But we have shown in Theorem 2.4 that V_r is ACD and so

$$V_r \# P = k(r)P \# l(r)Q;$$
$$k(r) = \tfrac{1}{3}r(r^2 - 6r + 11),\ l(r) = \tfrac{1}{3}(r - 1)(2r^2 - 4r + 3).$$

Thus $k' = k(r) - p$ and $l' = l(r) - b^-(V)$. Then in particular $k' \leqslant k(45p + 36)$ and $l' \leqslant l(45p + 36)$ and so the theorem follows.

Although these estimates give us a tractable bound on resolving numbers of complex surfaces we are still a long way from showing that all such surfaces are ACD. All progress in resolving Conjecture I′ has been made by considering families of surfaces which could be explicitly constructed. To quote Moishezon **[Msh]**, 'the "theoretical" Theorem 2.20 gives much weaker results than our "empirical knowledge". The interesting question is, how far can we move with such 'empirical achievements' in more general classes of simply-connected algebraic surfaces.'

CHAPTER 3. 4-MANIFOLDS AND THE CALCULUS OF LINKS

3.1 Framed links and the Kirby calculus. Having considered the question of decomposing algebraic surfaces via a combination of techniques from algebraic geometry and topology we discuss a more strictly topological method to obtain information about 4-manifolds. We recall that every PL-manifold admits a handlebody decomposition **[RS**, Chapter 6]. Thus to show that two manifolds are isomorphic we might try to show that they admit isomorphic decompositions. Kirby **[Kirb 1]** has developed a 'calculus' to manipulate handle decompositions of 4-manifolds which can be used to prove that two such manifolds are isomorphic. (See also **[FR]** and **[Sá]**.)

We recall that if M^4 is a connected 4-manifold then we can always decompose M^4 as

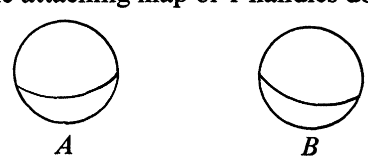
$$M^4 = H^0 \cup \bigcup H_i^1 \cup \bigcup H_j^2 \cup \bigcup H_j^3 \cup \bigcup_{k=1}^{n} H_k^4$$

where if $\partial M^4 = \varnothing$ we can assume $n = 1$ and if $\partial M^4 \neq \varnothing$ we may assume $n = 0$.

Each p-handle $H_i^p = D^p \times D^{4-p}$ has an attaching map f_i on $S^{p-1} \times D^{4-p}$ to $M^4 \cup$ {all handles preceding H_i^p}, with f_i an embedding, and the isomorphism class of our handlebody decomposition is determined by the isotopy classes of the f_i **[RS**, p. 71]. What are the various attaching maps we must worry about in the case of 4-manifolds?

The attaching maps for 1-handles are maps f: $S^0 \times D^3 \to \partial W$ (W will symbolize $H^0 \cup$ {other handles}). Essentially then we are just singling out neighborhoods of pairs of points as places where we will attach a 1-handle. All such f are clearly isotopic since W is connected and dim $W > 1$.

We can thus write the attaching map of 1-handles down in the form



in $S^3 = \partial H^0$ and our 2-handle will be an identification of the balls A and B in the above picture. [Note that $D^4 \cup \bigcup_{i=1}^{r} H_i^1$ is simply $\natural_{i=1}^{r} S^1 \times D^3$ with boundary $\#_{i=1}^{r} S^1 \times S^2$.]

Now let us consider 2-handles. Then our attaching maps $f\colon S^1 \times D^2 \to \partial W$ are precisely framings on embedded S^1's in ∂W. Now up to isotopy such framings are classified by $\pi_1(\mathrm{SO}^2) \simeq \mathbf{Z}$ so an attaching map of a 2-handle can be regarded as an embedded S^1 (i.e. a knot) with an integer attached to it representing the framing.

More generally, since by Principle 1 of Chapter 1 we can simultaneously attach all the handles to $\partial H^0 = S^3$ we make the following definition.

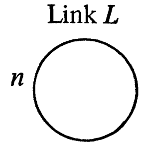
Definition 3.1 (Tentative). Let M be an oriented 3-manifold. L is a framed link in M if and only if L is a finite disjoint collection of smoothly embedded circles, $\gamma_1, \ldots, \gamma_r$, (knotted or unknotted), with an integer n_i associated with each γ_i. (Geometrically n_i means that the attaching map $f\colon S^1 \times D^2 \to S^3$ with $f(S^1 \times 0) = \gamma_i$ associated to (γ_i, n_i) is precisely one such that, for any $x \in D^2 - \{0\}$, $f(S^1 \times \{x\})$ has linking number n_i with γ_i. This means that the disc D^2 is twisted n-times in a *right-handed* direction as we traverse γ_i.) (We call $f(S^1 \times \{x\})$ a n_i-parallel curve for γ_i.)

If M is the boundary of an oriented 4-manifold V and M has the induced orientation we shall let V_L denote the manifold obtained by adding handles to V along M via the recipe given by the framed link L. We shall denote ∂V_L by $\chi_L(M)$. Note that the construction of V_L depends only on the orientation of V and not on orientations of components of L.

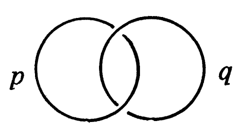
Example 1.

Some examples if $M = S^3$

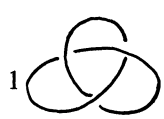
Link L

(1) 

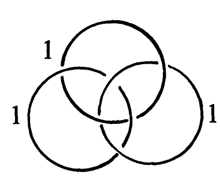
$\chi_L(M) =$ Lens space $L(n, 1)$

(2) 

$L(pq - 1, p) = L(pq - 1, q)$

(3) 

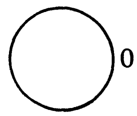
Dodecahedral space $P = S^3/G$
$G =$ binary dodecahedral group

(4) 

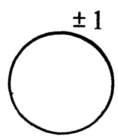
Dodecahedral space

EXAMPLE 2.

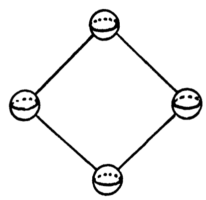
Some examples of B_L ($B = B^4$ with $\partial B^4 = S^3$)

(5) 

$S^2 \times D^2$

(6) 

$\pm CP^2 - B^4$

(7) 
$T^2 \times D^2$

(8) 

$P \# Q - B^4$

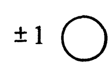
(9) 

$S^2 \times S^2 - B^4$

Now suppose L, L' are framed links in S^3. The framed link $L + L'$ will denote the disjoint union of the two links in S^3. Then using the above examples we see that if L' is the link in 5, 6, 8 or 9 above then

$B_{L+L'} = B_L \natural S^2 \times D^2$ if $L' =$ 

(10)

$B_{L+L'} = B_L \# \pm CP^2$ if $L' =$ 

(11)

$B_{L+L'} = B_L \# P \# Q$ if $L' =$ 

(12)

$B_{L+L'} = B_L \# S^2 \times S^2$ if $L' =$ 

(13)

where $\natural$ denotes the boundary connected sum and $\#$ is the connected sum in the interior of the manifold with boundary. Notice that $\chi_{L'}(\chi_L(M))$ is $\chi_L(M) \# S^2 \times S^1$, in case (10) and $\chi_{L'}(\chi_L(M)) = \chi_L(M)$ in cases (11), (12), (13).

There are two basic 'moves' introduced by Kirby in manipulating such framed link pictures. We assume henceforth that our 4-manifold B is simply B^4.

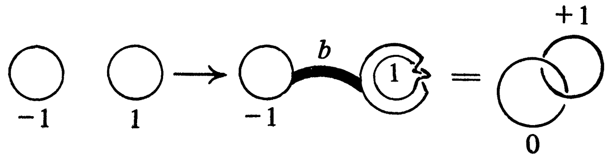
MOVE 1. $L \to L + \underset{+1}{\bigcirc}$. This move keeps $\chi_L(M)$ fixed but changes B_L to $B_L \# \pm \mathbf{C}P^2$ or in reverse changes $B_L \# \pm \mathbf{C}P^2$ to B_L.

MOVE 2. Given two circles γ_i and γ_j in L we "add" γ_i to γ_j as follows. Using the framing n_i of γ_i, let $\tilde{\gamma}_i$ be an n_i-parallel curve of γ_i. Now change L by replacing γ_j with $\gamma_j' = \tilde{\gamma}_i \#_b \gamma_j$, where b is a band connecting $\tilde{\gamma}_i$ and γ_j and missing the rest of L.

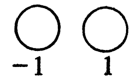
This move corresponds in B_L to adding (subtracting) the jth handle to (from) the ith handle. The new framing n_j' equals $\gamma_j + \gamma_i \pm 2a_{ij}$ where a_{ij} is the linking number of γ_i and γ_j (after they have been assigned orientations). (The linking number $lk(\xi, \eta)$ of disjoint knots ξ and η in S^3 can be defined as the image of the homology class $[\xi]$ in $H_1(S^3 - \eta) \approx \mathbf{Z}$, where S^3, ξ, η have been given fixed orientations.) The sign in the equations depends on whether or not the band b preserves orientation. Notice that Move 2 changes neither B_L nor $\chi_L(M)$. It simply provides a new handle decomposition for them.

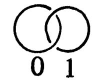
EXAMPLE 3.

(i)

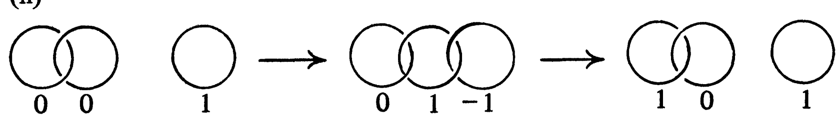


This shows that

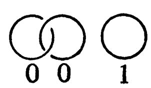


$= P \# Q - B^4 =$


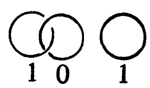
(ii)



This shows that



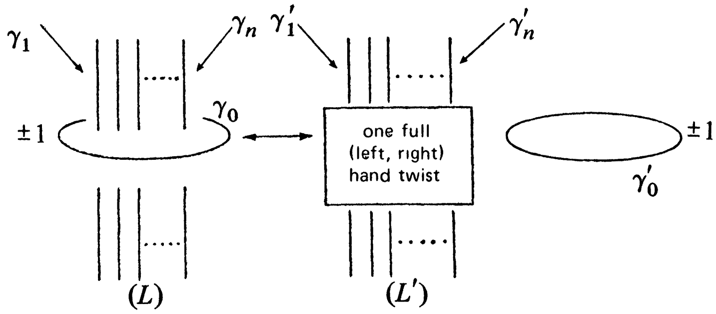
$= S^2 \times S^2 \# \mathbf{C}P^2 - B^4$

$=$


$= P \# Q \# \mathbf{C}P^2 - B^4 = 2P \# Q - B^4$

Thus one can replace Move 2 by the following generalized Move 2′ which it implies.

MOVE 2′. Let L be a link containing the portion (L) pictured below. Then go from L to L' or back where L' is the link identical to L except that the portion (L) of L has been changed to (L').

