IV. Some Hand-Calculations And ...

We shall match notation with Bar-Natan's two papers:

1. "On Khovanov's categorification of the Jones polynomial" Alg + Geom Topology, Vol. 2 (2002) 337–370.
2. "Khovanov's homology for tangles and cobordisms" Geom + Topology Vol. 9 (2005) 1443–1499

---

1. Version of bracket polynomial:

$$\langle \phi \rangle = 1, \quad \langle O K \rangle = (q + q^{-1}) \langle K \rangle$$

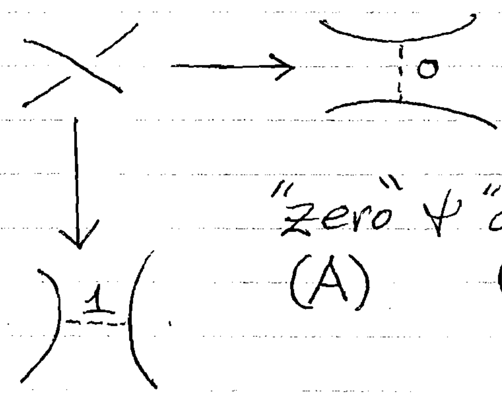
$$\langle \times \rangle = \langle \asymp \rangle - q \langle \supset\subset \rangle .$$

If K is oriented, let

$n_+ = \#\, \times$, $n_- = \#\, \times$

Then $J(K) = (-1)^{n_-} q^{n_+ - 2n_-} \langle K \rangle$

is invariant under all 3 R-moves + so is Jones poly up to change of variable.

2. 

"zero" + "one" smoothings.
(A) (B)

3. $W = \bigoplus_m W_m$ graded vector space with homog comps $\{W_m\}$.

$$q\dim W \underset{\text{def}}{=} \sum_m q^m \dim(W_m) .$$

• $\{l\}$ denotes degree shift operation on graded vector spaces.

e.g. $W = \bigoplus_m W_m$

then $W\{l\}_m \underset{\text{def}}{=} W_{m-l}$.

$\therefore \operatorname{qdim} W\{l\} = q^l \operatorname{qdim} W$

• $[s]$ denotes height shift operation on chain complexes. i.e. if

$$\overline{\mathcal{C}} : \cdots \longrightarrow \overline{\mathcal{C}}^r \xrightarrow{d^r} \overline{\mathcal{C}}^{r+1} \longrightarrow \cdots$$

a chain complex of possibly graded vector spaces; then if $\mathcal{C} = \overline{\mathcal{C}}[s]$ we mean

$$\mathcal{C}^r = \overline{\mathcal{C}}^{r-s}.$$

4°

In Dror's papers you will see a graded vector space V with basis $\{v_+, v_-\}$. We will identify V with our first example of a Frobenius algebra $\mathcal{A}$ with basis $\{1, X\}$. so $v_+ = 1$, $v_- = X$ and

$$1 \longleftrightarrow q, \quad X \longleftrightarrow q^{-1} \quad \text{in grading.}$$

$$\operatorname{qdim} V = q + q^{-1}.$$

We associate an ordering of the crossings to K and thence each state corresponds to a binary string of 0's and 1's. Call this string α. The height $|\alpha|$ is given by the formula

$$|\alpha| = \sum_i \alpha_i.$$

We take $V_\alpha(L) \underset{\text{def}}{=} V^{\otimes k}\{r\}$
where k = # loops in state corres to α (call it $S(\alpha)$) and $r = |\alpha|$.

The r-th chain group is denoted by $[\![K]\!]^r$ $\quad 0 \le r \le n$ (n = # crossings of K)
and is defined by

$$\boxed{[\![K]\!]^r \underset{\text{def}}{=} \bigoplus_{\alpha : |\alpha| = r} V_\alpha(L)}$$

We have already constructed

$$d^r : [\![K]\!]^r \longrightarrow [\![K]\!]^{r+1}$$

making this into a chain complex.

Note that

$$\text{qdim}\, V_\alpha(K) = q^r\, \text{qdim}\, V^{\otimes k}$$
$$= q^{|\alpha|}(q + q^{-1})^{\#\text{loops in } S(\alpha)}$$

Thus

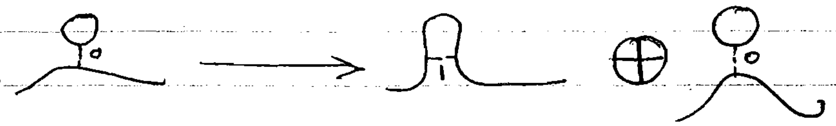
$$\sum (-1)^{|\alpha|} \text{qdim}\, V_\alpha(K) = \langle K \rangle$$
$$\| $$
$$\sum_r (-1)^r \text{qdim}\, [\![K]\!]^r = \chi_q [\![K]\!]$$

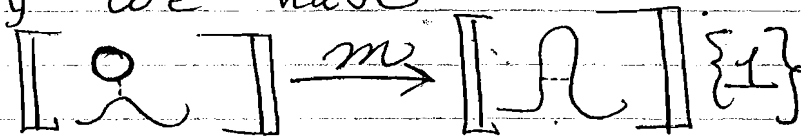
Similarly, we can let $\mathcal{C}(K) \underset{\text{def}}{=} [\![K]\!][-n_-]\{n_+ - 2n_-\}$

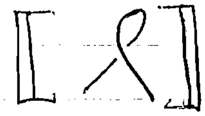
& ~~get~~ $J(K) = \chi_q(\mathcal{C}(K))$.

5.° We let $\mathcal{H}(K)$ denote the homology of the chain complex $[\![K]\!]$.

