may find something to interest them in these pages. But there is an age, between boyhood and manhood, when there is a natural tendency to look at life in a rather theoretical way and when such speculation is of real use and service. The native force of intellect is at that age perhaps as great as it is ever destined to be, but want of skill in handling the reason and inexperience of the deceptions to which it is subject render the effective power of the mind very inferior to that which is later developed. If a young man at that time of life will only acquire a distinct preliminary conception of the methods of thought and of inquiry by which alone the truth can be ascertained, though this theoretical knowledge will not make him a powerful thinker, it will form an admirable foundation upon which to build habits of effective thinking.

To such a young man, I offer an outline sketch of the whole method of reason. If he is disposed to accept what I say with implicit belief, I shall at least try not to abuse his confidence, so that he may not some day wake up to find that it was all an idle and delusive dream. If he feels strong enough to subject what I say to a critical examination, so much the better. I shall endeavor to make my statements and to give my reasons in such a way as to facilitate his investigation.

The name of the study which forms the subject of this volume is Logic. But logic as it is set forth in most books upon the subject, is a study far worse than useless. It tends to make a man captious about trifles, and neglectful of weightier matters. It condemns every inference which is really sound, and admits only those which [are] really childish. The reason is that this logic has been handed down from the Middle Ages, the ages of Faith, i.e. of Unreason, before modern science, physical and moral, had begun. The lore of those days was laborious nonsense, their law was regulated warfare, their medicine fetichism, their history fables, their astronomy astrology, their chemistry alchemy. They never, by any chance, themselves reasoned rightly. How then can they teach us to do so, or how can their theory of reasoning be of the slightest value?

This traditional logic did not, however, originate in the middle ages, but with Aristotle at a comparatively early period in the history of the Greek mind, when it was fully developed upon the artistic side, but before it had done much in the way of discovering truth. Aristotle's logic was a mere first essay; it did not fairly represent Aristotle's own methods of thought,—far less those of the early Greek mathematicians. It did not enjoy, in antiquity, the immense renown which has since attached to it; although it did faithfully mirror the distrust which the ancients generally entertained of Observation as a foundation for scientific truth. Those who did believe in observation, the Epicureans,—like Roger Bacon in the 13th Century,—were hostile to the Aristotelean logic.

In modern times, logic has very naturally had a bad reputation. First, there was in the 16th Century a pretended reform of the science in the interest mainly of literary elegance, by Peter Ramus. He imported the Dilemma from rhetoric into logic. Early in the 17th Century came two important books, the *Discours de la méthode* of Descartes, and the *Novum organum* of Francis Bacon. The former represents, in a very vague sketch, but with great perspicuity, the methods of thinking of modern metaphysicians, of those who draw their convictions from within. The latter is an eloquent and majestic assertion of the dignity of nature and the littleness of man, who can only attain knowledge by observation. All the greatest steps in the progress of modern science have involved improvements in the art of reasoning. Harvey's discovery of the circulation [of] the blood, Kepler's researches on the orbits of the planets, Galileo's development of the principle of inertia, and many other such works contained lessons in logic,—which were perhaps even more valuable to the world and contributed even more to the progress of civilization than their more special teachings. This has been particularly the case with discoveries in mathematics. Pure mathematics, indeed, is nothing but an art of drawing conclusions of a particular description. But the great discoveries in mathematics have carried with them very important improvements in the methods of thinking concerning nearly every subject. In ancient times, the discovery by Euclid that the whole of Geometry can be deduced from a few elementary principles suggested the idea of metaphysics and profoundly modified all subsequent science and philosophy. In modern times, several mathematical discoveries have added even more momentous consequences. The Coördinates of Descartes are now constantly applied to every subject of reasoning with great advantage; while the ideas of the infinitesimal calculus have penetrated everywhere, forming, for example, an important not to say the principal factor in Ricardo's political economy.

The doctrine of chances is a direct contribution made by modern mathematicians to the general principles of logic. It may be called the modern logic for it decidedly outweighs all that was known of reasoning before this invention.