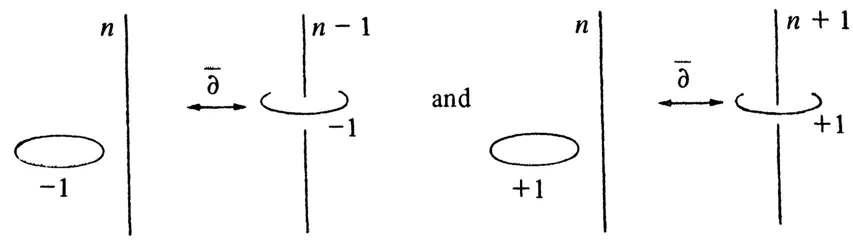


The linking matrix $\lambda_{L'}$ for L' is then given by

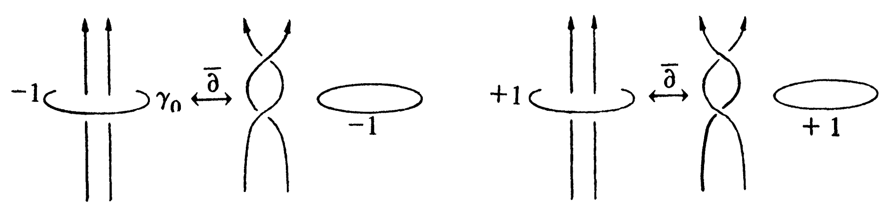
$$\begin{aligned} & lk(\gamma_i', \gamma_j') \quad \text{if } i \neq j \\ (\lambda_{L'})_{ii} &= (\lambda_L)_{ii} \mp [(\lambda_L)_{0i}]^2 \quad \text{if } i \neq 0 \\ &= \pm 1 \quad \text{if } i = j = 0, \end{aligned}$$

where λ_L is the linking matrix for L.

In case $n = 1$ or $n = 2$ the equivalences are pictured below.



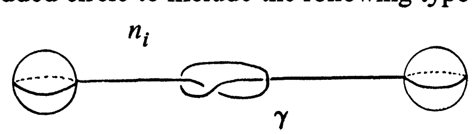
1-strand



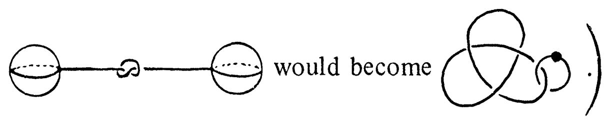
2-strands

Move 2′ proves very convenient when actually computing link equivalences.

Framed links as we have defined them only represent manifolds having a handle decomposition containing a 0-handle and some 2-handles. We thus define a generalized framed link by first adding in pairs of 3-balls representing the 1-handles. Note that if 1-handles are allowed we must broaden our definition of an embedded circle to include the following type of example



In this picture the curve γ with framing n_i really represents a circle embedded in the manifold ∂ (⊖ ⊖) $= \partial(D^4 + 1\text{-handle}) = S^1 \times S^2$ and homologous to $S^1 \times 0$! (We shall discuss a method of representing such an example by pairs of circles in S^3 later in this chapter



We thus redefine framed links to include pairs of embedded 3-balls representing 1-handles. To represent 3-handles we would have to draw in 2-spheres in our pictures. However as a consequence of a result of Montesinos and Laudenbach-Poeneru **[Mont 3]**, **[LP]** this is not necessary!

In fact we have the following theorem.

THEOREM 3.2 **[Mont 3]**. *Let M be a closed orientable* (PL) 4-*manifold with handle presentation $M = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$. Then M is completely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$.*

Thus the way the 3- and 4-handles are pasted in cannot affect the topology of M. We must therefore only keep track of the number, γ, of 3-handles we must add to a given link picture without worrying at all about their locations. Thus our present definition of a framed link allows us to take care of all possible handlebody decomposition of compact manifolds without boundary.

Given two framed links L_1 and L_2 we shall say $L_1 \sim_\partial L_2$ (i.e., L_1 is boundary equivalent to L_2) if and only if we can go from L_1 to L_2 by a sequence of moves of types 1 and 2. We then have:

THEOREM 3.3 (KIRBY) **[Kirb 1]** (SEE ALSO **[Crg]**). *$L_1 \sim_\partial L_2$ if and only if $\chi_{L_1}(S^3)$ is diffeomorphic to $\chi_{L_2}(S^3)$ (preserving orientations).*

If we are interested in the 4-manifolds B_L instead of their boundaries we must replace Move 1 by a move preserving B_L (instead of just its boundary).

The only geometric operation we have not covered is handle cancellation and introduction.

Thus Move 1′ will be the introduction of a cancelling pair of 1- and 2-handles or 2- and 3-handles.

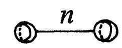
Now if $\bigcirc\ \bigcirc$ is a 1-handle then will be a cancelling pair consisting of a 1-handle and a 2-handle. [Note again that the horizontal line in really represents a circle in $\partial(B^4 \cup$ $)$ since the two 3-balls are identified via the 1-handle attaching them.]

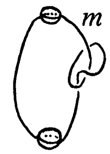
An alternate way of introducing such a cancelling pair is to note that if represents a 2-handle attached to some simply-connected manifold V giving $V' = V \natural S^2 \times D^2$ then surgering the 2-sphere S^2 above corresponds to attaching a 1-handle to V giving $\chi(V') = V \natural S^1 \times D^3$.

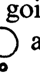
We can then thus represent a 1-handle as a surgered 2-handle which we write

(an unknot with a dot on it).

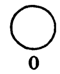
Notice that

n  is thus equivalent to 

while n  m would represent n 

(i.e. going through  is the same as going 'over' the 1-handle). To introduce a cancelling 2-3 pair we put down  and keep track of the complementary 3-handle without drawing it in.

Move 1′ is then:

Introduce  or  + (3-handle)

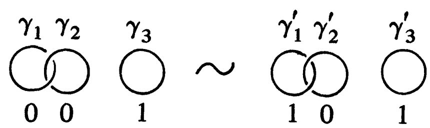
We say $L_1 \sim L_2$ if we can go between them by moves of type 1′ and type 2.

As an analogue of Theorem 3.3 we then have Theorem 3.3′ (Sá) **[Kirb 1, Sá]** $L_1 \sim L_2$ if and only if B_{L_1} is diffeomorphic to B_{L_2}. (See Sá **[Sá]** for a complete proof.)

Notice that if V_1 and V_2 are simply-connected 4-manifolds we can always write down link-pictures for $V_1' = V_1 - \{\text{4-ball}\}$ and $V_2' = V_2 - \{\text{4-ball}\}$. Clearly $V_1 = V_2$ if and only if $V_1' = V_2'$ and thus if $V_1' = B_{L_1}$ and $V_2' = B_{L_2}$ then we can show that V_1 is diffeomorphic to V_2 by showing that $L_1 \sim L_2$.

We apply these techniques in a few cases. (More applications will be found in later chapters.)

3.2 Handlebody decompositions of 4-manifolds. The simplest application of the link calculus arises in Example 3(2) above where



shows that $S^2 \times S^2 \# P = 2P \# Q$.

We note that in the above example the matrix (a_{ij}) where $a_{ij} = lk(\gamma_i, \gamma_j)$ and $a_{ii} = n_i$ is, in fact, the intersection matrix for the resulting 4-manifold.

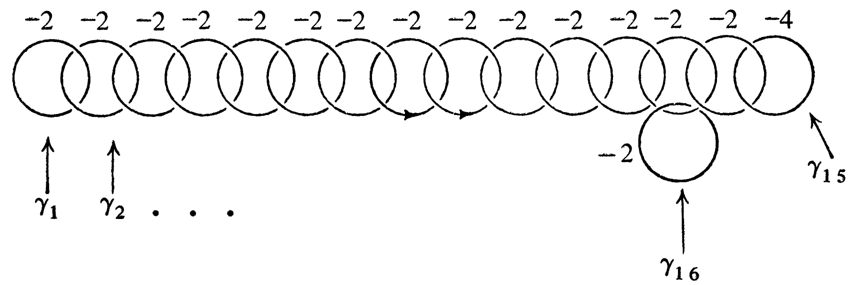
In fact this is a direct consequence of the definition of a handlebody decomposition. In general if L is a framed link with components γ_i with framing n_i, then setting $a_{ij} = lk(\gamma_i, \gamma_j)$, $a_{ii} = n_i$ gives us a matrix A_L representing the intersection form on B_L. We note that $H_1(\partial B_L) = 0$ if and only if A_L is unimodular. Thus to recognize a homology 3-sphere as the boundary corresponding to a link diagram we can simply construct the linking matrix A_L described above and compute its matrix. In particular if K is any knot then $(K, \pm 1)$ will be a homology sphere.

In Chapter 4 we shall discuss some of the 4-dimensional problems associated with homology 3-spheres and the 4-manifolds they bound. As preparation for some of the techniques used we present the following example of Kirby **[Kirb 1]**.

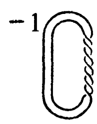
Suppose H is a homology 3-sphere which bounds an oriented 4-manifold M of type II, with $b_2(M) = (\sigma(M)) = 16$. If we could also show that H bounds another contractible 4-manifold X then $V = M \cup X$ would be the long sought after closed spin manifold with $b_2(M) = (\sigma(M)) = 16$. For example, let

$$\Sigma(a, b, c) = \{(x, y, z) \in \mathbf{C}^3 | x^a + y^b + z^c = 0\} \cap \{|x|^2 + |y|^2 + |z|^2 = \varepsilon\}.$$

Then $\Sigma(2, 7, 13)$ is a homology sphere which bounds an M as above. In fact, using some techniques from the theory of resolutions of singularities we can see that $\Sigma(2, 7, 13)$ is precisely $\chi_L(S^3)$ where L is the link

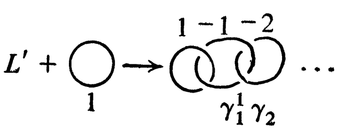


We claim that $\Sigma(2, 7, 13)$ is also the boundary of $\chi_{L'}(S^3)$.

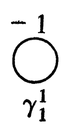
$L' =$ 

(This is the (2, 7) torus knot.)

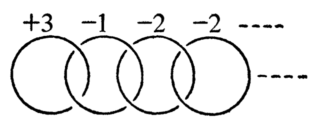
To show that $\chi_L(S^3) = \chi_{L'}(S^3)$ we can use moves 1 and 2 to obtain



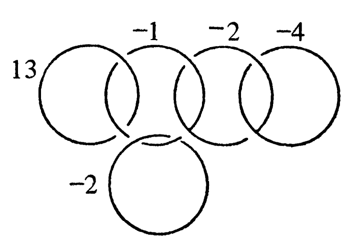
(rest of picture is the same).

