Adjoint Theorem and Cramer's Rule

A any $n \times n$ matrix.

$B = \mathrm{adj}(A)$: $B_{ij} = (-1)^{i+j} |M_{ij}(A)|$

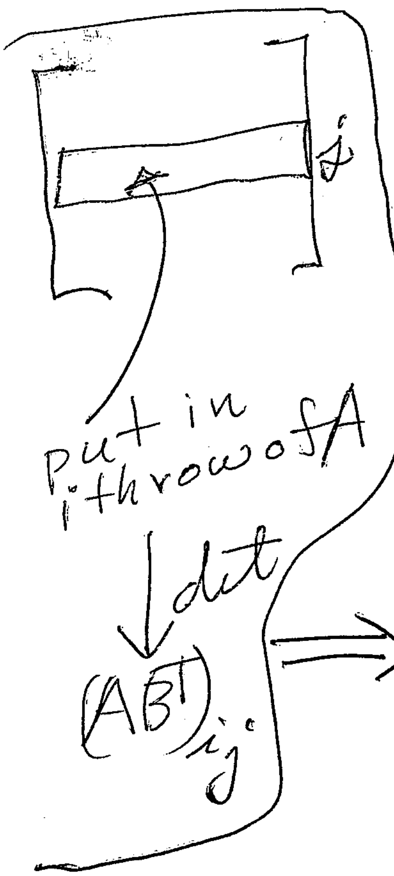
where $M_{ij}(A)$ is the ij minor of A.

Adjoint Theorem. $AB^T = \det(A)\, I$.

(Thus $\frac{1}{\det(A)} B^T = A^{-1}$ when $\det(A) \neq 0$.)

Proof. $(AB^T)_{ij} = \sum_k A_{ik} (B^T)_{kj} = \sum_k A_{ik} B_{jk}$ note

$$= \sum_k A_{ik} (-1)^{i+k} |M_{jk}(A)|$$



$=$ det [The result of putting the ith row of A into the jth row of A and leaving the rest of A alone.]

$$= \begin{cases} \det(A) & i = j \\ 0 & i \neq j \end{cases}$$ (det of a matrix with 2 equal rows is zero!)

$\Longrightarrow (AB^T) = \det(A)\, I$ //

Let A be any $n \times n$ matrix and suppose $\det(A) \neq 0$.

Consider the system of equations

$A\vec{x} = \vec{k}$ for some const vector $\vec{k}$.

Then $A^{-1}A\vec{x} = A^{-1}\vec{k}$

$$\Rightarrow \quad \vec{x} = A^{-1}\vec{k} = \frac{1}{|A|} B^T \vec{k}$$

$$\Rightarrow \quad x_i = \frac{1}{|A|}(B^T\vec{k})_i$$

$$= \frac{1}{|A|}\sum_k (B^T)_{ik} \vec{k}_k \qquad \left(\begin{array}{l} i^{th} \text{ row of } B^T \\ \text{dotted with } \vec{k} \end{array}\right)$$

$$= \frac{1}{|A|}\sum_k (-1)^{i+k} \vec{k}_k |M_{ki}(A)|$$

CRAMER'S RULE

$$\boxed{x_i = \frac{1}{|A|}\det(A(\vec{k}, i))}$$

where $A(\vec{k}, i)$ = the matrix obtained from A by replacing i^{th} col of A by $\vec{k}$.

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e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \Rightarrow x_1 = \begin{vmatrix} k_1 & b \\ k_2 & d \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$x_2 = \begin{vmatrix} a & k_1 \\ c & k_2 \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

e.g. $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix}$, $\vec{k}=\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$

$$A\vec{x}=\vec{k}$$

$$\Rightarrow x_1 = \begin{vmatrix} k_1 & b & c \\ k_2 & e & f \\ k_3 & h & l \end{vmatrix} \Big/ |A|$$

$$x_2 = \begin{vmatrix} a & k_1 & c \\ d & k_2 & f \\ g & k_3 & l \end{vmatrix} \Big/ |A|$$

$$x_3 = \begin{vmatrix} a & b & k_1 \\ d & e & k_2 \\ g & h & k_3 \end{vmatrix} \Big/ |A|$$

$A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad A\vec{x}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, Note $|A|=1$

$$\Rightarrow x_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -3 & -2 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3 & -2 \end{vmatrix} = 2-3=-1$$

$$x_2 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2-3=-1$$

$$x_3 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 3 \quad \checkmark$$

Characteristic Polynomial

CP1

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad tr(A) = a+d$$

$$B = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$|B| = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$C_\lambda(A) = \lambda^2 - tr(A)\lambda + |A|$$