The present century has produced important treatises upon the theory of logic, which have all come either from Germany or from Great Britain. Two different schools have prevailed in the two countries, which have not in the least understood one another. The German school, of which Hegel is the type, have approached the subject from the side of theology and metaphysics; the English school, represented, for example, by Mill, have viewed the matter from the side of modern science. But there is good reason to trust that the breach between those two schools, which is but the continuation of a dispute existing almost since the birth of philosophy, is now in the process of being closed and healed.

CHAPTER III
THE MODUS PONENS

When the validity of the simple consequence, "*P*, therefore, *C*," is challenged, we are led to search for the principle upon which the inference proceeds; and perhaps we may come to recognize this in the judgment that if *P* is true, *C* is true. It would be a mistake to suppose that is always even tacitly assumed in the simple consequence; for it may be the manner in which the premise, *P*, presents itself to my notice, and not its bare truth alone, which causes my judgment that *C* is true. [In] any case, in drawing the simple consequence, "*P*, therefore *C*," I do not consciously judge that "If *P*, then *C*." But in the after criticism of the inference, I shall be apt to recognize this proposition as its principle; and from that moment my reasoning ceases to be a simple consequence, because the proposition "If *P*, then *C*," now becomes a second premise. In short, I now reason in the form

> If *P* is true, *C* is true;
> But *P* is true;
> Hence, *C* is true.

This form of inference is called the *modus ponens* (or positing mode, to distinguish it from the *modus tollens*, or removing mode, which will be noticed hereafter). *P* is called the *antecedent*, *C* the *consequent*. The hypothetical proposition is called the *consequence*.

For example, a little girl might reason,

> If I am good, my mamma will love me;
> Now, I will be good;
> And so my mamma will love me.

We have seen that even in drawing a simple consequence, I have a vague consciousness of a body of possible inferences to which the inference actually drawn belongs. In the *modus ponens* the conception of the Possible emerges more clearly. We have here, as one of the premises, a judgment, "If *P*, then *C*," which is not *categorical*, that is, does not relate to the real world alone, but is *hypothetical*, that is, is meant to apply to a whole universe of possibilities. To say that a fact is *possible* means primarily that it is not known not to be true, that our knowledge leaves room for it. But there are, besides, kinds of possibility, which are determined not by the knowledge which we happen just now to have, but by every conceivable state of information. Thus, we say, Napoleon did not win the battle of Waterloo, but he might possibly have done so. We use the past tense, *might*, because the fact supposed is now no longer consistent with what we have learned; but we mean that a knowledge of all the previous conditions, capable of being known beforehand, would leave room for such an event. We say that it is physically possible that a needle should be so balanced as to rest upright upon its point. We mean that if we knew only the laws of physics and not various familiar facts, we should not know that this did not happen. We say that it is impossible for anything in nature to happen otherwise than it does happen. We mean that if we knew all past facts and all the laws of nature and all that could be deduced therefrom we should know everything that was about to happen. There is therefore a sense in which only the actual is possible. At the furthest extreme from that is the logically possible. A proposition is logically possible when if we knew no facts at all, but only the meanings of words, we could not reject the proposition. It is true that such a state of knowledge is itself in a certain sense impossible, like a geometrical line or surface; but it is none the less a very useful conception.

A supposed fact which would be true in *some or all* the states of things for which an assumed condition of knowledge leaves room is said to be *possible*; one which is true in none of these states is said to be *impossible*;

one which is true in all is said to be *necessary*; one which is true in a part or none is said to be *contingent*.

To say that "If *P* is true, *C* is true," means that in an assumed condition of information, every possible state of things in which *P* would be true is a state of things in which *C* would be true.