We can use the 

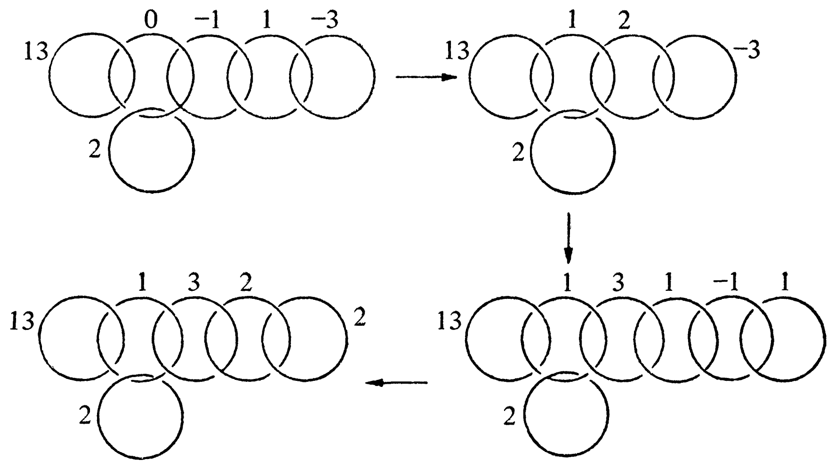
above to split off successive components and arrive at



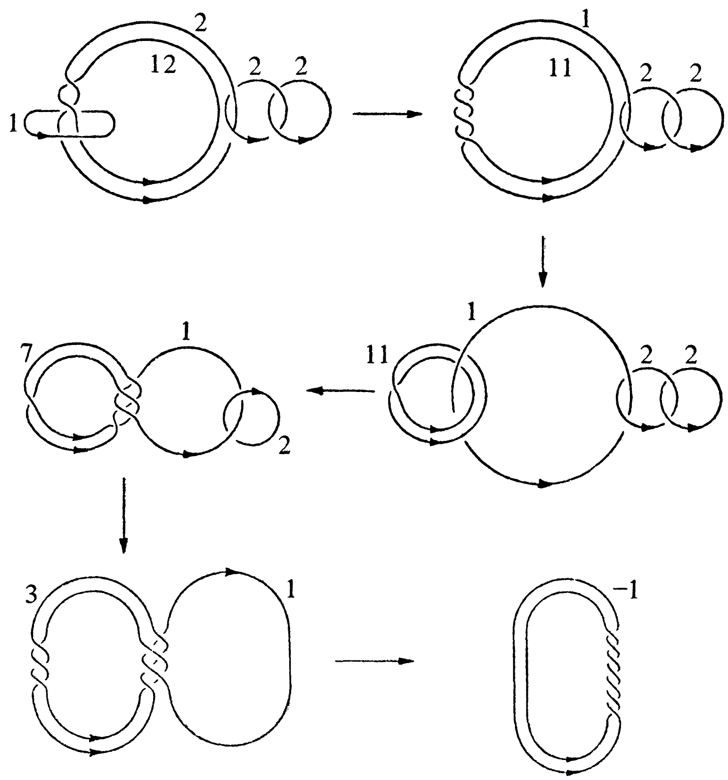
We iterate, obtaining first



and then by adding additional  we get



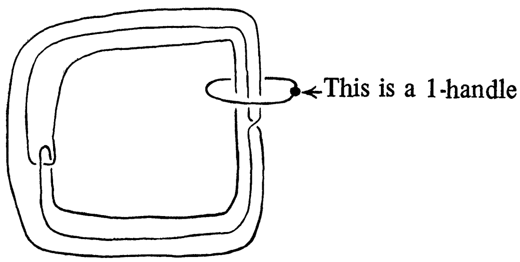
Now removing $+1$ circles successively we obtain as desired.



The problem then remains to show that $\Sigma(2, 7, 13)$ bounds a contractible manifold. One might try to solve this by constructing contractible 4-manifolds and examining their boundaries.

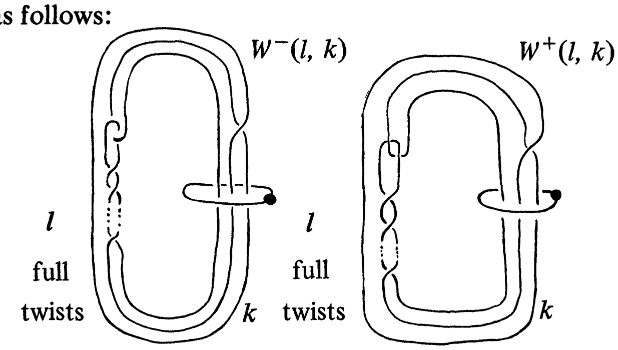
The first example of a compact contractible 4-manifold which is not D^4

was constructed by Mazur [**Mz**]. In our notation Mazur's manifold W is



[We have attached a 2-handle to $S^1 \times B^3$ along, a framed circle homologous to but not isotopic to $S^1 \times *$ in $S^1 \times S^2$.]

In [**AK 2**] Kirby and Akbulut defined Mazur manifolds $W^-(l, k)$, $W^+(l, k)$ as follows:



All of the manifolds can be seen to be contractible by simply noticing that $W^-(l, k) \times R \approx W^+(l, k) \times R \approx R^5$. [In fact, Mazur [**Mz**] showed that whenever one attaches a 2-handle to $S^1 \times B^3$ by a curve C in $\partial(S^1 \times B^3) = S^1 \times S^2$ such that C generates $H_1(S^1 \times S^2)$ one obtains a manifold W_C with $W_C \times I \approx R^5$. To arrange things so that $\pi_1(\partial W_C) \neq 0$ one must choose C so it is not of the same knot type as the standard $S^1 \times 1 \hookrightarrow S^1 \times S^2$.]

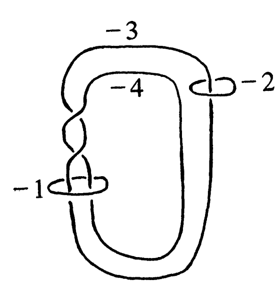
Then using the calculus of links one calculates $\partial W^-(l, k)$ and $\partial W^+(l, k)$.

The following results are obtained:

PROPOSITION 3.4.
(1) $\partial W^{\pm}(l, k) = \partial W^{\pm}(l + 1, k - 1)$,
(2) $\partial W^-(l, k) \approx \partial W^+(-l + 2, -k + 1)$,
(3) (a) $\partial W^+(0, 0) \approx \Sigma(2, 5, 7)$,
(b) $\partial W^+(-1, 0) \approx \Sigma(3, 4, 5)$,
(c) $\partial W^+(1, 0) \approx \Sigma(2, 3, 13)$.

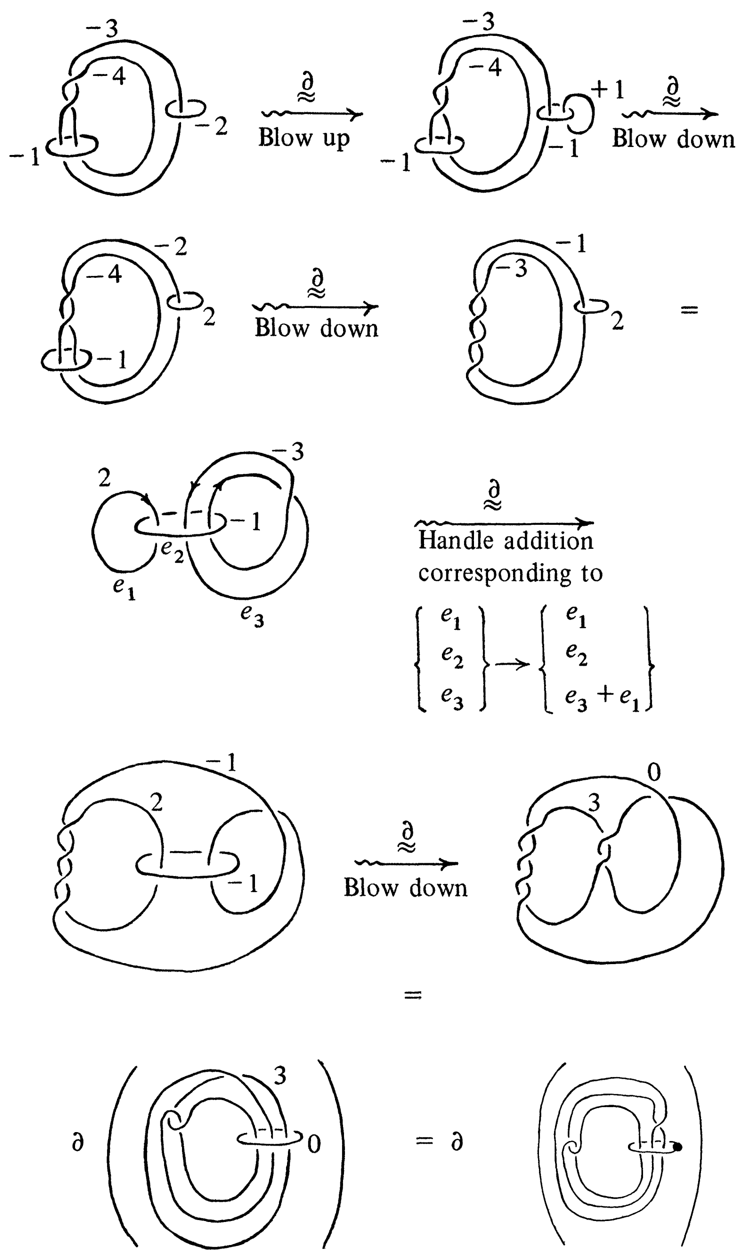
To further demonstrate the techniques of the link calculus we include the proof of (3)(a) (taken from [**AK 2**]).

We have

$\Sigma(2, 5, 7) =$ 

by definition.

Now



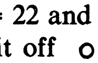
(where we have changed the 2-handle ○ ° to the 1-handle in the interior of the manifold)

$= \partial W^-(0, 3) \approx \partial W^-(1, 2) \approx \partial W^-(2,1)$. From (1) of Proposition 3.4
$= \partial W^+(0, 0)$ from (2) of Proposition 3.4.

In **[Ram]**, Ramanujam has constructed a nonsingular complex affine algebraic surface V^2 which is rational and contractible (∂V is a homology but not a homotopy 3-sphere) such that V^2 is not analytically equivalent to $\mathbf{C}^2$! Kirby **[Kirb 5]** has found a framed link L representing V and shown V to be essentially a 'Mazur' manifold. $V \times \mathbf{R}^2$ is homeomorphic to $\mathbf{R}^6$ but it is not known whether $V \times \mathbf{C}$ is analytically isomorphic to $\mathbf{C}^3$. The construction of V is as follows.

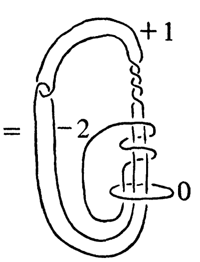
Let C_1 be a cubic curve in $\mathbf{C}P^2$ with a cusp. Let C_2 be a nondegenerate conic meeting C_1 at two distinct points P, Q of orders 5 and 1, respectively, such that P, Q are neither the cusp nor inflection point of C_2. (For example take C_1: $x^3 - y^2 = 0$ and C_2: $x^2 - \frac{8}{15}xy + \frac{1}{9}y^2 + \frac{1}{3}x - \frac{8}{9}y - \frac{1}{45}$ which intersect with multiplicity 5 at (1, 1) and multiplicity 1 at some other point Q.) Blow $\mathbf{C}P^2$ up at Q to get the variety F and let C_1', C_2' be the proper transforms of C_1, C_2. Then V is defined as $F - C_1' - C_2'$. For a proof that V has the requisite properties see **[Ram]** or **[Kirb 5]**.

Harer, Casson and Kaplan (**[HC]**, **[Kap]**) have used these methods to construct numerous other examples of homology spheres $\Sigma(a, b, c)$ which bound contractible or acyclic manifolds. As yet however no example has been found of such a homology sphere which bounds both an acyclic manifold on the one hand and a manifold with definite form of type II on the other.

