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Extreme coefficients of Jones polynomials and graph theory

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Abstract

We find families of prime diagrams of knots with arbitrary extreme coefficients in their Jones polynomials. Some graph theory is presented in connection with this problem, generalizing ideas by Yongju Bae and Morton [4] and giving a positive answer to a question in their paper.

1 Introduction

Let L be an oriented link, and $V_L(t)$ its Jones polynomial with normalization one. We are interested in exhibiting examples of links with arbitrary extreme coefficients in their Jones polynomials. Consider an unoriented diagram D of L. We denote by $\langle D\rangle$ its Kauffman bracket with normalization $\langle \bigcirc \rangle = 1$ (see [2]). Since $V_L(t) = (-A)^{-3w(D)}\langle D\rangle$ after the substitution $A = t^{-1/4}$, we have that $span(V_L(t)) = span(\langle D\rangle)/4$ and the coefficients of both polynomials are the same, maybe up to sign.

Hence we will work with the Kauffman bracket of unoriented diagrams. We recall the definition of this polynomial using the states sum:

$$\langle D\rangle = \sum_s \langle D, s\rangle$$

where the sum is taken over all states s of D and $\langle D, s\rangle = A^{a(s)-b(s)}(-A^{-2} - A^2)^{|s|-1}$.

A state s of D is a labelling of each crossing of D by either an A-chord or a B-chord. We write $a(s)$ (resp. $b(s)$) for the number of A-chords (resp. B-chords) of the state s, and $|s|$ for the number of components of the diagram sD, which is D after the s-smoothing of D. Precisely sD is obtained smoothing every

crossing in D according to the type of chord associated to the crossing by the state, as shown in Figure 1. We will draw a small chord with the letters A or B to remember which was the state. In this way we can reconstruct the diagram D from sD and the chords, by just reversing the smoothings shown in Figure 1. An example is shown in Figure 2. For a state s we denote by $max(s)$ (resp. $min(s)$)

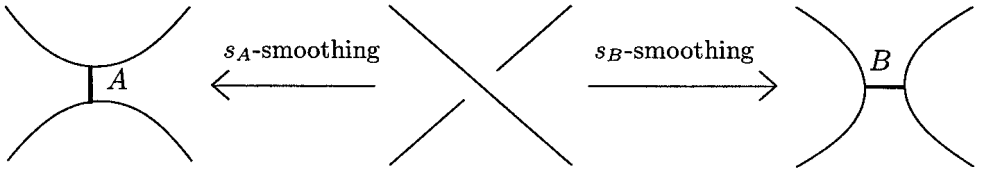

Figure 1: s_A and s_B-smoothing of a crossing and corresponding chords.

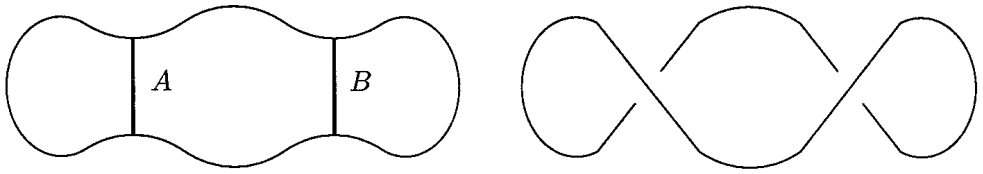

Figure 2: From sD and the chords to D.

the highest (resp. the lowest) degree of $\langle D, s\rangle$. The extreme states s_A and s_B are defined by the equalities $a(s_A) = c(D)$ and $b(s_B) = c(D)$ respectively, where $c(D)$ is the number of crossings of the diagram D. Write $m = min(s_B)$ and $M = max(s_A)$. Clearly $m = -c(D) - 2|s_B| + 2$ and $M = c(D) + 2|s_A| - 2$. These numbers m and M will be called the extreme states degrees of $\langle D\rangle$ and their corresponding coefficients a_m and a_M in $\langle D\rangle$ will be called the extreme states coefficients of $\langle D\rangle$. It turns out that (see Proposition 1)

$$\langle D\rangle = a_m A^m + a_{m+4}A^{m+4} + \cdots + a_{M-4}A^{M-4} + a_M A^M.$$

In this paper we deal with the question of finding arbitrary extreme coefficients for $\langle D\rangle$. Two different approachings will be given:

The first approaching follows Yonju Bae and Morton [4]. In their paper a connection between a_M and graph theory is given. The coefficient a_M appears to be, up to sign, the value $f(G_D)$ of a certain graph G_D, constructed from $s_A D$ and called the Lando's graph of sD. In general, for any graph G, $f(G) = \sum_C (-1)^{|C|}$ where C runs over all the independent subsets of vertices of G, where independent means that there is no edge joining two vertices of C. In [4] there arises the question of if any integer can be realized as $f(G)$ for some graph G_D. We have found a positive answer to this question. In fact, we will

exhibit examples (in fact complete families) of graphs, and from these we will reconstruct prime diagrams of knots with arbitrary extreme states coefficients a_m and a_M. This will be done in the second section. The third section will be dedicated to explain further constructions of graphs, which in many cases give easier examples of knots with the wanted extreme states coefficients. Here "easier" means much fewer crossings.

The second approaching is treated in the fourth and final section. Here we exhibit examples of prime diagrams of knots for which the extreme states coefficients a_m and a_M are zero, and the next coefficients a_{m+4} and a_{M-4} take arbitrary values. The idea is just to look at the more general circle graph defined by sD, rather than the Lando's graph G_D, and use a very simple trick for counting a_{M-4} in special cases. We do not know if there is a complete nice interpretation of a_{M-4} in terms of graph theory, parallel to that one in which a_M is described in terms of G_D in the second section. A really more interesting question is the following: is it possible to get any extreme coefficient when the spread of the Jones polynomial is previously fixed?

2 Extreme states coefficients of Jones polynomials and graph theory

We begin by recalling some very basic facts about the Kauffman bracket $\langle D\rangle$ of an unoriented diagram D.

Proposition 1 *(i) All degrees in $\langle D\rangle$ are congruent modulo four.*

(ii) $max(s) \leq M = max(s_A)$ with equality if and only if $|s| = |s_A| + b(s)$.

(iii) $min(s) \geq m = min(s_B)$ with equality if and only if $|s| = |s_B| + a(s)$.

(iv) The highest (resp. lowest) degree of $\langle D\rangle$ is less than (resp. greater than) or equal to M (resp. m).

(v) A state s contributes to a_M if and only if $s \in \Gamma_A = \{s/|s| = |s_A| + b(s)\}$. The contribution of $s \in \Gamma_A$ to a_M is $(-1)^{|s_A|-1}(-1)^{b(s)}$.