Every addition or improvement to our knowledge, of whatsoever kind, comes from an exercise of our powers of perception. In necessary inference my observation is directed to a creation of my own imagination, a sort of diagram or image in which are portrayed the facts given in the premises; and the observation consists in recognizing relations between the parts of this diagram which were not noticed in constructing it. Different persons no doubt construct their logical diagram in different ways; many probably very oddly; but every person must construct some kind of a diagram or its equivalent, or he could not perform necessary reasoning, at all. It is a part of the business of logic to teach useful ways of constructing such diagrams. The whole range of logical possibility may be represented by an imaginary sheet of paper occupying the whole field of vision. Every point on this sheet is to represent some conceivable state of things, of which the real state of things is one, undistinguished from the rest. Everything learned cuts off and removes from this sheet some part and leaves a range of possibility less than the whole range of logical possibility, but still containing the unknown point which represents the actual state of things. To form a diagram of the truth of the hypothetical proposition "If *P* is true, *C* is true," we must suppose that all the points which represent states of things in which *P* is true are gathered together in one area, and that a line is drawn around this area; and that the same is done with the points representing states of things in which *C* is true. If then the boundary of the area of the truth of *P* is imagined to lie wholly within the area of the truth of *C*, the hypothetical proposition is represented as true. If this happens before the original sheet has been cut down at all, the proposition is represented as true by logical necessity; but if the sheet has been cut down it is only represented to be true for some degree of knowledge not defined.

Let us now suppose that, in addition to the hypothetical proposition, we are certain that the antecedent, *P*, is true. Then, in our state of knowledge it is not possible for *P* to be false; and to represent our knowledge, we must cut down the sheet of possibility so as to leave only a part lying wholly within the area of *P*. We perceive that this will also lie within the area of *C*, and therefore we shall be certain that the consequent is true.

CHAPTER VI
THE LOGICAL ALGEBRA OF BOOLE

The actual state of things is represented by a single point in the field of possibility; and the reasonings above treated have been concerned with this single point. Categorical propositions, it is true, in one sense refer to whole classes of individuals, but they only take these classes *distributively*, that is to say, they only consider characters as belonging to any one or some one individual of such a class, not as relations between several individuals.

Every qualitative reasoning about single individuals may be analyzed into syllogisms and dilemmas; but it is beyond the power of ordinary language to state complicated arguments of this kind with perspicuity or precision. In the case of such complicated reasoning,—and indeed wherever the logic of our inferences requires to be analyzed,—there is a great saving of trouble and gain of accuracy in employing a logical "algebra," or perfectly systematic written language or body of symbols in which the premises being expressed, the conclusion may be obtained by transformations of the expressions according to formal rules.

I speak of a body of symbols, but in point of fact, syllogism and dilemma, every qualitative reasoning about individuals may be expressed by use of one single symbol besides the expressions for the facts whose relations are examined. This symbol must signify the relation of antecedent to consequent. In the form I would propose for it, it takes the shape of a cross placed between antecedent and consequent with a sort of streamer extending over the former. Thus, "If a, then b," would be written

$$\overline{a} \dashv b.$$

"From a, it follows that if b then c," would be written

$$\overline{a} \dashv \overline{b} \dashv c.$$

"From 'if a, then b,' follows c," would be written

$$\overline{a} \dashv \overline{b} \dashv c.$$

To say that a is false, is the same as to say that from a as an antecedent follows any consequent we like. This is naturally shown by leaving a blank space for the consequent, which may be filled at pleasure. That is, we may write "a is false"

$$\overline{a} \dashv \, ,$$

implying that from a every consequence may be drawn without passing from a true antecedent to a false consequent, since *a* is not true. Instead of the blank space we might use a special symbol, say a circle, and write

$$\overline{a \prec}\ \mathrm{o}.$$

By means of this sign of illation, or *copula*, we may express the most complicated relations without the slightest ambiguity. Thus, we may write

$$\overline{\overline{\overline{b \prec} c \prec} m \prec} n \prec \overline{\overline{\overline{\overline{a \prec} b \prec} l \prec} m \prec} \overline{\overline{\overline{a \prec} c \prec} l \prec} n$$

Let, for instance, *a* mean that any object is either Enoch or Elijah, *b* that that object is a man, *c* that that object is mortal, *l* that Enoch and Elijah were not snatched up to heaven, *m* that the bible errs, *n* falsity. Then what is written signifies that if it be false that all men's being mortal would imply that the Bible errs, and if the Bible would err were the fact of Enoch and Elijah being men to imply that they were not snatched up to heaven, then it is false that the fact of Enoch and Elijah being mortal implies that they were not snatched up to heaven. Thus, by some strain we can express such a proposition in ordinary language. But to express the *general* proposition about *a*, *b*, *c*, *l*, *m*, *n*, we should have to say

If {If [If (If *a*, then *c*), then *m*], then *n*},
then (if {if [if (if *a*, then *b*) then *l*], then *m*},
then if [if (if *a*, then *c*) then *l*], then *n*.).