3.3 Special handlebody decompositions. Another possible approach to finding a spin manifold M with $b_2(M) = |\sigma(M)| = 16$ is by directly constructing a handle decomposition of a spin manifold W with $b_2 = 22$ and $|\sigma| = 16$ and attempting to manipulate such a decomposition to split off  pairs. This would then correspond to decomposing W as $W = W' \# S^2 \times S^2$ with $\sigma(W') = \sigma(W)$ and $b_2(W') = 20$.

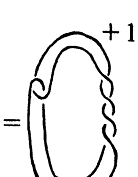
We first exhibit a (22, 16) manifold constructed by Kirby and Akbulut **[AK 4]** which has a handle decomposition with no 1- or 3-handles. The resultant manifold is homotopy equivalent to V_4 but it is not known whether it is diffeomorphic to it.

Let

$M_1^4 =$ 

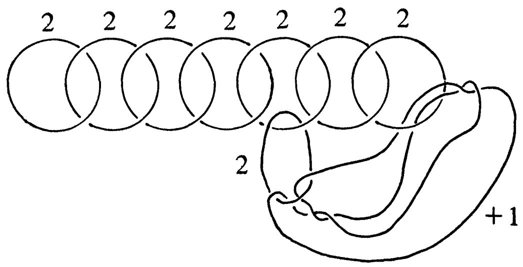
Then by the 'calculus' one shows that $\partial M_1^4 = S^3$.

Now let

$M_2^4 =$ 

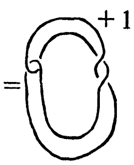
We note that $M_1^4 = M_2^4 \cup h_{11} \cup h_{22}$ where h_{11}, h_{22} are the 2-handles attached to the circles with framing 0 and −2 in the link diagram for M_1^4.

By successive moves on M_2^4 one obtains the manifold N_2^4 pictured below, preserving $\partial M_2^4 = \partial N_2^4$.



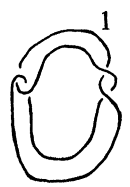
$= N_2^4$ (definition)

Let $M_3^4 =$

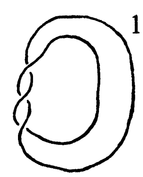


Then $N_2^4 = M_3^4 \cup h_{21} \cup \cdots \cup h_{28}$ where the handles h_{2i} are the handles attached to the $+2$ circles in the link diagram for N_2^4.

Now let $M_4^4 =$

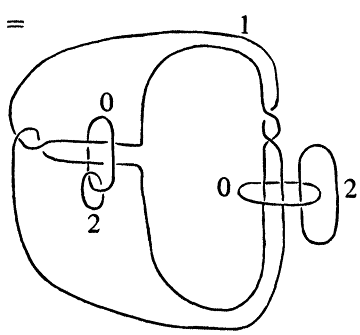


$=$



By successive moves on M_3^4 we get

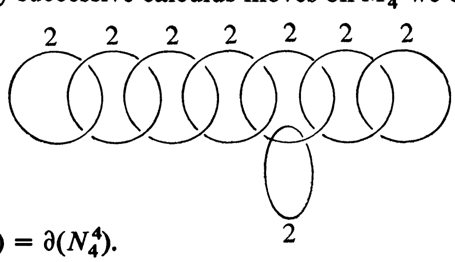
$N_3^4 =$



with $\partial M_3^4 = \partial N_3^4$.

We again note that $N_3^4 = M_4^4 \cup h_{31} \cup \cdots \cup h_{34}$ where the h_{3i} are the handles attached to the circles with framings 0 and 2 in the link diagram for N_3^4.

Lastly by successive calculus moves on M_4^4 we construct



$= N_4^4$ (definition)

with $\partial(M_4^4) = \partial(N_4^4)$.

We now let

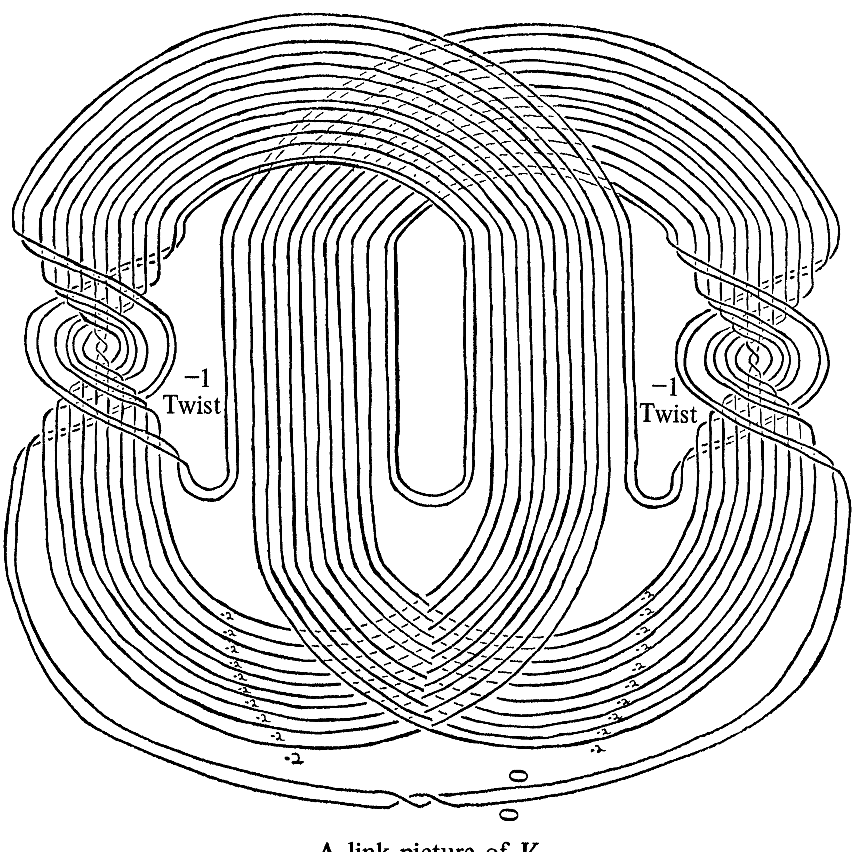
$$M^4 = N_4^4 \cup_{\partial N_4^4 = \partial M_4^4} \left(\bigcup_{i=1}^{4} h_{3i} \right) \cup_{\partial N_3^4 = \partial M_3^4} \left(\bigcup_{i=1}^{8} h_{2i} \right) \cup_{\partial N_2^4 = \partial M_2^4} \left(\bigcup_{i=1}^{2} h_{1i} \right).$$

Then $\partial M^4 \approx \partial M_1^4 \approx S^3$. Let $W = M^4 \cup_{\partial M^4} B^4$.

Then W is a simply-connected compact 4-manifold with intersection matrix determined by the linking matrix of our link diagram above. In particular we can compute A_L to be $A_L \approx E_8 \oplus E_8 \oplus 3U$.

Thus W is a spin manifold with $b_2(W) = 22$, $\sigma(W) = 16$. However, although this procedure gives us an explicit picture of W, all attempts to split off a sublink $\circ \bigcirc\!\!\bigcirc \circ$ representing a factor U in the decomposition of A_L have been unsuccessful.

Does there exist a decomposition with no 1- and 3-handles for V_4? In **[HKK]** Harer, Kas and Kirby answer this question in the affirmative by explicitly constructing such a decomposition. Their link picture is exhibited in Figure 3.2.



A link picture of V_4.

FIGURE 3.2

Now let M^4 be a simply-connected 4-manifold. M^4 will be said to admit a special handlebody decomposition if it admits a handle decomposition with one 0-handle, $b_2(M^4)$ 2-handles and one 4-handle. In this case M^4 can be completely represented by a framed link L [with no 1-handles (or 3-handles) in it]. It is an open question whether every simply-connected (PL) 4-manifold admits a special handlebody decomposition. In the algebraic case Rudolph **[Rd]** showed that every nonsingular hypersurface V_n of $\mathbf{C}P^3$ admits a decomposition with no 3-handles and **[HKK]** showed that V_4 admits a special decomposition. (V_1, V_2, V_3 clearly admit such decompositions since they are rational.) In **[Har]**, **[At 3]** it is shown that all $V_n \subset \mathbf{C}P^3$ admit such decomposition and in **[Man 3]** it is shown that all complete intersections and simply-connected elliptic surfaces (with no more than one multiple fiber) have special decompositions.

Casson **[Cas 3]** however has shown that there exists a compact simply-connected PL 4-manifold with $\partial M \neq 0$ requiring 1-handles in any handlebody decomposition. This result follows from the following observation.

Suppose M is a simply-connected 4-manifold with $\partial M \neq 0$ having a handle decomposition $M = D^4 \cup \bigcup_{i=1}^{n} h_i^{(2)}$. Then $\overline{M - D^4}$ gives a cobordism between S^3 and ∂M having only 2-handles. However adding a 2-handle to ∂M has the effect of adding one new generator and one new relation to $\pi_1(\partial M)$. Thus the existence of a cobordism between ∂M and S^3 having only 2-handles means that the group $\pi_1(\partial M)$ can be trivialized by adding the same number of generators and relations. Gerstenhaber and Rothaus **[GR]** have shown however that there exist finitely-presented groups $\{G\}$ which cannot be trivialized in this fashion, and Casson has shown that there exists a contractible 4-manifold M with $\pi_1(\partial M) \in \{G\}$. Thus M cannot have a decomposition with no 1-handles. (Note that in high dimensions such a counterexample is impossible as Wall **[Wa 3]** has shown that algebraic connectivity $\Rightarrow$ geometrical connectivity for manifold M with $\dim(\partial M) \geqslant 5$.)

We note that a special handlebody decomposition and the corresponding link picture is usually quite useful in proving that a given surface is almost completely decomposable. In Figure 3.3 the first step in a direct proof, using link calculus moves, that V_4 is ACD is shown.