(vi) A state s contributes to a_m if and only if $s \in \Gamma_B = \{s/|s| = |s_B| + a(s)\}$. The contribution of $s \in \Gamma_B$ to a_m is $(-1)^{|s_B|-1}(-1)^{a(s)}$.

(vii) $a_M = (-1)^{|s_A|-1}\sum_{s\in\Gamma_A}(-1)^{b(s)}$ and $a_m = (-1)^{|s_B|-1}\sum_{s\in\Gamma_B}(-1)^{a(s)}$.

Proof. We calculate the difference

$$\begin{aligned} max(s_A) - max(s) &= c(D) + 2|s_A| - 2 - a(s) + b(s) - 2|s| + 2 \\ &= 2b(s) + 2|s_A| - 2|s| \end{aligned}$$

Now the key point is that if two states s and s' differ in the label of only one crossing, then either $|s'| = |s| + 1$ or $|s'| = |s| - 1$, depending on whenever the two strings that appear after s-smoothing the crossing belong or not to the same component of sD. It follows that for an arbitrary state s we have that there are non-negative integers p and n such that $b(s) = p + n$ and $|s| = |s_A| + p - n$. Then $b(s) + |s_A| - |s| = p + n + |s_A| - |s_A| - p + n = 2n$ is an even number great or equal than zero. The completation of the proof is left to the reader. □

Remark In order to find sD, one can start with s_AD, choose an order in the set of the $b(s)$ crossings labelled with a B-chord in the state s and perform the opposite smoothing in these crossings following this order. In this way, one associates $+1$ (resp. -1) to each one of the $b(s)$ crossings if after the s_B-smoothing of this crossing we get one more (resp. fewer) component. By definition p (resp. n) is the number of associated $+1$ (resp. -1). This non-negative integer does not depend on the chosen order of the $b(s)$ crossings, although the association of $+1$ and -1 to each one of the $b(s)$ crossings does.

Now we recall the connection between a_M and graph theory given in [4]. In order to obtain the Lando's graph of D (see [1]) start with s_AD and delete the A-chords joining two different components. In this way we obtain a bipartite circle graph (BCG). Now define the Lando's graph of D as the graph that appears taking a vertex for every A-chord of the BCG, and joining two vertices with an edge if and only if the endpoints of the corresponding A-chords in the BCG alternate in the same component of the BCG. An example of s_AD, the corresponding BCG and the Lando's graph is shown in Figure 3. A subset

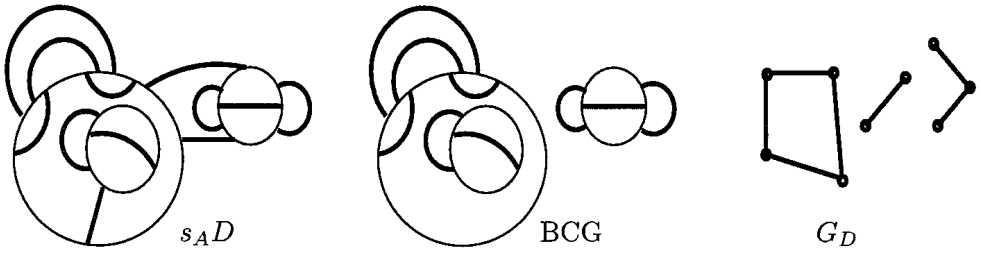

Figure 3: s_AD, the BCG and the Lando's graph G_D.

C of vertices of a graph G is said to be independent if there is no edge in G joining two vertices of C. We define $f(G) = \sum_C (-1)^{|C|}$ where C runs over the independent sets of vertices of G, the empty set included.

Then

$$\begin{aligned} a_M &= (-1)^{|s_A|-1} \sum_{s \in \Gamma_A} (-1)^{b(s)} \\ &= (-1)^{|s_A|-1} \sum_{C} (-1)^{|C|} \\ &= (-1)^{|s_A|-1} f(G_D). \end{aligned}$$

The first equality is given by Proposition 1 and the third equality is just the definition of $f(G_D)$. In order to check the second equality, we think of a state s as the set of $b(s)$ A-chords of $s_A D$ which correspond to the $b(s)$ B-chords of s. Then we have that $s \in \Gamma_A$ if and only if the two following conditions occur:

(1) The endpoints of every A-chord lies in the same component.

(2) The endpoints of two A-chords lying in the same component do not alternate.

The example in Figure 4 is exhibited in [4], where the corresponding Lando's graph G_D has $f(G_D) = 3$. The question formulated in [4] is if any integer n

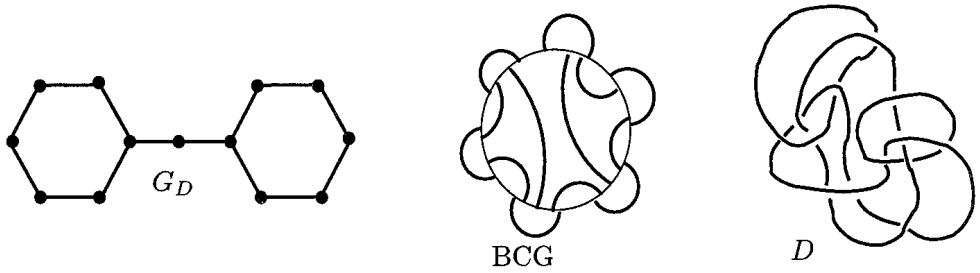

Figure 4: A Lando's graph G_D with $f(G_D) = 3$ and the corresponding BCG and link diagram.

can be realized as $f(G_D)$ for a graph G_D arising from a diagram D. Since these graphs arise from BCG, we will call them "graphs convertible in BCG".

Some graph theory

Since the graphs G_D must be always convertible in BCG, the examples shown in Figure 5 are not allowed. On the other hand the graphs in consideration are not necessarily planar graphs. Figure 6 exhibits such an example.

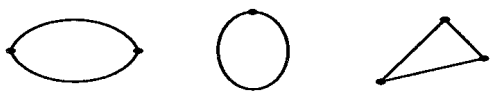

Figure 5: Examples of graphs non-convertible in BCG.

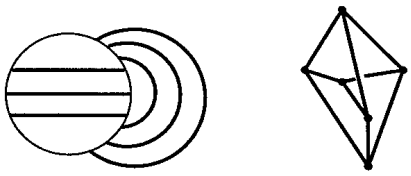

Figure 6: Non-planar Lando's graph.

Calculation of $f(G)$ can be simplified using some readily established properties, as explained in [4]. Here it is a very brief description of these properties:

• Law of recursion. If G is a graph and v is a vertex of G, we will denote by $G-v$ the graph obtained from G by deleting the vertex v and its incident edges. Let $\{v_1, \ldots, v_k\}$ be the set of the neighbour vertices of v in G (by definition these are the vertices of G joined to v by an edge). We will denote by $G - Nv$ the graph $(\ldots((G - v) - v_1)\ldots) - v_k$. Then the law of recursion says that $f(G) = f(G - v) - f(G - Nv)$.