But when we write the algebraic formula and desire to pronounce it, we may do so in this way,—the rule being to pronounce "if" at the beginning of each streamer and "then" at each cross. The braces, brackets, and parentheses, are quite unnecessary; and the *ifs* and *thens* may be pronounced rapidly and without pause, because the symbol exhibits the meaning.

Here then we have a written language for relations of dependence. We have only to bear in mind the meaning of the symbol $\overline{\;\prec}$ (not by translating it into *if* and *then*, but by associating it directly with the conception of the relation it signifies), in order to reason as well in this language as in the vernacular,—and, indeed, much better.

So far, we have a *language* but still no *algebra*. For an algebra is a language with a code of formal rules for the transformation of expressions, by which we are enabled to draw conclusions without the trouble of attending to the meaning of the language we use.

Our algebraic rules must enable us to prove the two propositions,

If $\overline{a \prec}\, b$, then if *a* then *b*; and
If from *a* follows *b*, then $\overline{a \prec}\, b$.

Any rules which will prove these propositions will evidently enable us [to] prove every conclusion. But that is not enough; for we require that the rules should enable us to dispense with all reasoning in our proofs except the mere substitution of particular expressions in general formulae. We do not as yet demand rules which shall enable us to dispense with difficult reasoning in discovering the truth and in inventing modes of proof,—that would be demanding more than an algebra, namely a *calculus*,—but we do require that in the proofs themselves nothing but simple substitutions shall be called for. The first of the above propositions, however, namely, that if a and $\overline{a} \prec b$ then b, being nothing but the *modus ponens*, might stand as a fundamental rule. The second proposition may be divided into two, each of which follows from it while it follows from them dilemmatically. They are

If not a, then $\overline{a} \prec b$; and
If b, then $\overline{a} \prec b$.

In making our code of laws, we shall first want a rule to show what substitutions can be made; next, we must give the properties of denial; and finally we need a general test of necessary truth.

I. The general rule of substitution is that if $\overline{m} \prec n$, then n may be substituted for m under an even number of streamers (or under none), while under an odd number m may be substituted for n.

This may be proved true by the peculiar kind of reasoning invented by Fermat. Any expression in which m is under an even number of streamers may be written in one of the forms of the two infinite series here exhibited

$$
\begin{array}{ll}
m & \overline{s} \prec m \\
\overline{\overline{m} \prec A} \prec a & \overline{\overline{\overline{s} \prec m} \prec A} \prec a \\
\overline{\overline{\overline{\overline{m} \prec A} \prec a} \prec B} \prec b & \overline{\overline{\overline{\overline{\overline{s} \prec m} \prec A} \prec a} \prec B} \prec b \\
\text{etc.} & \text{etc.}
\end{array}
$$

The rule holds for the first form of the first series by the *modus ponens*, and for the first form of the second series by syllogism. If in either series the rule holds from the first member up to any form for which we may write M, then let this form after the substitution of n for m become N. Then according to the rule, $\overline{M} \prec N$. The next form of the series after M will be say $\overline{\overline{M} \prec I} \prec i$. Now by dilemmatic reasoning, we have

But this conclusion is the result of substituting n for m in $\overline{\overline{M} \prec I} \prec i$. Thus we see that the rule is true of the first form of each series, and if of any form true of the next in the same series; consequently, it is true of all forms which can be built up in this way.