"characteristic poly of A".

$$(\lambda-2)(\lambda-3) = \lambda^2 - 5\lambda + 6$$

$$e.g.\ A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\text{or} \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} = A$$

Claim: $A^2 = 5A - 6I$

compare: $\lambda^2 = 5\lambda - 6$

or $\lambda^2 - 5\lambda + 6 = 0$

$C_\lambda(A) = 0$

$$A^2 = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -10 \\ 5 & 14 \end{pmatrix}$$

$$5A - 6I = \begin{pmatrix} 5 & -10 \\ 5 & 20 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -10 \\ 5 & 14 \end{pmatrix} \checkmark$$

Thm. If A is 2×2.

then $\boxed{A^2 = tr(A)A - det(A)I}$

Proof. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$tr(A) = a+d$

$det(A) = ad - bc$

...

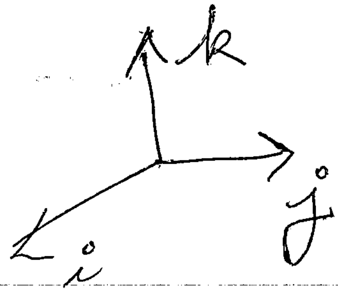
$\boxed{\text{You do this!}}$

//

Cayley Hamilton Theorem says "Any matrix A is a "root" of its own characteristic polynomial."

VC1

Vector Cross Product



$$i = \begin{bmatrix}1\\0\\0\end{bmatrix} = e_1$$

$$j = \begin{bmatrix}0\\1\\0\end{bmatrix} = e_2$$

$$k = \begin{bmatrix}0\\0\\1\end{bmatrix} = e_3$$

$$\boxed{\begin{aligned} i\times j &= k\\ j\times k &= i\\ k\times i &= j \end{aligned}}$$

$$\boxed{i\times i = j\times j = k\times k = 0}$$

Each of $x, y = i, j, k$

$$\boxed{x\times y = -y\times x}$$

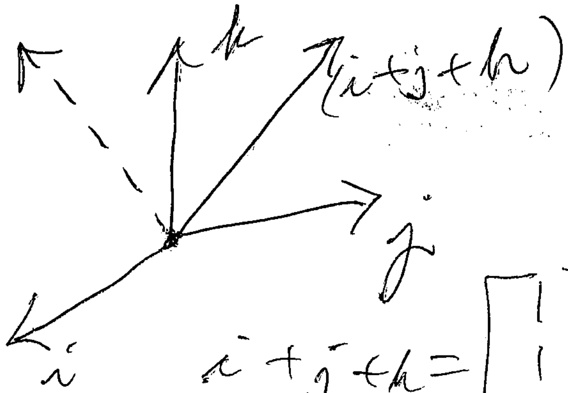
$(a+b)\times c$
$= a\times c + b\times c$

$$(i+j+k)\times j = i\times j + j\times j + k\times j$$

$$= k + 0 + (-i)$$

$$= k - i$$

$$= -i + k = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$



$i+j+k = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $j = \begin{bmatrix}0\\1\\0\end{bmatrix}$

V(2)

$i \times j = k$
$j \times k = i$
$k \times i = j$

$v = ai + bj + ck$
$w = di + ej + fk$

$$v \times w = (ai + bj + ck)(di + ej + fk)$$

$$= ae\, i \times j + af\, i \times k$$
$$bd\, j \times i \qquad + bf\, j \times k$$
$$cd\, k \times i + ce\, k \times j$$

$$= (ae - bd)k + (cd - af)j + (bf - ce)i$$

$$= (bf - ce)i + (cd - af)j + (ae - bd)k$$

$$= \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} \Rightarrow w \times v = -v \times w$$

$$\& \; v \times v = 0$$

etc.