• Law of multiplication. If a graph G is the disjoint union of two graphs G_1 and G_2, then $f(G) = f(G_1)f(G_2)$.

• Law of duplication. Suppose that v and w are two non-neighbour vertices of a graph G, and the set of neighbour vertices of v is included in the set of neighbour vertices of w. Then we say that G is a duplication of $G - w$ and we have $f(G) = f(G - w)$.

We consider examples that will be used later. Let L_n be the graph with n vertices shown in Figure 7. Clearly $f(L_2) = -1$. By using duplication in the third vertex and then multiplication we get the formula $f(L_n) = -f(L_{n-3})$. Let C_n be the polygon with n vertices, $n \geq 3$. By recursion we obtain $f(C_n) =$

Figure 7: Graph L_n.

$f(L_{n-1}) - f(L_{n-3})$. In particular, for the hexagon $H = C_6$ we have $f(H) = 2$. We now explain how to find different families $\{G_r\}_{r\in\mathbf{Z}}$ of graphs with $f(G_r) = r$ for any integer r. From now on, we will write G^v to mean the pair (G, v), where G is a graph and v is a particular vertex of G.

Definition Let G be a planar graph convertible in BCG and let v be a vertex of G. We say that G^v if a brick of type (n, k) if $f(G) = n$ and $f(G - v) = k$.

Our main example will be the hexagon H with any arbitrary vertex chosen, which is a brick of type $(2, 1)$. It is a graph convertible in BCG, as we show in Figure 8. Now we describe the basic construction that will be used for giving

Figure 8: The hexagon H is convertible in BCG.

our examples: Let G_1 and G_2 be two graphs and v_1 and v_2 two vertices of G_1 and G_2 respectively. Then $G_1^{v_1} * G_2^{v_2}$ will denote the new graph obtained from the disjoint union of G_1 and G_2 by joining v_1 and v_2 with an extra edge (see Figure 9). Note that $G_1^{v_1} * G_2^{v_2}$ is convertible in BCG if both G_1 and G_2 are.

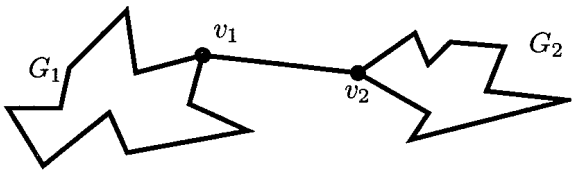

Figure 9: Graph $G_1^{v_1} * G_2^{v_2}$.

Lemma 1 *Let G be a graph and v a vertex of G. If G^v is a brick of type (n,k), then (G',v') is a brick of type $(n+k,k)$, where $G' = G^v * H^w$ and v' is a vertex of H adjacent to w (see Figure 10).*

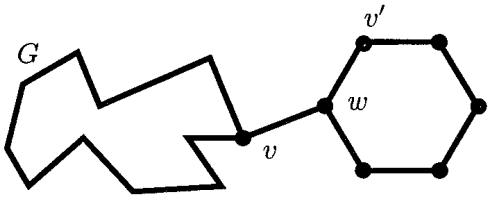

Figure 10: Graph (G',v') arising from (G,v).

Proof. We have to show that $f(G') = n + k$ and $f(G' - v') = k$.

We have $f(G' - v') = (f(L_2))^2 f(G - v) = (-1)^2 k = k$ where we have used duplication, first in the vertex v_1 and then in the vertex v, multiplication and the fact that $f(L_2) = -1$ (see Figure 11).

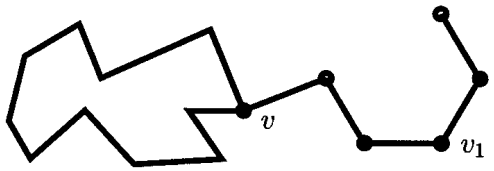

Figure 11: $G' - v'$.

On the other hand

$$\begin{aligned} f(G') &= f(G' - v') - f(G' - Nv') && (recursion) \\ &= k - f(L_3)f(G) && (multiplication) \\ &= k - (-1)n && (f(L_3) = -1) \\ &= n + k. \end{aligned}$$

Finally, G' is convertible in BCG since G and H are. □

Lemma 2 *Let G be a graph. Then $f(G^-) = -f(G)$ where $G^- = G^v * L_3^w$, v being any vertex of G and w being an extreme vertex of L_3. Moreover, G^- is convertible in BCG if G is (see Figure 12).*

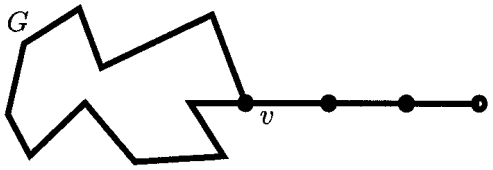

Figure 12: Graph G^-.

Proof. We have

$$\begin{aligned} f(G^-) &= f(G^- - w) && (duplication) \\ &= f(L_2)f(G) && (multiplication) \\ &= -f(G) && (f(L_2) = -1). \end{aligned}$$

Finally, G^- is convertible in BCG since G and L_3 are. □

Theorem 3 *For any integer n there is a planar graph G_{n-1} convertible in bipartite circle graph such that $f(G_{n-1}) = n$.*

Proof. Consider the brick H^v of type $(2,1)$ and apply r times Lemma 1 to get a planar graph G_{r+1} convertible in BCG such that $f(G_{r+1}) = r+2$ (see Figure 13).

On the other hand L_5, L_1 and L_2 are planar graphs convertible in BCG with $f(L_5) = 1$, $f(L_1) = 0$ and $f(L_2) = -1$ respectively.

Finally, for all integer $r \geq 1$ we have that G^-_{r+1} is a planar graph convertible in BCG with $f(G^-_{r+1}) = -(r+2)$ according to Lemma 2 (see Figure 14). □

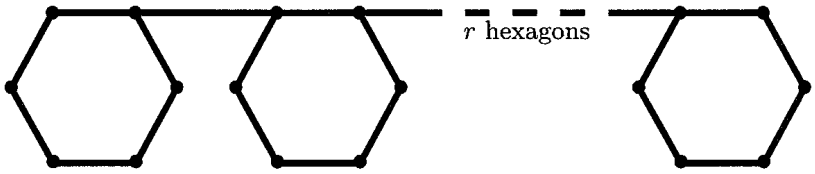

Figure 13: The graph G_r.