In precisely the same way, any proposition in which n is under an odd number of streamers, may be written in one of the forms

$$\overline{n} \prec p \qquad\qquad \overline{\overline{s} \prec n} \prec p$$

$$\overline{\overline{\overline{n} \prec p} \prec A} \prec a \qquad\qquad \overline{\overline{\overline{\overline{s} \prec n} \prec p} \prec A} \prec a$$

$$\overline{\overline{\overline{\overline{\overline{n} \prec p} \prec A} \prec a} \prec B} \prec b \qquad\qquad \overline{\overline{\overline{\overline{\overline{\overline{s} \prec n} \prec p} \prec A} \prec a} \prec B} \prec b$$

etc. etc.

The rule holds of the first form of the first series by syllogism. It holds of the first form of the second series, because

$$\overline{\overline{\overline{s} \prec m} \prec \overline{s}} \prec m$$

is true identically; and taking this as a premise of a syllogism of which the other is $\overline{m} \prec n$, we have the conclusion

$$\overline{\overline{\overline{s} \prec m} \prec \overline{s}} \prec n,$$

and this taken with $\overline{\overline{s} \prec n} \prec p$ gives by another syllogism

$$\overline{\overline{s} \prec m} \prec p.$$

If the rule holds for any form N, let this after the substitution be M, and we have by the rule $\overline{N} \prec M$. The next form of the same series is $\overline{\overline{N} \prec I} \prec i$ and by the same dilemma as before we prove that $\overline{\overline{M} \prec I} \prec i$, and consequently that the rule is correct.

II. For the circle which is the symbol of falsity (or that all propositions are true), we have the general formula

$$\overline{\circ} \prec a$$

whatever a may be. The falsity of a, usually written $\overline{a} \prec$ is really equivalent to $\overline{a} \prec \circ$.

The formula of identity $\overline{a} \prec a$, for which we may write ∞ for short, has properties conjugate to those of the circle. Namely, we have the general formula

$$\overline{b} \prec \overline{a} \prec \overline{a} \prec \quad \text{or} \quad \overline{b} \prec \infty,$$

for this is a case under the general principle that if B, then $\overline{A} \prec B$, since $\overline{a} \prec a$ is necessarily true. The truth of b may be written

$$\overline{\infty} \prec b;$$

for if b is true, so is this by the general principle just cited, and if this is true, so is b by the *modus ponens*.

III. If a proposition is true when the circle is substituted for one of its letters or terms, and also when ∞ is substituted for the same term, then the proposition is true in its original form; for the term replaced must be either true or false.

Besides these fundamental rules, there are a number of others which may be deduced from them. Among these are the following:

IV. The circle may be substituted for any term under an odd number of streamers, and ∞ for any term under an even number or none.

V. If a proposition has its antecedent false or its consequent true, it is true; and conversely, if its antecedent is true while its consequent is false, the proposition is false. That is to say,

If b, then $\overline{a \prec}\, b$
If $\overline{a \prec}$, then $\overline{a \prec}\, b$
If $\overline{a \prec}\, b$, then either $\overline{a \prec}$ or b.*

(*It must be carefully borne in mind that all our discourse in this part of the book is about individuals. Give $\overline{a \prec}\, b$ a categorical form, and the proposition stated seems false. Namely, it does not follow that because all men are mortal, therefore either every object is a non-man or else that every object is mortal; but it does follow that each single individual is either not a man or is a mortal.)

The first proposition follows by the rule of substitution from $\overline{a \prec}\, \infty$ and $\overline{\infty \prec}\, b$, which are true by Rule II. The second proposition follows from $\overline{a \prec}\, \circ$ and $\overline{\circ \prec}\, b$. The third may be proved by Rule III, as follows. First, put $a = \infty$, $b = \infty$, and the proposition becomes

If $\overline{\infty \prec}\, \infty$, then either $\overline{\infty \prec}$ or ∞, i.e. $\overline{a \prec}\, a$

Now $\overline{a \prec}\, a$ is proved by Rule III, for both

$\overline{\infty \prec}\, \infty$ and $\overline{\circ \prec}\, \circ$

are true by Rule II. Second, put $a = \circ$, $b = \circ$, and the proposition becomes

If $\overline{\circ \prec}\, \circ$, then either $\overline{\circ \prec}$ (that is, $\overline{\circ \prec}\, \circ$) or $\circ$,

and the first alternative is true, by Rule II. Third, put $a = \circ$, $b = \infty$, and the proposition becomes

If $\overline{\circ \prec}\, \infty$, then either $\overline{\circ \prec}\, \circ$ or $\overline{a \prec}\, a$.