Recently Akbulut **[At 3]** has developed new techniques for producing special decompositions of branched cyclic covers. Using these techniques he and Kirby **[AK 3]** show that direct proofs of almost complete decomposability can be demonstrated using the 'calculus'. For example they show

THEOREM 3.5. *Let V be a 2-fold cyclic covering of $\mathbf{C}P^2$ branched over a curve of degree $2n$. Then V is ACD.*

Cyclic covers are, of course, not sufficient to generate all oriented 4-manifolds. In dimension 3 Hilden and Montesinos **[Hd]**, **[Mont 1]**, **[Mont 4]** have shown that every oriented 3-fold M^3 is a 3-fold dihedral covering manifold of S^3. In **[Mont 2]**, Montesinos has shown that every orientable 4-manifold V having a handle decomposition with no 3- or 4-handles arises as a 3-fold dihedral cover of S^4. Berstein and Edmonds **[BE]** have shown that $T^4 = S^1 \times S^1 \times S^1 \times S^1$ cannot be represented as a 3-fold covering of S^4. (More precisely they have shown that if an orientable n-manifold M is a p-fold

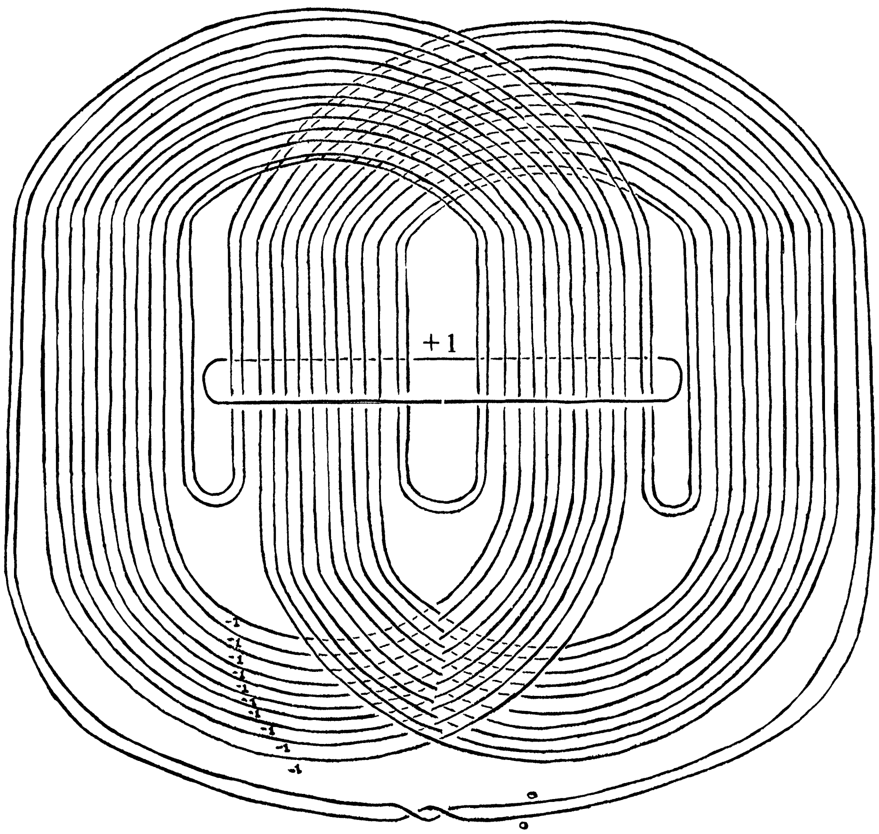


FIGURE 3.3 A link diagram for $V_4 \# \mathbf{C}P^2$.

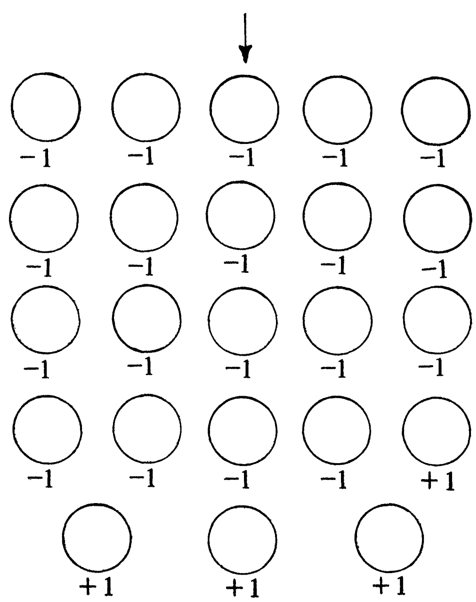


FIGURE 3.3a The result of type 2 moves on Figure 3.3

covering of S^4 then $p \geq$ cup length (M), where the cup length of a manifold M is the maximum integer r such that there exist $\gamma_1, \ldots, \gamma_r \in H^*(M, \mathbf{Z})$ with $\gamma_1 \cup \cdots \cup \gamma_r \neq 0$. By a theorem of Alexander [**Ax 1**] every orientable n-manifold M *is* a p-fold covering of S^4 for some finite p.)

The construction of dihedral covering manifolds for a given 4-manifold is quite a bit more complicated than the construction of cyclic covers. In [**CS 4**] Cappell and Shaneson have indicated a novel approach to the construction of dihedral covers M of B^4 such that ∂M is a dihedral covering manifold of S^3 branched over a knot and shown how the μ-invariant of ∂M can be calculated. (We discuss μ-invariants next.) Akbulut [**At 3**] has noted that their method can be adapted to give link diagrams of more general dihedral covers.

In accordance with the results mentioned obtained above Montesinos and Edmonds have conjectured:

CONJECTURE. Let M be a simply-connected 4-manifold. Then M admits a representation as a 3-fold branched cover of S^3.

CONJECTURE. Let M be an oriented n-manifold. Then M admits a representation as an n-fold branched cover of S^n.

Before closing this section we point out that it is not always possible to show that two link diagrams give equivalent manifolds without moves of type I. Thus simply showing that by adding and subtracting link components one cannot transform link L_1 to link L_2 does not suffice to conclude that $B_{L_1} \neq B_{L_2}$ or $\partial B_{L_1} \neq \partial B_{L_2}$. In fact Akbulut has shown that if K, R are the framed links of Figure 3.4 then K is not equivalent to R (a direct computation shows that K has signature -2 and R has signature zero) but $B_K = B_R$ [**At 1**].

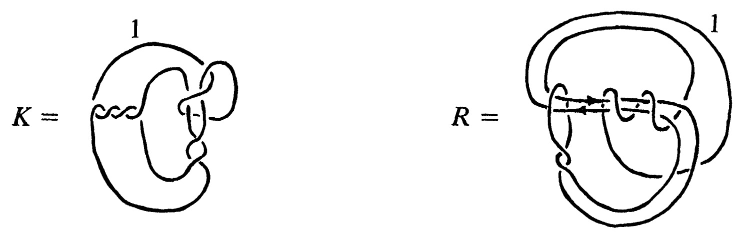


FIGURE 3.4

CHAPTER 4. THE μ-INVARIANTS, HOMOLOGY SPHERES AND FAKE 4-MANIFOLDS

4.1 The μ-invariant. One of the only known numerical invariants interrelating 3- and 4-dimensional topology is the μ-invariant [**Hirz 2**], [**EK**] which we now discuss.

Suppose M^n is a manifold with tangent bundle TM. A fixed trivialization $\mathcal{F}\colon TM \to M \times \mathbf{R}^n$ of TM will be called a framing on M and $(M, \mathcal{F})$ will be called a framed manifold. We recall that the stable tangent bundle T^sM of M can be geometrically realized as $TM \oplus \varepsilon^r$ for r sufficiently large ($\varepsilon^r = M \times \mathbf{R}^r$) and thus a stable framing on M is a fixed trivialization of some $TM \oplus \varepsilon^r$. Lastly, an almost framing on TM is a framing on $T(M - \{\mathrm{pt}\})$. We have:

THEOREM 4.1 (MILNOR [**M 2**]); ALSO [**Kap**]. *Let* $(M^3, \mathcal{F})$ *be a closed oriented stably framed* 3-*manifold. Then there exists a compact stably framed* 4-*manifold* $(W^4, \mathcal{G})$ *with* $\partial W = M$ *so that* $\mathcal{G}$ *and* $\mathcal{F}$ *agree on* $M - \{\mathrm{pt}\}$. (W^4 *can, in fact, always be found with* $\pi_1(W^4) = 0$.)

PROOF. We shall later outline Kaplan's proof of the above result.

A result of Whitehead [**M 3**, §4] shows that stably framed implies almost framed and that for connected manifolds with boundary, framed = almost framed = stably framed. Thus W above will in fact be framed. We also recall that a framed (stably framed) manifold is sometimes called parallelizable (stably or S-parallelizable or π-manifold) and an almost framed manifold is called almost parallelizable.

We note that since W is framed the form L_W must be even. Thus if $(W', \mathcal{G}')$ is any other 4-manifold satisfying the conclusion of Theorem 4.1 we can form $V = W \cup W'$ and using our stable framings show that V is also stably framed. Then using Novikov additivity [**Hirz 2**] we find $\sigma(W) - \sigma(W') = \sigma(V)$ and by Rohlin's theorem $\sigma(V) \equiv 0 \pmod{16}$.

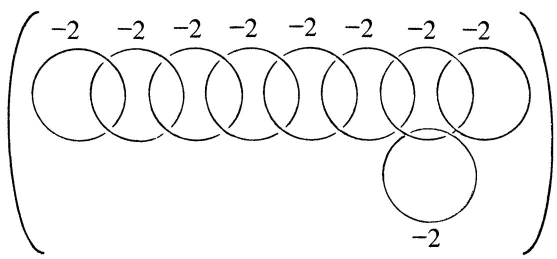
We are thus led to the following.

DEFINITION 4.2. Let $(M, \mathcal{F})$ be a stably framed 3-manifold and suppose $(W, \mathcal{G})$ is a framed 4-manifold with $\partial W = M$ and $\mathcal{F} = \mathcal{G}$ on $M - \{\mathrm{pt}\}$.

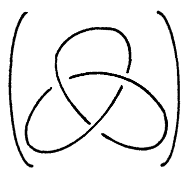
Then set $\mu(M, \mathcal{F}) = \sigma(W) \bmod 16$ (by our discussion above $\mu(M, \mathcal{F})$ is clearly well defined).

We note that to define μ it is enough to specify some almost framing on M^3. However by obstruction theory since $\pi_2(\mathrm{SO}(3)) = 0$ we find that the almost framings of M^3 will be in 1-to-1 correspondence with the elements of $H^1(M, \pi_1(\mathrm{SO}(3))) = H^1(M; \mathbf{Z}_2)$. Thus if M^3 is a $\mathbf{Z}_2$-homology sphere it has a unique almost framing. Thus for $\mathbf{Z}_2$-homology spheres M one can speak of $\mu(M)$ without referring to an almost framing of M. In this case it can further be shown that $\mu(M)$ is in fact an invariant of the h-cobordism class of M.

Now let $\mathcal{H}^3(G)$ be the abelian group of h-cobodism classes of G-homology 3-spheres. (The group structure is given by the connected sum operation. $[S^3]$ is the identity and $[-M^3]$ is the inverse of M^3, where $-M$ is M with orientation reversed.) Then it is not difficult to show that μ: $\mathcal{H}^3(\mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}_{16}$ is a homomorphism. It is clear that $\operatorname{Im} \mu$ is contained in a subgroup of $\mathbf{Z}_{16}$ isomorphic to $\mathbf{Z}/8\mathbf{Z}$ and by calculation of μ-invariants of lens spaces one can show that μ is in fact onto this subgroup. If M is a $\mathbf{Z}$-homology sphere then for W defined as in the theorem we have $\sigma(W) \equiv 0 \pmod 8$. Thus in fact μ: $\mathcal{H}^3(\mathbf{Z}) \to \mathbf{Z}_8$ is onto a subgroup $\mathbf{Z}_2 \approx \mathbf{Z}/2\mathbf{Z}$. (To see that μ: $\mathcal{H}^3(\mathbf{Z}) \to \mathbf{Z}_2$ is onto we simply note that if P is Poincaré's homology 3-sphere then if

$W =$ 

clearly $\sigma(W) = 8$ and using Kirby moves one can show that

$P = \partial W = \partial$ 

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Thus $\mu(P) = 8$ which generates the $\mathbf{Z}_2$ above.

We consider some applications of the μ-invariant. We see immediately that by Alexander duality if $M \in \mathcal{H}^3(\mathbf{Z}/2\mathbf{Z})$ is embedded in $\mathbf{R}^4$ then $\mu(M) = 0$.