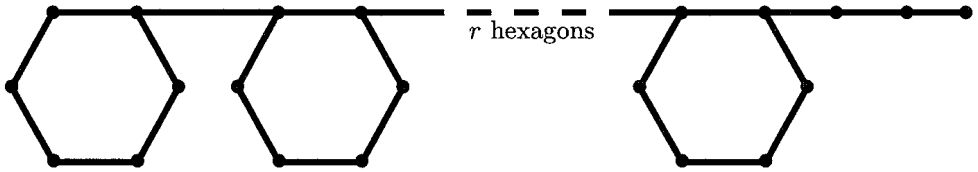

Figure 14: The graph G^-_r.

We will now construct a prime diagram of a knot with arbitrary extreme states coefficients, starting with the graphs G_r. This process is illustrated in figures 15, 16, 17 and 18. We first reconstruct the associated bipartite circle graph (Figure 15). Changing every A-chord $\mathbb{I}$ to a crossing $\mathbb{X}$ we see that this BCG is $s_A D'_r$ where the (3−components) diagram D'_r is shown in Figure 16. Figure 17 shows $s_B D'_r$, proving that D'_r is quite far from being a minus-adequate diagram (the coefficient a_m of $\langle D'_r \rangle$ is not ± 1 in general since there are other states in Γ_B apart from s_B). Because of this we modify D'_r to produce the diagram D_r in Figure 18.

Theorem 4 *$\langle D_r \rangle = A^m + \cdots + (r+1)A^M$ where $m = -24r - 4$ and $M = 12r$. In particular span$(\langle D_r \rangle) = 4(9r+1)$. Moreover, D_r is a prime diagram of a knot with $12r$ crossings.*

Proof. First we fix our attention in D_r after s_A-smoothing. Note that $|s_A D_r| = 1$ hence $M = c(D) = 12r$. On the other hand the Lando's graph of D_r is seen to be a duplication of G_r, hence $a_M = (-1)^{|s_A|-1} f(G_r) = r+1$.

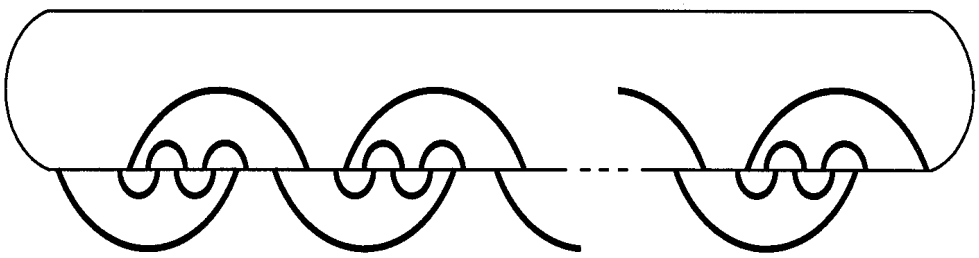

Figure 15: The BCG obtained from G_r.

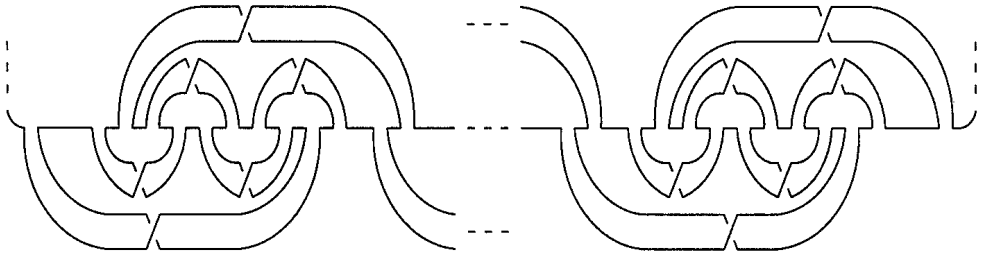

Figure 16: The diagram D'_r.

Now apply the s_B-smoothing to D. We have $|s_B D_r| = 6r+3$ ($6r$ components are given by the small circles $\overline{\infty}$ and the other three components are those appearing in Figure 17), hence $m = -12r - 2(6r + 3) + 2 = -24r - 4$. On the other hand the Lando's graph (respect to the s_B-smoothing) is the empty set, hence $a_m = (-1)^{|s_B|-1} f(\emptyset) = 1$. □

We now join in a very particular way D_r and $\bar{D}_s$ (the mirror image of D_s) in order to control simultaneously both extreme states coefficient (we avoid the obvious solution given by the appropriate connected sum in order to get a prime diagram):

Theorem 5 *Let D_{rs} be the diagram shown in Figure 19. Then $\langle D_{rs} \rangle = (s + 1)A^m + \cdots + (r+1)A^M$ where $m = -24r - 12s - 6$ and $M = 12r + 24s + 6$. In particular, span$(\langle D_{rs} \rangle) = 36(r + s) + 12$. Moreover, D_{rs} is a prime diagram of a knot with $12(r + s) + 2$ crossings.*

Proof. Note first that $c(D_{rs}) = c(D_r) + c(\bar{D}_s) + 2 = 12r + 12s + 2$.

Now, the very special way in which we join D_r and the mirror image $\bar{D}_s$ of D_s gives the equalities $|s_A| = |s_A(\bar{D}_s)| = |s_B(D_s)| = 6s + 3$ and the fact that the Lando's graph is still a duplication of G_r. It follows that $M = 12r + 24s + 6$ and $a_M = (-1)^{|s_A|-1} f(G_r) = r + 1$.

Analogously, $|s_B| = |s_B(D_r)| = 6r + 3$ and the Lando's graph is still a duplication of G_s, hence $m = -24r - 12s - 6$ and $a_m = (-1)^{|s_B|-1} f(G_s) = s + 1$. □

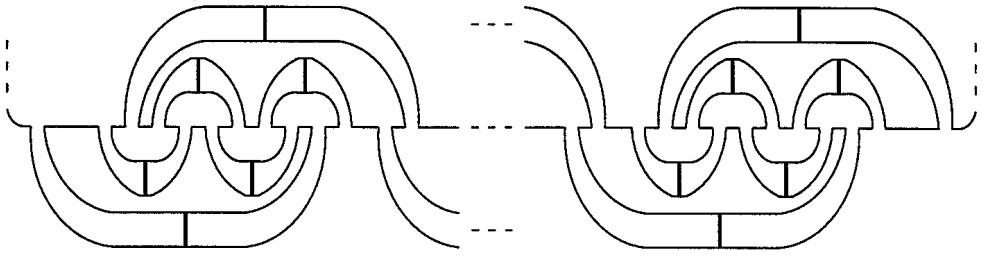

Figure 17: $s_B D'_r$: D'_r is not minus-adequate.