Vector Cross Product is not associative

$$i \times (i \times j) = i \times k = -j$$

$$\neq$$

$$(i \times i) \times j = 0 \times j = 0$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad \vec{c} = i c_1 + j c_2 + k c_3$$

$$\Rightarrow (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \overset{\text{why?}}{=} \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

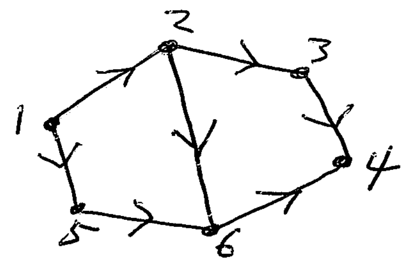
$$= (\vec{c} \times \vec{a}) \cdot \vec{b}$$

$$= (\vec{b} \times \vec{c}) \cdot \vec{a}$$

Exercise: Show that

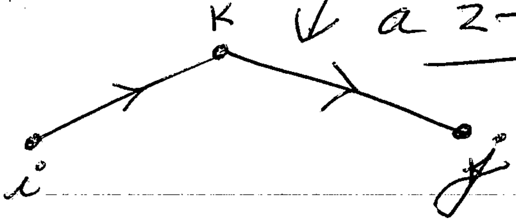
$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}.$$

Some Theory



(1)

1° $\displaystyle (A^2)_{ij} = \sum_k \underbrace{A_{ik}A_{kj}}$ $\updownarrow$ a 2-step path



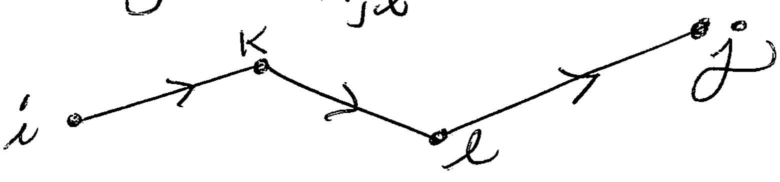
Thus if $1, 2, \ldots, n$ are the nodes of a graph (1) and

$$A_{ij} = \begin{cases} 1 & \text{if there is } i \to j \text{ (an edge from } i \text{ to } j) \\ 0 & \text{if there is no edge from } i \text{ to } j \end{cases}$$

Then $\displaystyle (A^2)_{ij} = \sum_{\substack{\text{2 step paths} \\ \text{from } i \text{ to } j}} 1 \leftarrow$ for each path

$= \#$ of 2-step paths.

$\displaystyle (A^3)_{ij} = \sum_{k,l} A_{ik}A_{kl}A_{lj}$ } 3 step paths



etc.

2° **Theorem.** A $n \times n$. The following conditions are equivalent.

(a) A is non-singular.

(b) $A\vec{x} = \vec{0}$ has only trivial soln $\vec{0}$.

(c) A is row equiv to $I = I_n$.

Proof. Recall that A non-singular means that there exists A^{-1} such that $A^{-1}A = AA^{-1} = I$.

(i) Suppose A is nonsingular.

If $A\vec{x} = \vec{0}$ then $A^{-1}A\vec{x} = A^{-1}\vec{0} = 0$

$\Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = 0$.

$\therefore \boxed{(a) \Rightarrow (b)}$

Now suppose (b).

Let R = row reduced echelon form of A. If $R \neq I$, then $A\vec{x} = \vec{0}$ has non-zero solutions. So

$\boxed{(b) \Rightarrow (c)}$.

Now suppose (c). But we know that if $R = I$ then there are invertible elem matrices s.t. $E_k E_{k-1} \cdots E_1 A = R = I$.

$\therefore (E_k E_{k-1} \cdots E_1) = A^{-1}$

$\therefore \boxed{(c) \Rightarrow (a)}$ So

$$\begin{tikzcd} & (a) \arrow[dr, Rightarrow] & \\ (c) \arrow[ur, Rightarrow] & & (b) \arrow[ll, Rightarrow] \end{tikzcd}$$

//

Corollary. The system of n linear equations in n unknowns $A\vec{x} = \vec{b}$ has a unique solution if and only if A is non-singular.

Proof. If A non-singular with inverse A^{-1}. Then, given

$$A\vec{x} = \vec{b}$$
$$\Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\Rightarrow I\vec{x} = A^{-1}b \Rightarrow \vec{x} = A^{-1}b.$$

So $A\vec{x} = \vec{b}$ has a unique soln.

Now suppose A singular. Then $A\vec{x} = \vec{0}$ has a non-zero solution $\vec{z}$. Suppose $\vec{w}$ is any soln to $A\vec{x} = \vec{b}$. So $A\vec{w} = \vec{b}$.

But then
$$A(\vec{z}+\vec{w}) = A\vec{z} + A\vec{w} = \vec{0} + \vec{b} = \vec{b}.$$

So $A\vec{x} = \vec{b}$ has many solns.