Fourth, put $a = \infty$, $b = \circ$. In this case, the proposition becomes

Now if $\overline{\infty} \prec \mathrm{o}$ then $\overline{\infty} \prec \mathrm{o}$ is a case under $\overline{a} \prec a$.

VI. We have necessarily

$$\overline{\overline{a} \prec \overline{b} \prec c} \prec \overline{b} \prec \overline{a} \prec c\,,$$

so that, by the *modus ponens*,

$$\overline{a} \prec \overline{b} \prec c = \overline{b} \prec \overline{a} \prec c$$

and antecedents can be transposed. This [is] proved by Rule III, for

$\overline{\overline{\mathrm{o}} \prec \overline{\mathrm{o}} \prec c} \prec \overline{\mathrm{o}} \prec \overline{\mathrm{o}} \prec c$ is true, since $\overline{\mathrm{o}} \prec c$ is true;
$\overline{\overline{\mathrm{o}} \prec \overline{\infty} \prec c} \prec \overline{\infty} \prec \overline{\mathrm{o}} \prec c$ is true for the same reason;
$\overline{\overline{\infty} \prec \overline{\mathrm{o}} \prec c} \prec \overline{\mathrm{o}} \prec \overline{\infty} \prec c$ is true, since $\overline{\mathrm{o}} \prec \overline{\infty} \prec c$ is true;
$\overline{\overline{\infty} \prec \overline{\infty} \prec c} \prec \overline{\infty} \prec \overline{\infty} \prec c$ is true by identity.

VII. We have necessarily

$$\overline{\overline{\overline{\overline{a} \prec b} \prec c} \prec d} \prec \overline{a} \prec \overline{\overline{b} \prec c} \prec d,$$

so that by the *modus ponens*,

If $\overline{\overline{\overline{a} \prec b} \prec c} \prec d$, then $\overline{a} \prec \overline{\overline{b} \prec c} \prec d$,

and the ends of two streamers terminating together can be cut off to any point together. This is conveniently proved by a *reductio ad absurdum* by means of Rule V. The proposition can only be false if

(1) $\overline{\overline{\overline{a} \prec b} \prec c} \prec d$ is true, while
(2) $\overline{a} \prec \overline{\overline{b} \prec c} \prec d$ is false.

The second can be false only if

(3) a is true, while
(4) $\overline{\overline{b} \prec c} \prec d$ is false.

The fourth can be false only if

(5) $\overline{b} \prec c$ is true, while
(6) d is false.

But if (1) is true while d is false,

(7) $\overline{\overline{a} \prec b} \prec c$ is false, and therefore
(8) $\overline{a} \prec b$ is true, while
(9) c is false.

But if c is false, while (5) is true,

(10) b is false.

And if b is false, while (8) is true, a is false, which contradicts (3).

VIII. Suppose two propositions such that when any term or terms are replaced by the circle, one proposition becomes false, while when the same terms are replaced by identity, $\overline{a}\dashv a$, the other proposition becomes true, then the latter follows from the former, both in their original forms. For let A be the proposition which becomes ○, and let A' be what it becomes when identity is substituted. Let B be the proposition which becomes true, and let B' be what it becomes when the circle is substituted. Then the proposition $\overline{A}\dashv B$, becomes in the two cases $\overline{\bigcirc}\dashv B'$ and $\overline{A'}\dashv \infty$, both of which are true; so that $\overline{A}\dashv B$ is true by Rule III.