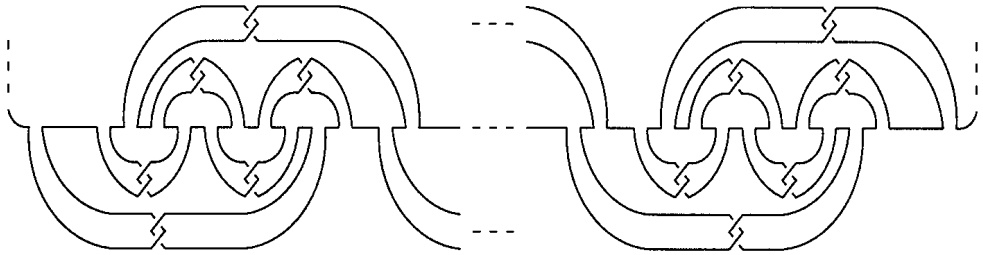

Figure 18: The diagram D_r.

A small refinement of the last result allows us to modify the signs of the extreme states coefficients:

Theorem 6 *Let α be an odd integer great than 1. Let D^{α}_{rs} be the diagram D_{rs} with a modification in the way in which D_r and $\bar{D}_s$ are joined on the left, as shown in Figure 20. Then $\langle D^{\alpha}_{rs}\rangle = -(s+1)A^m + \cdots + (r+1)A^M$ where $m = -24r - 12s - \alpha - 8$ and $M = 12r + 24s + 3\alpha + 4$. In particular $span(\langle D^{\alpha}_{rs}\rangle) = 36(r+s) + 4\alpha + 12$. Moreover, D^{α}_{rs} is a prime diagram of a knot with $12(r+s) + 2 + \alpha$ crossings.*

Proof. We have that $c(D^{\alpha}_{rs}) = c(D_{rs}) + \alpha$, $|s_A| = |s_A(D_{rs})| + \alpha - 1 = 6s + \alpha + 2$ and $|s_B| = |s_B(D_{rs})| + 1 = 6r + 4$. In addition the s_A and s_B Lando's graphs are still the duplications of G_r and G_s respectively. □

3 More graph theory

Let us consider the specific example provided by Theorem 4 for the extreme state coefficient $a_M = 41$. The knot diagram D_{40} has $12 \times 40 = 480$ crossings! If we take the non minus-adequate link diagram with three components D'_{40}, we still have $6 \times 40 = 240$ crossings! In this section we develop other graph constructions, providing easier examples for many possible extreme states coefficients. Here "easier" means always that the diagram has fewer crossings. In many cases, easier means as well that the corresponding span is lower.

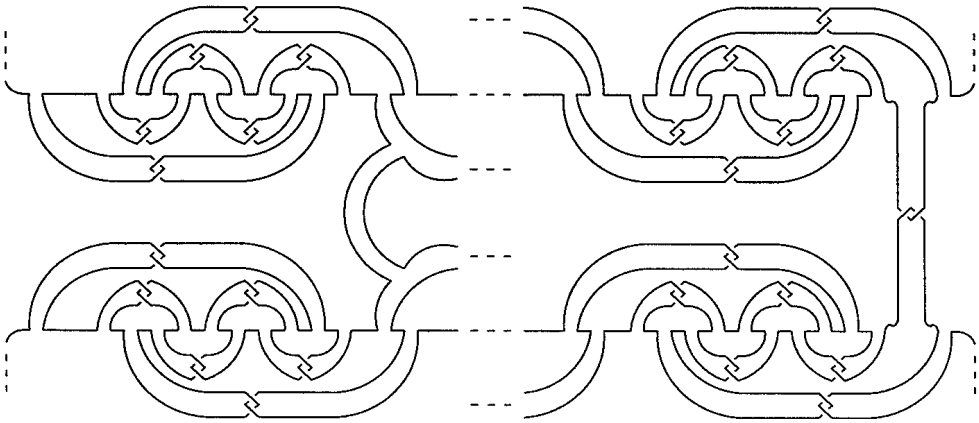

Figure 19: The diagram D_{rs}.

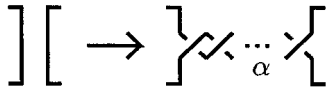

Figure 20: A small modification of D_{rs} produces D^{α}_{rs}.

Building with bricks

Let $\{G_1^{v_1}, \ldots, G_k^{v_k}\}$ be a set of graphs with a chosen vertex $v_i \in G_i$ for every $1 \leq i \leq k$. We denote by $S = S(G_1^{v_1}, \ldots, G_k^{v_k})$ the graph shown in Figure 21a, and we call it "simple building" constructed with the bricks $G_1^{v_1}, \ldots, G_k^{v_k}$. Precisely, S can be defined in steps using the operation $*$ introduced in the second section. Let w be the only vertex of L_1. Then

$$S = ((\ldots((L_1^w * G_1^{v_1})^w * G_2^{v_2})^w * \ldots)^w) * G_k^{v_k}.$$

Other related construction can be obtained introducing k extra vertices $w_1, \ldots, w_k$, one in every edge joining v_i and w in the graph S. The new graph is denoted by $C = C(G_1^{v_1}, \ldots, G_k^{v_k})$ and it is shown in Figure 21b. We will call it "complicated building" constructed with the bricks $G_1^{v_1}, \ldots, G_k^{v_k}$. Its precise definition using the operation $*$ is left to the reader. In both constructions the vertex w is called the central vertex. In the second construction, the vertices w_i are called the intermediate vertices.

Lemma 7 *Let $G_i^{v_i}$ be a brick of type (n_i, m_i) for every $i \in \{1, \ldots, k\}$. Let $S = S(G_1^{v_1}, \ldots, G_k^{v_k})$ and $C = C(G_1^{v_1}, \ldots, G_k^{v_k})$. Then:*

(1) S^w is a brick of type $(\prod_{i=1}^k n_i - \prod_{i=1}^k m_i, \prod_{i=1}^k n_i)$, where w is the central vertex of S.

(2) S^{v_j} is a brick of type $(\prod_{i=1}^k n_i - \prod_{i=1}^k m_i, m_j \prod_{j \neq i=1}^k n_i - \prod_{i=1}^k m_i)$.

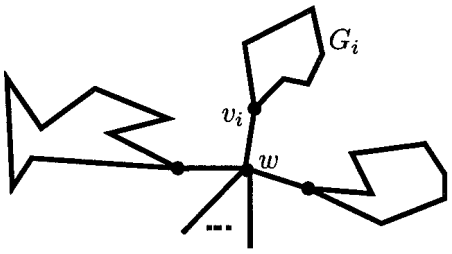

Figure 21a: Simple building.

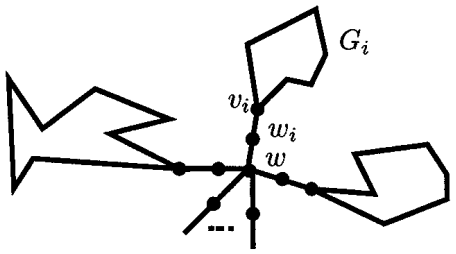

Figure 21b: Complicated building.