$\therefore$ If $A\vec{x} = \vec{b}$ has a unique soln, then A is non-singular. //

Remark. If E is an $n\times n$ elementary matrix and A is any $n\times n$ matrix then $\det(EA) = \det(E)\det(A)$ by direct calculation.

Example. We use 2×2 matrices.

(a) $\det\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \det\left[\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right] = bc-ad$

$= -(ad-bc) = -\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$ ✓

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\det\left[\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}A\right] = \det\left[\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$

$= \det\left[\begin{pmatrix} a & b \\ ra+c & rb+d \end{pmatrix}\right]$

$\boxed{\det\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = 1}$

$= a(rb+d) - b(ra+c)$

$= ad-bc + (arb-bra)$

$= ad-bc = \det(A).$

(c) More generally

$$\det\left[\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ s & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & & 1 \end{pmatrix}\begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{bmatrix}\right] = \det\begin{bmatrix} \vec{r}_1 \\ s\vec{r}_1+\vec{r}_2 \\ \vec{r}_3 \\ \vdots \\ \vec{r}_n \end{bmatrix} = \det\begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{bmatrix}$$

Why?

3''

(d) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(A) = ad - bc$

$E = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ $\det(E) = r$

$$\begin{aligned}
\det(EA) &= \det\left(\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \det\begin{bmatrix} ra & rb \\ c & d \end{bmatrix} \\
&= r \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= r \det(A) \\
&= \det(E)\det(A).
\end{aligned}$$

We know that if $\det(A) \neq 0$ (A any $n\times n$ matrix) then $A = E_1 E_2 \cdots E_K$, a product of elem matrices.

∴ If B is any $n\times n$, then

$$\begin{aligned}
\det(AB) &= \det(E_1 E_2 \cdots E_K B) \\
&= \det(E_1)\det(E_2 E_3 \cdots E_K B) \\
&= \det(E_1)\det(E_2)\det(E_3 \cdots E_K B) \\
&= \cdots \\
&= \underbrace{\det(E_1)\det(E_2)\cdots\det(E_K)}\det(B) \\
&= \det(A)\det(B).
\end{aligned}$$

Why?

<u>Theorem</u>. $n \times n$ matrix A is singular $\Longleftrightarrow \det(A) = 0$.

<u>Proof</u>. $R = E_k E_{k-1} \cdots E_1 A$ row reduced echelon form of A.

Then $\det(R) = \det(E_k E_{k-1} \cdots E_1 A)$

$= \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A)$

(We know that if E elem matrx then $\det(EA) = \det(E)\det(A)$.)

Thus $\det(A) = 0$
$\Longleftrightarrow \det(R) = 0$.

A singular $\Rightarrow R$ has a zero row.
$\Rightarrow \det R = 0 \Rightarrow \det A = 0$.

A non-sing $\Rightarrow R = I \Rightarrow \det R = 1$
$\Rightarrow \det A \neq 0$. //

<u>Note</u>: The argument in this proof also shows that

$A_{n \times n}, B_{n \times n} \Rightarrow \det(AB) = \det(A)\det(B)$.

Vector Spaces

First the standard example.

$$\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \vec{a} \;\middle|\; a_i \text{ are real numbers} \right\}$$

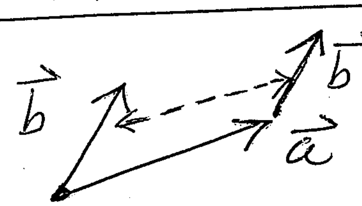
$$(\vec{a}+\vec{b})_i = a_i + b_i$$

$$(c\vec{a})_i = ca_i$$

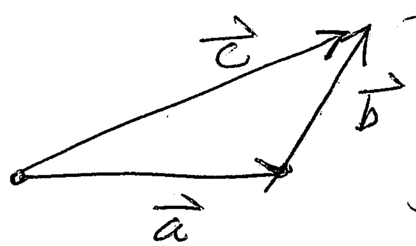
$$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{1 in the } i\text{th place}$$

Familiar Geometry

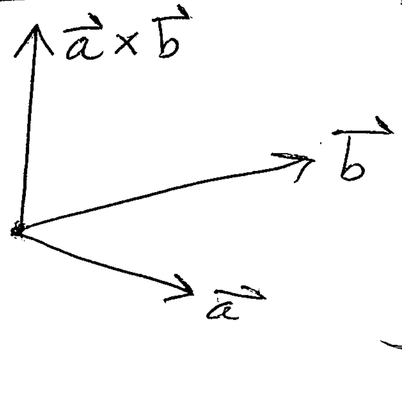


"A vector is a quantity with <u>magnitude</u> and <u>direction</u>."