IX. We have necessarily

$$\overline{\overline{A}\dashv\overline{B}\dashv C}\dashv\overline{\overline{C}\dashv L}\dashv\overline{A}\dashv\overline{B}\dashv L,$$

so that from

$$\overline{A}\dashv\overline{B}\dashv C \text{ follows } \overline{\overline{C}\dashv L}\dashv\overline{A}\dashv\overline{B}\dashv L.$$

For if either A or B is replaced by the circle the proposition is true. If both are replaced by ∞ and c by the circle, $\overline{A}\dashv\overline{B}\dashv C$ is false, and again the proposition is true. If C is ∞ and L is ○, $\overline{C}\dashv L$ is false and the proposition is true, while if L is true the proposition is true.

X. Any two premises a and b may at once be united in the form $\overline{\overline{a}\dashv\overline{b}\dashv}\dashv$. For if the two premises are true, this form becomes $\overline{\overline{\infty}\dashv\overline{\infty}\dashv\bigcirc}\dashv\bigcirc$ which is true because $\overline{\infty}\dashv\overline{\infty}\dashv\bigcirc$ is false.

Rule I

If a and b, then $\overline{\overline{a}\dashv\overline{b}\dashv}\dashv$.

This is the *rule of combination.* It is plain that we cannot reason, unless we can combine different premises; and $\overline{\overline{a}\dashv\overline{b}\dashv}\dashv$ expresses no more than that that a and b are both true at once. To show this, we use a diagram invented by Mr. Venn [Fig. 1].

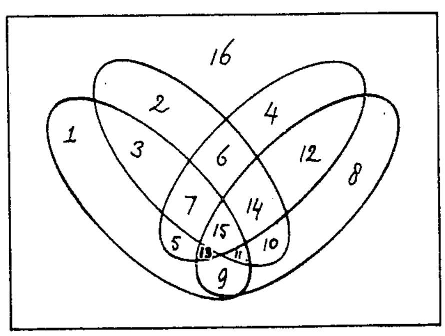

Fig. 1

The rectangle represents the whole field of possibility. The oval containing the odd numbers represents the area of possibility of *a*. The oval containing the even numbers not divisible by 4 and the odd numbers one less than [a multiple of] 4 (2, 6, 10, 14, and 3, 7, 11, 15) represents the area of possibility of *b*. The oval containing the numbers 4 to 7 and 12 to 15 inclusive represents the area of possibility of another fact *c*; and finally the oval containing the numbers from 8 to 15 inclusive represents the area of possibility of a fourth fact *d*. The proposition $\overline{b \prec}\, c$ is true wherever *b* is not true or *c* is true, that is everywhere except in the compartments 2, 3, 10, 11. The proposition $\overline{a \prec}\, \overline{b \prec}\, c$ is true wherever $\overline{b \prec}\, c$ is true or *a* is not true, that is, everywhere except in compartments 3 and 11.

Finally, the proposition $\overline{\overline{a \prec}\, \overline{b \prec}\, c \prec}\, d$ is true wherever $\overline{a \prec}\, \overline{b \prec}\, c$ is not true or [*d*] is true, that is, in compartment 3 and all those numbered from 8 to 15 inclusive. Now the last expression $\overline{\overline{a \prec}\, \overline{b \prec}\, c \prec}\, d$ becomes $\overline{\overline{a \prec}\, \overline{b \prec} \prec}$ on substituting o for *c* and *d*. That is, we must erase the ovals *c*, and *d*. Then there remain only the compartments 1, 2, 3, 16, of which 3 is that where *a* and *b* are both true, and also that where $\overline{\overline{a \prec}\, \overline{b \prec} \prec}$ is true.

Rule II

$\overline{a \prec}\, \overline{b \prec}\, b$ is necessarily true.

This is the *rule of identity*. To prove it, we note as above that in the diagram the proposition $\overline{a \prec}\, \overline{b \prec}\, b$ is true everywhere except in the compartments 3 and 11. Now make $c = b$, that is, erase all compartments where either extend beyond the other, that is, compartments 2 to 5 and 10 to 13. Then we have erased 3 and 11 and consequently all where $\overline{a \prec}\, \overline{b \prec}\, c$ is not true. So that $\overline{a \prec}\, \overline{b \prec}\, b$ is true everywhere.