(3) C^w *is a brick of type* $(\prod_{i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i, \prod_{i=1}^k (n_i - m_i))$, *where* w *is the central vertex of* C.

(4) C^{v_j} *is a brick of type* $(\prod_{i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i, -m_j \prod_{j \neq i=1}^k n_i)$.

(5) C^{w_j} *is a brick of type* $(\prod_{i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i, n_j \prod_{j \neq i=1}^k (n_i - m_i) - \prod_{i=1}^k n_i)$, *where* w_j *is any intermediate vertex of* C.

Proof. We prove *(1)* and leave the other proofs to the reader:

$$\begin{aligned} f(S - w) &= \prod_{i=1}^k f(G_i^{v_i}) \qquad (multiplication) \\ &= \prod_{i=1}^k n_i \end{aligned}$$

and

$$\begin{aligned} f(S) &= f(S - w) - f(S - Nw) \qquad (recursion) \\ &= \prod_{i=1}^k n_i - \prod_{i=1}^k f(G_i - v_i) \qquad (multiplication) \\ &= \prod_{i=1}^k n_i - \prod_{i=1}^k m_i. \end{aligned}$$

□

Let us come back to the example with $a_M = 41$. Note that $41 = 5 \times 5 \times 2 - 3 \times 3 \times 1$. Hence $S = S(G_1^{v_1}, G_2^{v_2}, H^v)$ has $f(S) = 41$ if $G_1^{v_1}$ and $G_2^{v_2}$ are two copies of a brick of type $(5, 3)$. Two different bricks of type $(5, 3)$ are shown in Figure 22. It follows from *(1)* in above lemma that the planar graph S convertible in BCG shown in Figure 23 has $f(S) = 41$. The corresponding diagram with $a_M = 41$ in its Kauffman bracket has 43 crossings.

Finally in this section we present a sequence $\{F_r\}_{r \in \mathbf{N}}$ of planar graphs, all of them convertible in BCG, such that the corresponding sequence of integers $\{f(F_r)\}_{r \in \mathbf{N}}$ is the Fibbonachi sequence $2, 3, 5, 8, 13, 21, \ldots$. The graph F_r is shown in Figure 24. If we compare the link diagram arising from F_{r+1} and F_r, we note that while a_M increases by $f(F_{r-1})$, the number of crossings of the diagram increases only by 7.

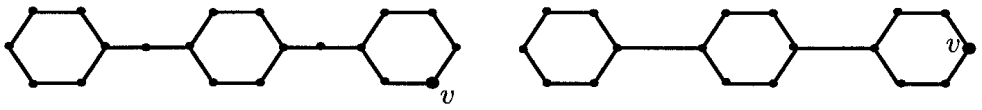

Figure 22: Two different bricks of type $(5,3)$.

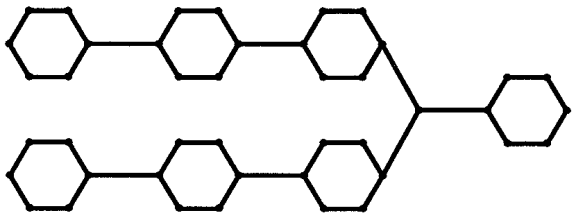

Figure 23: A graph G with $f(G) = 41$.

4 Non-states extreme coefficients of Jones polynomials

In this final section we will exhibit examples of prime diagrams of knots for which the extreme states coefficients a_m and a_M are zero and the next coefficients a_{m+4} and a_{M-4} take arbitrary values. The idea is just to look at the more general bipartite circle graph defined by sD, rather than the Lando's graph G_D, and use a very simple trick for counting a_{M-4} in special cases.

We start with the natural enlargement of proposition given in the second section. Recall that $\langle D\rangle$ denotes the Kauffman bracket of an unoriented diagram D with normalization one.

Proposition 2 *(i) A state s contribute to a_{M-4} if and only if either $s \in \Gamma_A = \{s/|s| = |s_A| + b(s)\}$ or $s \in \Gamma_A^1 = \{s/|s| = |s_A| + b(s) - 2\}$. The contribution of $s \in \Gamma_A$ to a_{M-4} is $(-1)^{|s_A|-1}(-1)^{b(s)}(|s_A| + b(s) - 1)$. The contribution of $s_1 \in \Gamma_A^1$ to a_{M-4} is $(-1)^{|s_A|-1}(-1)^{b(s_1)}$.*

(ii)

$$a_{M-4} = (-1)^{|s_A|-1}\left[(|s_A|-1)\sum_{s\in\Gamma_A}(-1)^{b(s)} + \sum_{s\in\Gamma_A}(-1)^{b(s)}b(s) + \sum_{s_1\in\Gamma_A^1}(-1)^{b(s_1)}\right].$$

(iii) If $a_M = 0$ then the whole contribution of Γ_A to a_{M-4} is given by

$$(-1)^{|s_A|-1}\sum_{s\in\Gamma_A}(-1)^{b(s)}b(s).$$

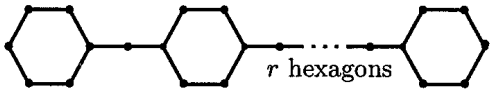

Figure 24: The graph F_r.

Proof. A state s contributes to a_{M-4} if and only if $max(s) \geq M-4$, hence if s contributes to a_{M-4} but is not in Γ_A, we have that $max(s) = M-4$ by *(i)* in Proposition 1. Now the statements follow from the fact that $\{s_1/max(s_1) = M-4\} = \{s_1/|s_1| = |s_A| + b(s_1) - 2\}$. □

Consider now the generalized bipartite circle graph s_AD obtained from an unoriented diagram D by s_A-smoothing. Suppose that locally this has the very special aspect shown in Figure 25, and suppose that only the A-chords $x_1, \ldots, x_k$ $(k > 1)$ have their endpoints in the same component. The Lando's graph is then given by k parallel A-chords, hence by duplication we have that $a_M = 0$ in $\langle D \rangle$. From now on, we will identify a state s with the set of the $b(s)$ A-chords in s_AD

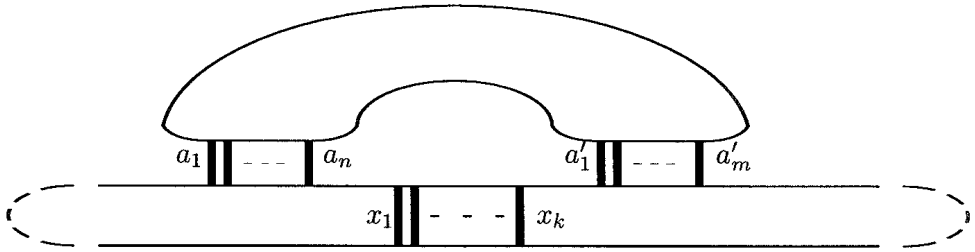

Figure 25: A very special s_AD.

that correspond to the $b(s)$ crossings of D in which the label associated by s is a B-chord. Let $\mathcal{X} = \{x_1, \ldots, x_k\}$, $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $\mathcal{A}' = \{a'_1, \ldots, a'_m\}$ (see Figure 25).