$$\vec{c} = \vec{a} + \vec{b}$$

In $\mathbb{R}^3$:



$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Fact: $\vec{a} \in \mathbb{R}^n \Longrightarrow$

$x \in S$ means "x is a member of S"

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$$

Proof. $a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$

$$= a_1\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + a_3\begin{bmatrix}0\\0\\1\\0\\\vdots\\0\end{bmatrix} + \cdots + a_n\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}a_1\\0\\0\\\vdots\\0\end{bmatrix} + \begin{bmatrix}0\\a_2\\0\\\vdots\\0\end{bmatrix} + \begin{bmatrix}0\\0\\a_3\\0\\\vdots\\0\end{bmatrix} + \cdots + \begin{bmatrix}0\\0\\\vdots\\0\\a_n\end{bmatrix}$$

$$= \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix} = \vec{a} \checkmark \quad //$$

Fact. $\sum_{K=1}^{n} a_i e_i = \vec{0} \Rightarrow a_1 = a_2 = \cdots = 0.$

Proof. $\sum_{K=1}^{n} a_i \vec{e}_i = \vec{a}$. So $\sum_{K=1}^{n} a_i e_i = \vec{0}$

$\Rightarrow \vec{a} = \vec{0} \Rightarrow a_i = 0$ for each $i = 1, 2, \ldots, n$. //

Vector Space Axioms

V is a set with operations
+ , scalar multiplication.

Scalars $= \mathbb{R} =$ the real numbers.
(later we will use other scalars.)

$x, y \in V \Longrightarrow x + y \in V.$ [C2]

$\alpha \in \mathbb{R}, x \in V \Longrightarrow \alpha x \in V.$ [C1]

A1. $x + y = y + x \quad \forall x, y \in V$

A2. $(x+y)+z = x+(y+z) \quad \forall x, y, z \in V.$

A3. $\exists\, 0 \in V$ s.t. $x + 0 = x \quad \forall x \in V.$

A4. $x \in V \Longrightarrow \exists -x \in V$ s.t. $x + (-x) = 0.$

A5. $\alpha(x+y) = \alpha x + \alpha y \quad \forall \alpha \in \mathbb{R}; x, y \in V.$

A6. $(\alpha+\beta)x = \alpha x + \beta x \quad \forall \alpha, \beta \in \mathbb{R}; x \in V.$

A7. $(\alpha\beta)x = \alpha(\beta x) \quad \forall \alpha, \beta \in \mathbb{R}; x \in V.$

A8. $1 . x = x \quad \forall x \in V.$

Examples. (a) $\mathbb{R}^n \ni \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and usual operations.

(b) $\{n \times m$ matrices$\} = \mathcal{M}(n, m).$
$+ =$ addition of matrices
scalar x matrix as usual $(\alpha A)_{ij} = \alpha A_{ij}.$

(c) $C[a,b] =$ all real valued continuous functions $f : [a,b] \longrightarrow \mathbb{R}.$
$(f+g)(x) = f(x) + g(x).$
$(\alpha f)(x) = \alpha f(x).$

Remarks

(a) $\mathbb{R}^n = V$

vector space over $\mathbb{R}$.

$\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ with $a_i \in \mathbb{R}$.

$\vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$

$\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n$ <u>dot product</u>

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{\vec{a} \cdot \vec{a}}$ is the length of the vector $\vec{a}$.

We say $\vec{a} \perp \vec{b}$ ($\vec{a}$ is perpendicular to $\vec{b}$) when $\vec{a} \cdot \vec{b} = \vec{0}$.

(b) $\mathcal{M}(n, m) \longleftrightarrow \mathbb{R}^{n+m}$

e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow (a, b, c, d)$

This correspondence is an example of an <u>isomorphism</u> of <u>vector spaces</u>.

(c) $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous} \}$

Think of f as $\{\{f(x) \mid x \in [a,b]\}\}$ but written "in order". Then you see that f is a generalization of a vector in $\mathbb{R}^n$.

(d) Let $V_n = \{ f : \{1, 2, \ldots, n\} \longrightarrow \mathbb{R} \}$

$(f+g)(x) = f(x) + g(x) \quad x = 1, 2, \ldots, n$

$(kf)(x) = kf(x) \quad x = 1, 2, \ldots, n.$

Show V_n is a vector space isomorphic to $\mathbb{R}^n$.