Now since μ is a diffeomorphism invariant if we could find a homotopy $\mathbf{Z}$-sphere M with $\mu(M) \neq 0$ then M would be a counterexample to the Poincaré conjecture in dimension 3. In **[CS 3]**, Cappell and Shaneson have developed a formula for computing the μ-invariant for any p-fold dihedral cover M of S^3, branched over a knot such that M is a $\mathbf{Z}/2\mathbf{Z}$ homology-sphere. Since by **[Mont 1]**, **[Mont 4]**, **[Hd]** every orientable 3-manifold can be represented as such a cover it might be possible to find a counterexample to the Poincaré conjecture by constructing the p-fold dihedral covering manifolds $M_p(K)$ associated with all the knots K in the Conway **[Con]** tables, computing their fundamental groups $\pi_1(M)$ and μ-invariants if $\pi_1(M) = 0$. (This might profitably be done by computer.)

A possible application in dimension 4 of the μ-invariant is to the problem of finding simply-connected 4-manifolds which are spin and have definite forms. In particular it would be desirable to find out when elements $M \in \mathcal{H}^3(\mathbf{Z})$ bound contractible or acyclic 4-manifolds. Work in this direction can be found in **[HC]**, **[Kap]**. Gordon has devised a construction in **[Gor 1]** which associates a contractible 4-manifold with boundary a $\mathbf{Z}$-homology sphere to any slice knot $K \subset S^3$. (A knot $K \subset S^3$ is a slice knot if there exists a (PL) 2-disc $D^2 \subset B^4$ such that D^4 intersects $\partial B^4 = S^3$ transversely in K.) It would be of interest to determine which of Gordon's homology spheres also bound simply-connected 4-manifolds M with definite even-forms, or even with even-forms for which $b_2(M) - |\sigma(M)|$ is as small as possible. It is thus of interest to have a more constructive proof of Theorem 4.1 above. (Milnor's proof involves the Thom construction **[Thm]** and does not give a wholly clear picture of the manifold W constructed.) In **[Kap]** Kaplan proves Theorem 4.1 by constructing an explicit handlebody decomposition for W.

We again recall that if L is a bilinear form $V \times V \to \mathbf{Z}$ on the module V then $x \in V$ is characteristic for L if and only if $L(x, y) + L(y, y) \equiv 0 \pmod 2$ for all $y \in V$. Now if J is a framed link in S^3 with associated 4-manifold B_J ($\partial B_J = M_J$) then to each sublink R of J there corresponds a homology class $[R]$ in $H_2(B_J; \mathbf{Z})$. $[R]$ is representable as the union of the core of the handle over R and the cone on R in B^4. By a result of Thom **[Thm]**, $[R]$ is in fact also always representable by a smooth oriented 2-manifold. We shall call R a characteristic sublink if $j_*[R]$ is characteristic for the homology pairing on $H_2(B_J, M_J; \mathbf{Z})$ dual to L_{B_J} ($j_*\colon H_2(B_J) \to H_2(B_J, M_J)$), or equivalently if any smooth oriented manifold F representing $[R]$ is characteristic in the sense of Chapter 1. (Alternatively we define the linking matrix λ for the link J by

$$\lambda_{ij} = \begin{cases} lk(K_i, K_j) & \text{if } i \neq j, \\ \text{framing } K_i & \text{if } i = j, \end{cases} \qquad 1 < i, j < n,$$

where $K_1, \ldots, K_n$ are the components of J and $lk(K_i, K_j)$ is the algebraic linking number of the knots K_i, K_j. Note that λ will always be a representative matrix for the cup-product form L_{B_J}. We have that R is a characteristic sublink if and only if it is characteristic for the bilinear pairing induced by λ on $\bigoplus \mathbf{Z} K_i$.)

We note that if a 4-manifold W has characteristic submanifold F then a procedure whereby W could be modified so as to 'kill' F would produce a new 4-manifold W' which would necessarily have $w_2(W') = 0$ and thus be spin or equivalently (in this dimension) almost parallelizable. This idea is the core of Kaplan's construction. That is, given $(M^3, \mathfrak{F})$ one can always find a framed link representation **[Lick]** for it and thus one knows $M^3 = \partial(B_J)$ for some framed link J. One then identifies a characteristic sublink R of J and shows how one can kill R using Kirby moves. This produces $B_{J'}$ with $\partial(B_{J'}) = M$ and $w_2(B_{J'}) = 0$ as desired.

More concretely if $M = \partial(B_J)$ then corresponding to each 2-handle of B_J there is a dual circle (the b-circle in the terminology of Chapter 1) in M, the attaching circle of the dual handle. The dual circles generate $H_1(M; \mathbf{Z}_2)$ and so the almost framing for M is determined by the framing induced on the tubular neighborhoods of the dual circles. Now the framed link J also induces a framing on the neighborhoods of the dual circles and the difference between the two framings gives a map $H_1(M; \mathbf{Z}_2) \xrightarrow{T} \mathbf{Z}_2$, i.e. an element of $H^1(M; \mathbf{Z}_2)$. Then finding a characteristic sublink can be reduced to finding a sublink R of J such that $T(K_i) = 1 \in \mathbf{Z}_2$ for all the components K_i of R.

In Figures 4.1–4.3 **[Kap]** we give examples of characteristic sublinks of given links. Killing these characteristic sublinks by Kirby moves then gives the desired manifold $B_{J'}$. We note that the parallelizable manifold $B_{J'}$ in Figure 4.3 produced by killing the characteristic sublink has $b_2(B_{J'}) = 22$, $\sigma(B_{J'}) = 16$. Furthermore $\partial(B_{J'}) = S^3$ so $W = B_{J'} \cup D^4$ gives another example of an almost framed 4-manifold with $b_2 = 22$ and $\sigma = 16$ having a handle decomposition with no 1- or 3-handles. W is homotopy equivalent to $-V_4$ but it is not known if it is diffeomorphic to it.

(It is actually not strictly necessary to follow this procedure if one is only interested in computing the μ-invariant of $(M^3, \mathfrak{F})$. The following formula of Cappell and Shaneson **[CS 2]**:

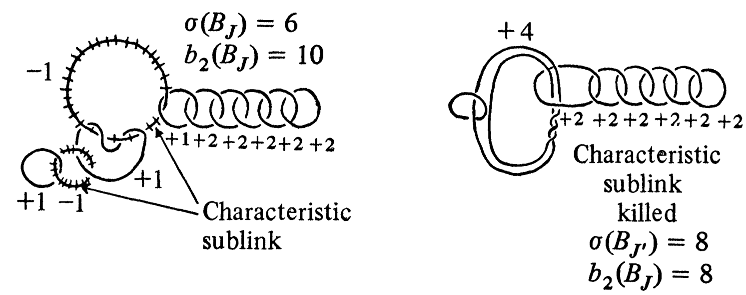
$$\mu(M^3, \mathfrak{F}) \equiv \sigma(L_{B_J}) + R \circ R + 8 \operatorname{Arf}(J) \pmod{16},$$

where $M^3 = \partial B_J$, R is the characteristic sublink of J and $R \circ R$ is the self-linking of R given by $L_{B_J}(R, R)$ gives μ directly.)

Our description above of the relationship between characteristic sublinks and almost framings shows that each almost framing of $M = \partial(B_J)$ determines a unique characteristic sublink of J. Consider for example, the

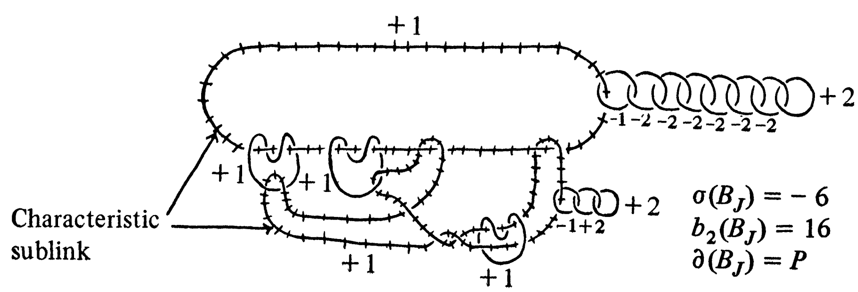
3-torus T^3 with framed link picture representation L as in Figure 4.4 (i.e., $\partial B_L = T^3$). T^3 has eight distinct almost framings and the corresponding characteristic sublinks are simply the eight distinct $\mathbf{Z}_2$-linear combinations of the components of L. It can be shown **[Kap]** that any proper characteristic sublink L_i of L induces an almost framing $\mathcal{F}_i$ with $\mu(T^3, \mathcal{F}_i) = 0$. However if L is itself characteristic then the almost framing $\mathcal{F}_0$ induced by L satisfies $\mu(T^3, \mathcal{F}_0) = 8$. The above behavior of T^3 is, in fact, typical of a large class of 3-manifolds, and Kaplan has shown

THEOREM 4.3 **[Kap]**. *Suppose M is a closed, connected oriented 3-manifold which bounds a framed manifold of index σ. Suppose further that there exist elements $X_i \in H^1(M; \mathbf{Z}_2)$, $i = 1, 2, 3$, with $X_1 \cup X_2 \cup X_3 \neq 0$. Then M also bounds a framed manifold of index $\sigma + 8$.*



$\partial B_J = \partial B_{J'} = P$

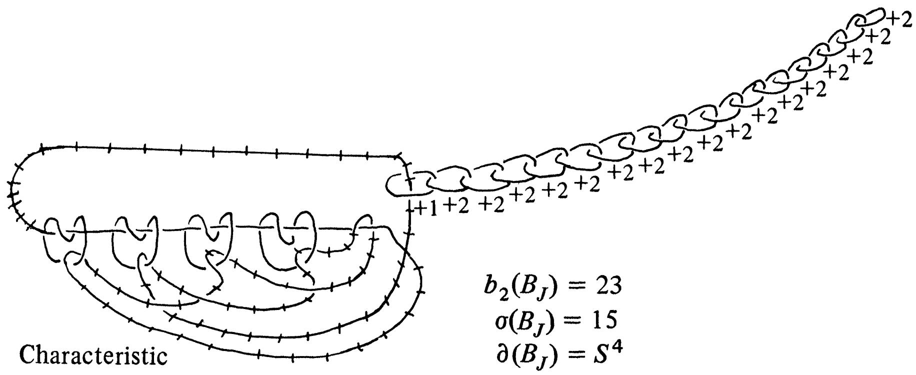
FIGURE 4.1



J' can be obtained by 'blowing down' the characteristic sublink indicated in the picture. The manifold $B_{J'}$ obtained has $\sigma(B_{J'}) = -8$; $b_2(B_{J'}) = 14$ and $\partial(B_{J'}) = P$.

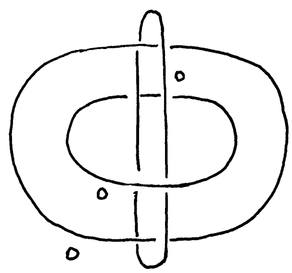
Gluing together the $B_{J'}$ of Figures 1 and 2 gives a closed spin manifold K with $\sigma(K) = 16$ and $b_2(K) = 22$.