The states in Γ_A are the subsets of $\mathcal{X}$, hence by *(iii)* in Proposition 2 we have that the contribution of Γ_A to a_{M-4} is given up to sign by $\sum_{s \in \Gamma_A} (-1)^{b(s)} b(s) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j j$, which is 0 when $k > 1$.

We now analyze the contribution to a_{M-4} of the states in Γ_A^1 which contain at least one A-chord of $\mathcal{A} \bigcup \mathcal{A}'$. Note first that these states are necessarily contained in $\mathcal{X} \bigcup \mathcal{A} \bigcup \mathcal{A}'$.

Suppose first that in s_1 there are no A-chords of both sets $\mathcal{A}$ and $\mathcal{A}'$. Then the union of $s_1 \setminus \mathcal{X}$ and any subset of $\mathcal{X}$ is a state which lies in Γ_A^1, and all of their contributions to a_{M-4} cancel.

By the contrary, suppose that s_1 has at least one A-chord of $\mathcal{A}$ and one of

$\mathcal{A}'$. Then $s_1 \bigcap \mathcal{X}$ is the empty set, and the contribution of all these states is $(-1)^{|s_A|-1}$ in total.

As a consequence we have found a geometrical reason for some of the results algebraically obtained in [3] about the Kauffman bracket of pretzel link diagrams. Let P be the pretzel link diagram $P(a, b_1, \ldots, b_s)$ where $a \geq 2$, and $b_i \leq -2$ for any $i = 1, \ldots, s$ with s at least 2 . Let μ be the cardinal of the set $\{i/b_i = -2\}$ and assume that $\mu \geq 1$. Then P is a minus-adequate diagram, $a_M = 0$ and $a_{M-4} = \mu$ up to sign.

Now, in order to get a prime diagram of a knot with extreme states coefficients equal to zero and arbitrary values for a_{m+4} and a_{M-4}, we manipulate two pretzel link diagrams $P(2, \underbrace{-2, \ldots, -2}_{s})$ and $P(\underbrace{2, \ldots, 2}_{r-1}, -2, 2)$. We join these two diagrams in the way shown in Figure 26, using two extra columns with α and β crossings respectively, both α and β great or equal than two. We will denote by $D(r, s; \alpha, \beta)$ the obtained diagram. Note that $D(r, s; \alpha, \beta)$ is the pretzel link

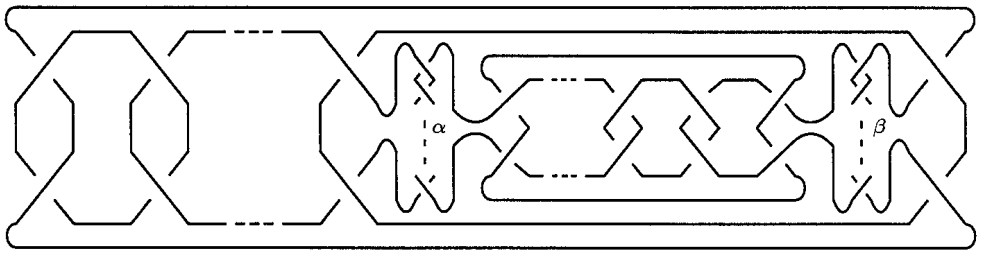

Figure 26: The link diagram $D(r, s; \alpha, \beta)$.

diagram $P(2, \underbrace{-2, \ldots, -2}_{s-2}, -\alpha, \underbrace{2, \ldots, 2}_{r-2}, -2, \beta)$ with two extra trivial knots placed in a very special way, as shown in Figure 27.

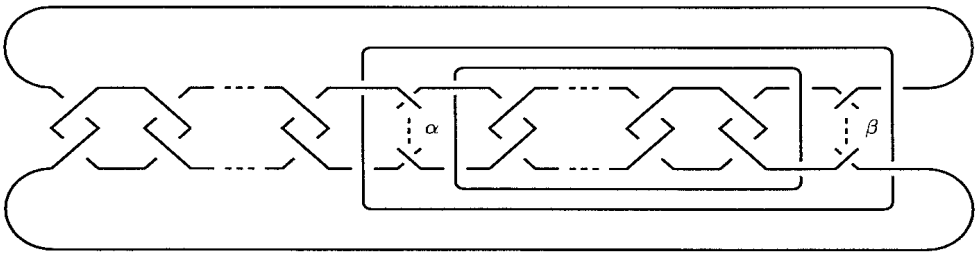

Figure 27: Other view of $D(r, s; \alpha, \beta)$.

Theorem 8 *Let $D=D(r,s;\alpha,\beta)$ be the link diagram shown in Figure 27. Then*

$$\langle D\rangle = (-1)^{r+s+\beta-1}rA^{-4r-4s-\alpha-3\beta+2} + \cdots + (-1)^{r+s+\alpha-1}sA^{4r+4s+3\alpha+\beta-2}.$$

In particular span$(\langle D\rangle)$ $= 8(r+s)+4(\alpha+\beta)-4$. The highest degree of this polynomial is $M-4$, and the lowest degree is $m+4$. Moreover, D has $r+s+2-\frac{1-(-1)^{\alpha}}{2}-\frac{1-(-1)^{\beta}}{2}$ components.

Proof. Note first that $c(D) = 2+2s+2+2r+\alpha+\beta$.

Figure 28 shows $D(r,s;\alpha,\beta)$ after s_A-smoothing. We have $|s_A| = r+s+\alpha$ (hence $M = 4r+4s+3\alpha+\beta+2$), and since the Lando's graph is given by two parallel A-chords, we have $a_M = 0$. The above discussion applied to Figure 28 gives $a_{M-4} = (-1)^{|s_A|-1}s = (-1)^{r+s+\alpha-1}s$. Analogously, after s_B-smoothing

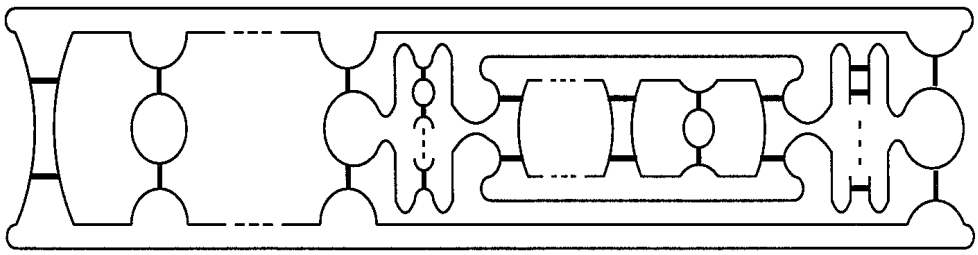

Figure 28: $D(r,s;\alpha,\beta)$ after s_A-smoothing.