Rule III

$\overline{a \prec}\, \overline{b \prec}\, c = \overline{b \prec}\, \overline{a \prec}\, c$.

This is the *rule of commutation*. We have seen that $\overline{a \prec}\, \overline{b \prec}\, c$ is true everywhere except in compartments 3 and 11, which are symmetrically placed with reference to the ovals of *a* and *b*. Consequently, *a* and *b* can be transposed in the proposition without change of meaning.

Rule IV

If $\overline{\overline{a \prec}\, \overline{\overline{a \prec}\, c \prec} \prec}$, then *c*.

This is the rule of the *modus ponens*. The proof of the rule of combination, by putting $\overline{a \prec}\, c$ in place of *b*, shows that $\overline{\overline{a \prec}\, \overline{\overline{a \prec}\, b \prec} \prec}$ expresses the conjoint truth of a and $\overline{a}\,\rceil\!+ c$. Consequently, by the *modus ponens*, c follows from it.

Rule V

If $\overline{\overline{\overline{a}\,\rceil\!+ b}\,\rceil\!+ \overline{\overline{b}\,\rceil\!+ c}\,\rceil\!+}\,\rceil\!+$, then $\overline{a}\,\rceil\!+ c$.

This is the *rule of syllogism*. It is proved simarly to Rule IV.

Rule VI

If $\overline{b}\,\rceil\!+$, then $\overline{b}\,\rceil\!+ c$.

This is the *rule of contradiction*. It serves to state that negation is equivalent to saying that every consequent follows from the fact denied. It is proved by the diagram which shows that $\overline{b}\,\rceil\!+ c$ is true over the whole field of possibility except a part of the area of b. If therefore the actual state of things is a point outside of b, it is within the area of the truth of $\overline{b}\,\rceil\!+ c$.

Rule VII

If $\overline{\overline{\overline{a}\,\rceil\!+ \overline{b}\,\rceil\!+ c}\,\rceil\!+ \overline{\overline{a}\,\rceil\!+ c}\,\rceil\!+}\,\rceil\!+$, then c.

This is the *rule of dilemma*. It is proved like Rules V and VI.

We now have a complete algebra for qualitative reasoning concerning individuals. But it is not yet a very commodious calculus. To render it so, we introduce certain abbreviations which make it identical with the logical algebra of Boole as modified by Jevons and Mitchell.* (*Other modifications by Mr. Mitchell relate to the logic of relatives.) Namely, we first separate the streamer of the sign of illation from the cross, and in place of

$$\overline{a}\,\rceil\!+ b \qquad \text{write} \qquad \overline{a} + b.$$

Second, wherever the sign of illation is followed by a blank we omit the cross, and thus

in place of $\overline{a}\,\rceil\!+$, write $\overline{a}$.

Third, as the sign of the simultaneous truth of a and b, instead of writing $\overline{\overline{a}\,\rceil\!+ \overline{b}\,\rceil\!+}\,\rceil\!+$, or $\overline{\overline{a} + \overline{b}}$, we write simply ab. This last is a superfluous sign, adopted for the sake of abbreviation. Our Seven Rules now take the following form.

I. *Rule of combination*. If a and b, then $\overline{\overline{a} + \overline{b}} = ab$.

II. *Rule of identity*. $\overline{b} + \overline{a} + a$.

III. *Rule of commutation.* $\overline{a} + \overline{b} + c = \overline{b} + \overline{a} + c$, or $\overline{a} + \overline{b} = \overline{b} + \overline{a}$.
IV. *Rule of the modus ponens.* If $a(\overline{a} + c)$, then c.
V. *Rule of Syllogism.* If $(\overline{a} + b)(\overline{b} + c)$, then $\overline{a} + c$.
VI. *Rule of contradiction.* If $\overline{b}$, then $\overline{b} + c$.
VII. *Rule of dilemma.* If $(a\overline{b} + c)(\overline{a} + c)$, then c.