FIGURE 4.2



Blowing down the characteristic sublink in this picture produces a manifold V with $b_2(V) = 22$ $\sigma(V) = 16$ with $\partial V = S^3$ and L_V of type II

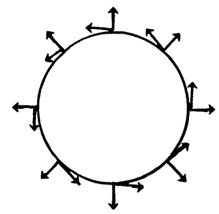
FIGURE 4.3



A framed link L with $\partial(B_L) = T^3$

FIGURE 4.4

(Another way to establish the existence of an (almost) framing $\mathfrak{F}_0$ on T^3 with $\mu(T^3, \mathfrak{F}_0) = 8$ (see also **[CS 2]**) is to consider the rational elliptic surface W_0 we discussed in Chapter 2. $W_0 \xrightarrow{\pi} S^2$ is an elliptic fiber space with $W_0 = P \# 9Q$. We can assume without loss of generality that the fiber of W_0 over $0 \in S^2$ is nonsingular, and we let $T_0 = \pi^{-1}(D_0)$ where D_0 is a disc about $0 \in S^2$ containing no critical values. Then $T_0 \approx T^2 \times D_0$ and setting $W_0^* = \overline{W_0 - T_0}$ we have by Novikov additivity that $\sigma(W_0^*) = \sigma(W_0) = -8$. W_0^* is in fact easily seen to be parallelizable (we have killed the second Steifel-Whitney class of W_0 by removing T_0) and thus its framings restricts to a framing $\mathfrak{F}$ on $\partial W_0^* = \partial(T^2 \times D_0) = T^3$. Then clearly $\mu(T^3, \mathfrak{F}) = 8 \bmod 16$ and $\mathfrak{F}$ thus must coincide with $\mathfrak{F}_0$ as above. If we let f be the stable framing on S^1 indicated in Figure 4.5 then one can check that $\mathfrak{F}_0 = f \times f \times f$ as a framing on $T^3 = S^1 \times S^1 \times S^1$.)



The stable framing f on S^1 exhibited here is the unique framing not extendable to a (stable) framing of D^2

FIGURE 4.5

4.2 Fake 4-manifolds. An extremely beautiful use of the existence of this 'exotic' framing on T^3 has been made by Cappell and Shaneson in their construction of a compact 4-manifold Q^4, simple homotopy equivalent to $\mathbf{R}P^4$ but not diffeomorphic to it.

We shall indicate the geometric construction of Q^4 but suppress most of the algebra involved in proving it is not diffeomorphic to $\mathbf{R}P^4$. More precisely we have

THEOREM 4.4 (CAPPELL AND SHANESON) **[CS 2]**. *Let $(X, \partial X)$ be a compact smooth connected 4-manifold with (possibly empty) boundary. Suppose $\pi_1(X)$ has an orientation reversing element of order 2. Then there is a manifold $(Q^4, \partial Q)$ and a simple homotopy equivalence $f\colon (Q, \partial Q) \to (X, \partial X)$ with $f|\partial Q\colon \partial Q \to \partial X$ a diffeomorphism such that f is not homotopic to a diffeomorphism or* PL *homeomorphism.*

COROLLARY 4.5. *Let $X = \mathbf{R}P^4$. Then there exists a 4-manifold Q, simple homotopy equivalent but not* PL *homeomorphic or even* PL *s-cobordant to X. Furthermore for any $k \geqslant 0$, $Q \# k(S^2 \times S^2)$ is not* PL *s-cobordant to $X \# k(S^2 \times S^2)$. However if V is any manifold with* $\dim V \geqslant 1$ (($\dim V \geqslant 2$) *if $\partial V \neq \varnothing$) then $Q \times V$ is topologically homeomorphic to $X \times V$.*

PROOF. The construction of Q^4 proceeds as follows.

Let $C \subset X$ be an embedded circle in X representing an order 2 orientation reversing element of $\pi_1(X)$.

Let T_C be a tubular neighborhood of C. Then $T = T_C$ is the unique nontrivial nonorientable orthogonal D^3-bundle over S^1. Set $X_0 = \overline{X - T_C}$ and $\partial X_0 = H$. H is thus a nontrivial S^2-bundle over S^1.

To construct Q we construct a new manifold M_0 with $\partial M_0 = H$ and define $Q = X_0 \cup_H M_0$. (More accurately we construct an infinite family of manifolds $M_0(A)$ and set $Q(A) = X_0 \cup_H M_0(A)$.)

Ideally we would construct Q by taking M_0 to be an 'exotic' D^3-bundle over S^1 which agrees with the standard nonorientable D^3-bundle near the boundary. (Note that in dimension n, $n \geqslant 5$, exotic D^3-bundles over T^{n-3} do exist! See **[HS 1]**, **[HS 2]**, **[Sh 2]**.) Unfortunately we do not know if such a bundle exists. Instead we will take M_0 to be a punctured T^3-bundle over S^1 which is $\mathbf{Z}$-homology equivalent to T_C.

Thus let $T^3 = S^1 \times S^1 \times S^1$ be the 3-torus and suppose $A \in \mathrm{GL}(3, \mathbf{Z})$ with $\det A = -1$ and $\det(I - A^2) = \pm 1$. A induces a diffeomorphism ϕ_A:

$T^3 \to T^3$ with $(\phi_A)_*: \pi_1 T^3 \to \pi_1 T^3$ equal to A. Clearly we can isotope ϕ_A to a diffeomorphism $\phi: T^3 \to T^3$ with a fixed point $* \in T^3$.

[For example letting $S^1 = \mathbf{R}/\mathbf{Z}$, define ϕ by $\phi(\theta_1, \theta_2, \theta_3) = (\theta_2, \theta_3, -\theta_1 + \theta_2)$. Then $\phi(0, 0, 0) = (0, 0, 0)$ and

$$\phi_* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

satisfies the hypothesis.]

Now let $e^3 \subset T^3$ be a smooth closed 3-cell about $*$ and set $T_0^3 = T^3 - e^3$. Then since $\det A = -1$ by a further isotopy fixing $*$ we may assume ϕ_0: $T_0^3 \to T_0^3$, $\phi_0 = \phi | T_0^3$, is a diffeomorphism with $\phi_0 | \partial e^3$ an orientation reversing orthogonal map. We now let $M_0 = M_0(A)$ be the mapping torus of ϕ_0. That is $M_0 = T_0^3 \times I/(X, 0) \sim (\phi_0(X), 1)$.

Note that ∂M_0 is, in fact, the nonorientable S^2-bundle over S^1 and so $\partial M_0 = H$. Thus the construction of Q is complete.

To construct f: $(X, \partial X) \to (Q, \partial Q)$ we must construct a map h: $M_0 \to T_C$ such that $h | \partial M_0 \to \partial T_C$ is the identity with $f = \mathrm{id}_{X_0} \cup h$ the requisite simple homotopy equivalence. Now a straightforward calculation using $\det(I - A^2) = \pm 1$ shows that M_0 is a $\mathbf{Z}$-homology 1-sphere and T_C is homotopy equivalent to S^1. Thus the obstructions to extending the identity $\partial M_0 \to \partial T_C$ to a map h: $M_0 \to T_C$ all vanish and h exists.

What must still be shown is

(1) f is a simple homotopy equivalence;

(2) f is not homotopic to a PL-homeomorphism.

The proof of (1) is a rather straightforward homotopy theoretic calculation. The proof of (2) uses the difference between the exotic framing on T^3 and the other framings to show that the 'PL normal invariant' of f is nonzero and thus f cannot be homotopic to the identity. We shall define 'normal invariants' and comment further on this proof in Chapter 6.

To prove the first part of the corollary (the last part will also be discussed in Chapter 6) it must only be shown that any homotopy equivalence of $\mathbf{R}P^4$ with itself is homotopic to the identity. But calculating $[\mathbf{R}P^4, \mathbf{R}P^4]$ one finds any map inducing the identity on $\pi_1(\mathbf{R}P^4)$ is homotopic to the identity map. Thus the only homotopy equivalence in $[\mathbf{R}P^4, \mathbf{R}P^4]$ is the class of the identity map, and so the map f constructed above cannot be a self-homotopy equivalence. Q must in fact not be PL-homeomorphic to $\mathbf{R}P^4$. It is still possible that Q is homeomorphic to $\mathbf{R}P^4$ since the possibility that f is homotopic to a topological homeomorphism has not been ruled out. (f has zero topological normal invariants.) We note that different choices of A give possibly different Q's. Thus there could conceivably be many distinct diffeomorphism classes of fake $\mathbf{R}P^4$'s.

Now suppose $Q(A)$ is a fake $\mathbf{R}P^4$. Let Σ_A be its universal covering space. Then Σ_A is a homotopy 4-sphere and one can ask whether it is homeomorphic to S^4. We note immediately that the existence of fake $\mathbf{R}P^4$'s implies

THEOREM 4.6 [CS 2]. *There is a smooth free involution on a homotopy 4-sphere Σ^4 which has no equivariant* PL-*homeomorphism with a linear action on S^4.*

In **[AK 1]** Kirby and Akbulut claimed to have shown that for at least certain matrices A, such that $Q(A)$ is a fake $\mathbf{R}P^4$, Σ_A is PL homeomorphic to S^4. In particular, there would then have existed an 'exotic' involution on S^4 itself, rather than just on a homotopy 4-sphere Σ^4. The proof in **[AK]** is however wrong. (See **[Rb]**.)

The key idea of that proof was an explicit construction of a handlebody picture for Σ_A and the use of Kirby moves to show that Σ_A is just S^4. We recall that $Q(A)$ was constructed by writing $X = \mathbf{R}P^4$ as the union of two pieces, X_0 = the normal D^2-bundle of $\mathbf{R}P^2$ in $\mathbf{R}P^4$ which we shall also denote by $\mathbf{R}P^2 \tilde{\times} D^2$ and T_C the nontrivial D^3-bundle over S^1 written as $S^1 \tilde{\times} D^3$.

Then $S^1 \tilde{\times} D^3$ was replaced by M_0 = mapping torus of $\phi_A | T_0^3$. Thus the construction of Σ_A is just $\Sigma_A = \widetilde{\mathbf{R}P^2 \tilde{\times} B^2} \cup \tilde{M}_0$, where the $\sim$'s indicate 2-fold cover. But $\widetilde{\mathbf{R}P^2 \tilde{\times} B^2}$ is just $S^2 \times B^2$ and $\tilde{M}_0$ is the mapping torus of $\phi_A^2 | T_0^3$.

Let

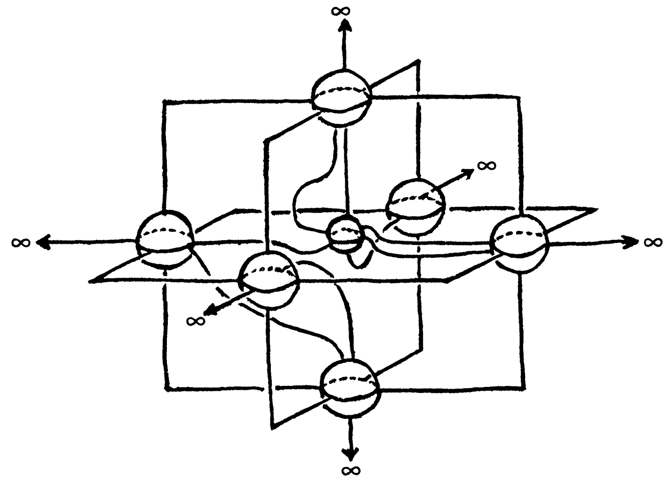
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

be the matrix of our previous example. Then a framed link picture of the mapping torus of ϕ_A^2 is exhibited in Figure 4.6A. (A general procedure for constructing handlebody pictures of mapping tori of diffeomorphisms of 3-manifolds can be found in **[Mont 2]**.) Finally, the $S^2 \times B^2$ must be added.