(see Figure 29) we count $|s_B| = r+s+\beta$ (hence $m = -4r-4s-\alpha-3\beta-2$), and since the Lando's graph is given by two parallel B-chords, we have $a_m = 0$. Finally, from the above discussion applied to Figure 29 we deduce that $a_{m+4} = (-1)^{|s_B|-1}r = (-1)^{r+s+\beta-1}r$. □

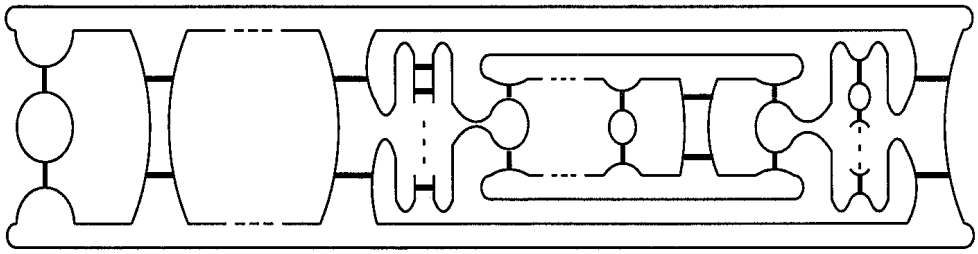

Figure 29: $D(r,s;\alpha,\beta)$ after s_B-smoothing.

Finally, we modify the link diagram $D(r,s;\alpha,\beta)$ in order to get a prime diagram $K(r,s;\alpha,\beta)$ of a knot . This is shown in Figure 30.

Theorem 9 *Let $K=K(r,s;\alpha,\beta)$ be the link diagram shown in Figure 30. Then*

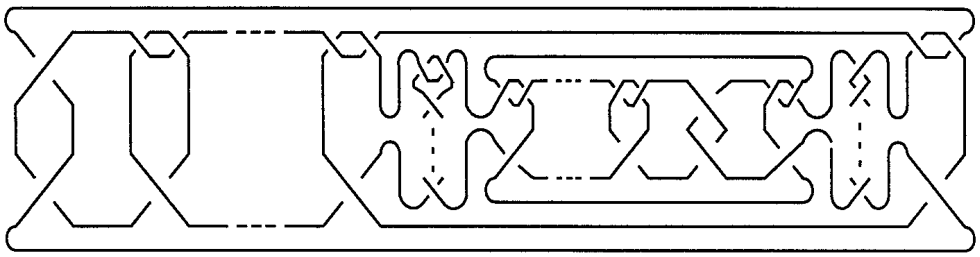

Figure 30: The knot diagram $K(r,s;\alpha,\beta)$.

$$\langle K\rangle = (-1)^{r+\beta} r A^{-5r-7s-\alpha-3\beta-1} + \cdots + (-1)^{s+\alpha-1} s A^{7r+5s+3\alpha+\beta-1}.$$

In particular $span(\langle K\rangle) = 12(r+s)+4(\alpha+\beta)+8$. *The highest degree of this polynomial is* $M-4$, *and the lowest degree is* $m+4$. *Moreover,* K *is a prime diagram of a knot.*

Proof. We first consider the modification $D' = D'(r,s;\alpha,\beta)$ of $D = D(r,s;\alpha,\beta)$ shown in Figure 31. Looking at the differences between D and D' after s_A and

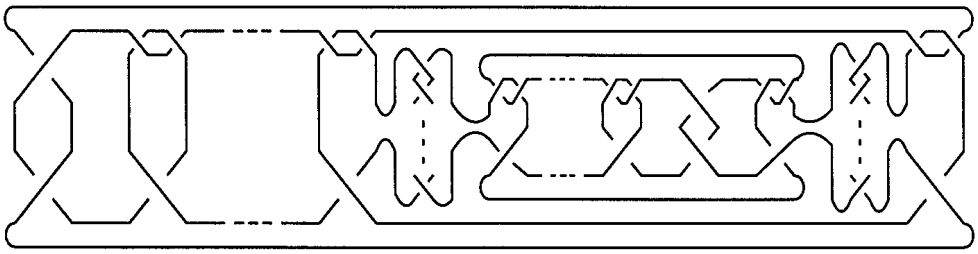

Figure 31: The link diagram $D' = D'(r,s;\alpha,\beta)$.

s_B-smoothings, we can prove that $\langle D'\rangle = (-1)^{r+\beta-1} r A^{-5r-7s-\alpha-3\beta+2} + \cdots + (-1)^{s+\alpha-1} s A^{7r+5s+3\alpha+\beta-2}$ where the highest and lowest degrees are $M-4$ and $m+4$ respectively.

But D' is not a knot diagram when the parity of α and β is the same. For this reason we introduce a small change in D' in order to get the prime diagram K of a knot. This change is shown in Figure 32. Note then that $c(K) = c(D')+1$ and $|s_A|$ does not change, hence $M(K) = M(D')+1 = 7r+5s+3\alpha+\beta+3$. Respect to the coefficients we have $a_M = 0$ and $a_{M-4} = (-1)^{|s_A|-1} s = (-1)^{s+\alpha-1} s$.

But after s_B−smoothing we have $|s_B| = |s_B(D')|+1$, hence $m(K) = m(L')-3 = -5r-7s-\alpha-3\beta-5$. Respect to the coefficients we have $a_m = 0$ and $a_{m+4} = (-1)^{|s_B|-1} r = (-1)^{r+\beta} r$. □

As we said in the introduction, we do not know if there is a nice interpretation of a_{M-4} in terms of graph theory, parallel to that one in which a_M is given in

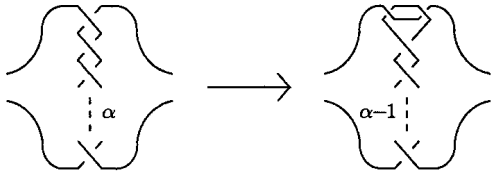

Figure 32: A small change in D' produces K.

terms of the graph G_D as described in the second section. Apart from the examples, this last section can be seen as a partial interpretation of a_{M-4} in terms of graph theory. More important, it remains to be answered the following question: how arbitrary can be the extreme coefficients of the Jones polynomial when the value for the spread is previously fixed?

Acknowledgements

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