This can be done in two distinct ways. (See the discussion in §7.4 and the references there.) Unfortunately as J. H. Rubenstein noted **[Rb]** the way $S^2 \times B^2$ was added to the link picture in 4.6A (see 4.6B) does not correspond to the universal covering space of $Q(A)$. Thus the link moves of **[AK 2]** do not prove that Σ_A is homeomorphic to S^4. By choosing the correct gluing map for $S^2 \times B^2$ one can in fact use link calculus techniques to construct a framed link picture of Σ_A **[AK 5]**. However, despite a good deal of recent activity, it is as yet still unknown whether Σ_A is or is not homeomorphic to S^4 and thus whether S^4 admits an 'exotic' involution.

The manifold Σ pictured in Figure 4.6B, though not Σ_A, is nevertheless a homotopy 4 sphere and it is still instructive to recall how Akbulut and Kirby showed it to be homeomorphic S^4. One begins by sliding handles until all the 1-handles in this figure are cancelled by complementary 2-handles. This is indicated in Figure 4.7 for the one handle corresponding to the $0 \times B^3 \cup \infty \times B^3$ and in Figure 4.8 we show what is left after all the 1-handles are cancelled. However examining Figure 8 carefully we see that it is just the 3-component unlink and thus is complementary to the three 3-handles coming from the mapping torus of $\phi_A^2 | T_0^3$. Thus what is represented by Figure 8 is precisely S^4!

We note that the homotopy 4-spheres Σ_A provide, at present, the most plausible candidates for counterexamples of the 4-dimensional Poincaré conjecture. We leave as a not unrewarding challenge to the reader to determine which, if any, of the Σ_A's are homeomorphic to S^4 or to show that some Σ_A is, in fact, not homeomorphic to S^4.



There is an additional ball centered at ∞ and connected to the other 1-handles in the picture by the link components with arrows pointing towards ∞.

FIGURE 4.6A

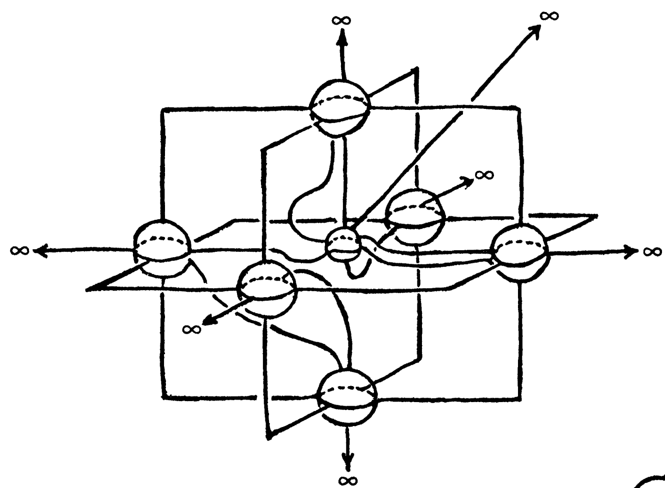


FIGURE 4.6B

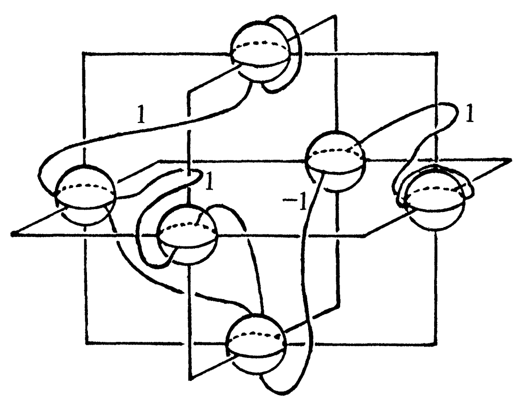


FIGURE 4.7

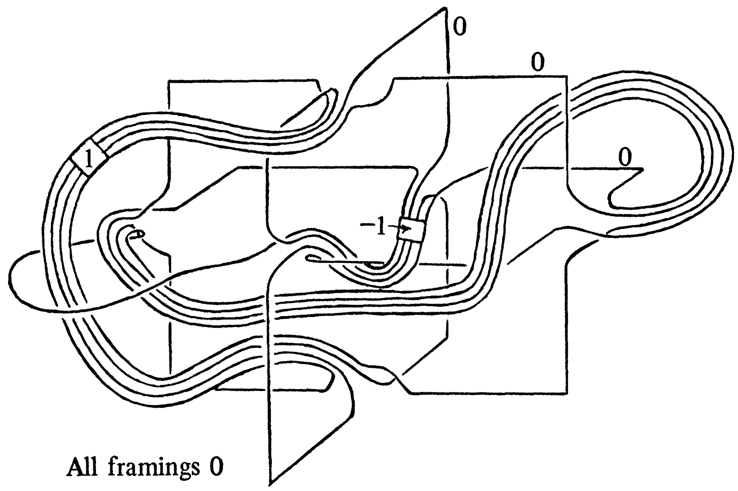


FIGURE 4.8

4.3 Triangulating high-dimensional manifolds. The relationship between 3- and 4-manifolds captured by the μ-invariant leads to interesting consequences for higher-dimensional manifold theory as well. Returning to the map μ: $\mathcal{H}^3(\mathbf{Z}) \to \mathbf{Z}_2$ let $\partial\mathcal{H}^3(\mathbf{Z})$ be the subgroup of elements of $\mathcal{H}^3(\mathbf{Z})$ which bound acyclic (PL) 4-manifolds. Since $\mu(\partial\mathcal{H}^3(\mathbf{Z})) = 0$ letting $\theta_3^H = \mathcal{H}^3(\mathbf{Z})/\partial\mathcal{H}^3(\mathbf{Z})$ we see that μ induces a homomorphism $\bar{\mu}$: $\theta_3^H \to \mathbf{Z}_2$ which remains onto. What is the structure of θ_3^H and the nature of $\bar{\mu}$? Essentially nothing other than the surjectivity of $\bar{\mu}$ is known. In particular it is not even known whether θ_3^H is finitely generated! Among the more well-known conjectures regarding θ_3^H are

CONJECTURE I. There exists an element of order 2 in θ_3^H.

CONJECTURE II. Ker $\bar{\mu} = 0$.

The extreme importance of θ_3^H and the significance of Conjecture I becomes apparent as a result of the following.

THEOREM 4.7 (GALEWSKI-STERN, SEE ALSO MATUMOTO, **[GS 1]**, **[GS 2]**, **[Mat]**). *Suppose H is a $\mathbf{Z}$-homology 3-sphere with $\mu(H) \neq 0$ such that $H \# H$ bounds an acyclic* (PL) *4-manifold. Then every topological m-manifold M ($m \geqslant 5$ if $\partial M = \varnothing$ and $m \geqslant 6$ otherwise) can be triangulated as a simplicial complex!*

(Note that by the work of Kirby-Siebenmann there exist m-manifolds which cannot admit the structure of a PL-manifold. We already mentioned that $M_1^4 \times S^n$ for $n \geqslant 1$ will be a topological manifold admitting no PL-structure [M_1^4 as in Chapter 1]. These manifolds cannot be triangulated *combinatorially*. [We say a triangulation is combinatorial if it is PL-homogeneous. That is if M is a manifold with triangulation K (so $|K| = M$) then for K to be a combinatorial triangulation we must have that for every $x, y \in M$ there exists a piecewise-linear homeomorphism h (PL relative to K of course) such that $h(x) = y$.] However, they still might have noncombinatorial triangulations. (See **[Sb 2]**.) [That not all triangulations are combinatorial was demonstrated by R. D. Edwards **[Ed 1]**, **[Ed 2]** who showed that S^n ($n \geqslant 5$)

always has noncombinatorial triangulation. (Proof: We have $\Sigma^{n-3}P$ is topologically homeomorphic to S^n and clearly the triangulation of S^n defined by $\Sigma^{n-3}P$ is not combinatorial)].)

We note that the problem of the structure of θ_3^H is unique to the relationship between 3- and 4-manifolds in the following sense.

Let $\mathcal{H}^{4k-1}(\mathbf{Z})$ be the group of (DIFF) h-cobordism classes of (DIFF) $\mathbf{Z}$-homology spheres which bound almost parallelizable *smooth* $4k$-manifolds and let $\partial\mathcal{H}$ be the subgroup bounding smooth contractible $4k$-manifolds. Let θ_{4k-1}^H be $\mathcal{H}^{4k-1}(\mathbf{Z})/\partial\mathcal{H}^{4k-1}$. Then there exists an isomorphism λ: $\theta_{4k-1} \to \mathbf{Z}_t$ (where $t = I_k$, I_k as in Theorem 1.15) and $\lambda[M] = \sigma(W_M)$ mod t, W_M an almost parallelizable $4k$-manifold with $\partial W_M = M$. (See **[M 2]**.)

CHAPTER 5. SURGERY THEORY AND ITS APPLICATIONS

5.1 Surgery theory in higher-dimensions. Throughout many of the preceding sections of this paper we have alluded to surgery and its implications. In particular there are the fundamental results of **[Br 1]**, **[N]** on the classification of manifolds of a given homotopy type which we mentioned in Chapter 1 and the surgery-type calculations used to show that the homotopy $\mathbf{R}P^4$ constructed in **[CS 2]** is indeed fake. Unfortunately the full power of higher-dimensional surgery theory is not available in low-dimensional topology. In the following chapters we will illustrate what goes wrong in dimension 4 (and 5) and present some examples of what can be salvaged. In order to do this we use this chapter to review some of the basic ideas, methods and constructions of high-dimensional surgery theory.

We begin by reviewing the essential ideas of surgery theory. Thus let us consider the following questions. (We assume that we are working in some fixed category either DIFF or PL or TOP.)

(1) Suppose X is a CW-complex, M^m a manifold and ϕ: $M^m \to X$ a map. When can we modify M to get a new manifold M' and map ϕ': $M' \to X$ with ϕ' a (simple) homotopy equivalence?

(2) Suppose M, M' are manifolds and f: $M \to M'$ is a (simple) homotopy equivalence. When is f homotopic to an isomorphism?

The basic geometrical construction used in solving these problems is the same, that of surgering M. We recall its definition: **[M 3]**, **[KM 2]**.

Suppose f: $S^r \times D^{m-r} \to V^m$ is an orientation preserving embedding into the oriented manifold V^m. (We will in the sequel work in the smooth category. Analogous results occur in the PL locally flat and TOP locally flat categories.) Form the disjoint sum $V^m - f(S^r \times 0) \amalg D^{r+1} \times D^{-r-1}$ and let V' be the quotient manifold obtained by identifying $f(u, rv)$ with (ru, v) for $u \in S^r$, $v \in S^{m-r-1}$ and $0 < r \leqslant 1$. We say V' is obtained by surgery on an r-sphere in V and write $V' = \chi(V, f)$ to indicate the dependence on f in this process. An alternative way, already mentioned in Chapter 1, to visualize V' is as follows. Let $W = V \times I$ and attach an $(r + 1)$-handle H^{r+1} to the cobordism W by means of the attaching map f: $S^r \times D^{m-r} \to V^m \times \{1\}$. This gives a new cobordism $W' = V \times I \cup H^{r+1}$ with ends V and $\tilde{V} = \partial W' - V$. It is readily seen that $\tilde{V}$ is diffeomorphic to V' defined above and thus V' is of course cobordant to V. It is thus clear that talking about a surgery on