Now consider $\mathcal{H}($  $)$. We have

d: 

and more generally we have a chain mapping 

We can describe the chain complex for  by a general construction on chain complexes (the mapping cylinder) that associates to a map $C \xrightarrow{f} C'$ of chain complexes a new chain complex

$$C'' \underset{def}{=} [C \xrightarrow{f} C'] \quad \text{as follows:}$$

$$\| def$$

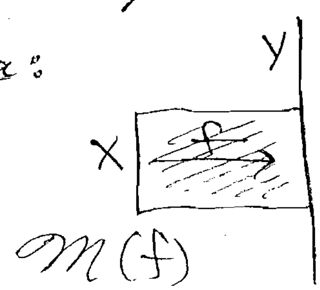
$$\underline{M(f)}$$

$$(C'')_{n+1} = MC_n \oplus C'_{n+1} \qquad (\partial : C_n \to C_{n-1} \text{ etc})$$

we write elements $M\alpha \in MC_n$ corresponding to elements $\alpha \in C_n$.

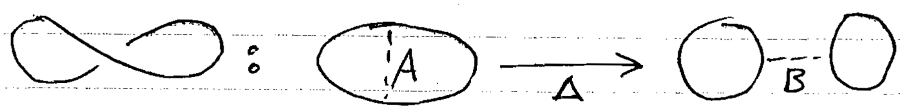
$$\partial M\alpha \underset{def}{=} \alpha - f\alpha + M(\partial\alpha), \quad \partial\beta \text{ usual for } \beta \in C'.$$

<u>The idea:</u>



$f : X \to Y$
Attach $X \times I$ to Y
via $X \times 1 \longleftrightarrow Y$ via f.
Take chain complex of $\mathcal{M}(f)$.

In any case, we can work examples and see what happens. In doing these examples we will work <u>mod 2</u> and we will use homology grading where $C_n \leftrightarrow$ states with n - A smoothings so that $\partial : C_n \longrightarrow C_{n-1}$. We then expect that there will be grading shifts when we calculate the effects under Reidemeister moves.



$$\phi \longrightarrow V \xrightarrow{\Delta = \partial} V \otimes V \longrightarrow \phi$$

$$1 \longmapsto 1 \otimes X + X \otimes 1$$

$$X \longmapsto X \otimes X$$

$V \otimes V$ gen by $\{1 \otimes 1, 1 \otimes X, X \otimes 1, X \otimes X\}$. So

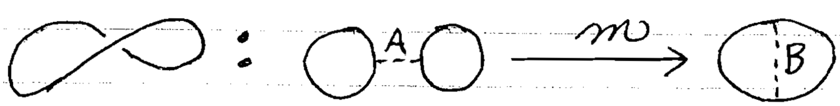
$H_0 \cong \langle 1 \otimes 1, 1 \otimes X \rangle$

$H_1 \cong 0$.

Compare with $\bigcirc$: $\phi \longrightarrow \bigcirc \longrightarrow \phi$

$$V$$

with $H_0 \cong V = \langle 1, X \rangle$.



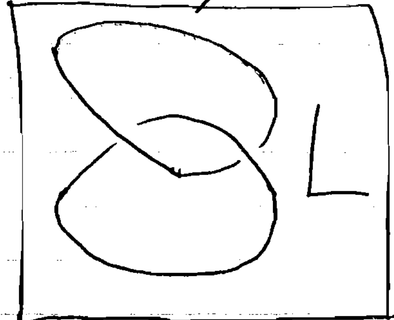
$$\phi \longrightarrow V \otimes V \xrightarrow{m = \partial} V \longrightarrow \phi$$

$$\left.\begin{array}{l} 1 \otimes 1 \longmapsto 1 \\ 1 \otimes X \longmapsto X \\ X \otimes 1 \longmapsto X \\ X \otimes X \longmapsto 0 \end{array}\right\} \begin{array}{l} \partial \text{ is surjective,} \\ \text{so } H_0 \cong 0 \end{array}$$

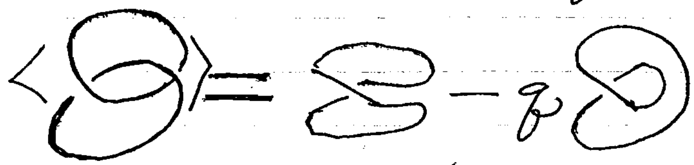
$\mathrm{Ker}\,\partial = \langle 1 \otimes X + X \otimes 1, X \otimes X \rangle$

$\Rightarrow H_1 \cong \mathrm{Ker}\,\partial$ is <u>2 dim'l</u>.

Now lets do an example and keep track of the cohomology grading.



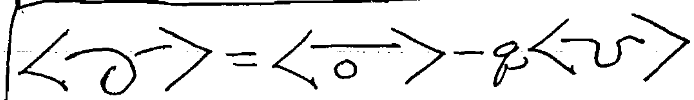
First compute the q-bracket:



$= q^{-1}(q+q^{-1}) - q(q^{2})(q+q^{-1})$

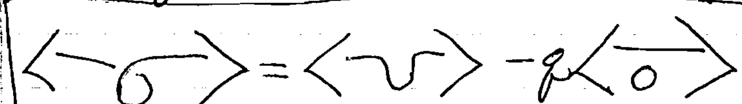
$= (1+q^{-2}) + q^{2}(q^{2}+1)$

$= 1 + q^{-2} + q^{4} + q^{2}$



$= ((q+q^{-1}) - q)\langle - \rangle$

$= q^{-1}\langle - \rangle$

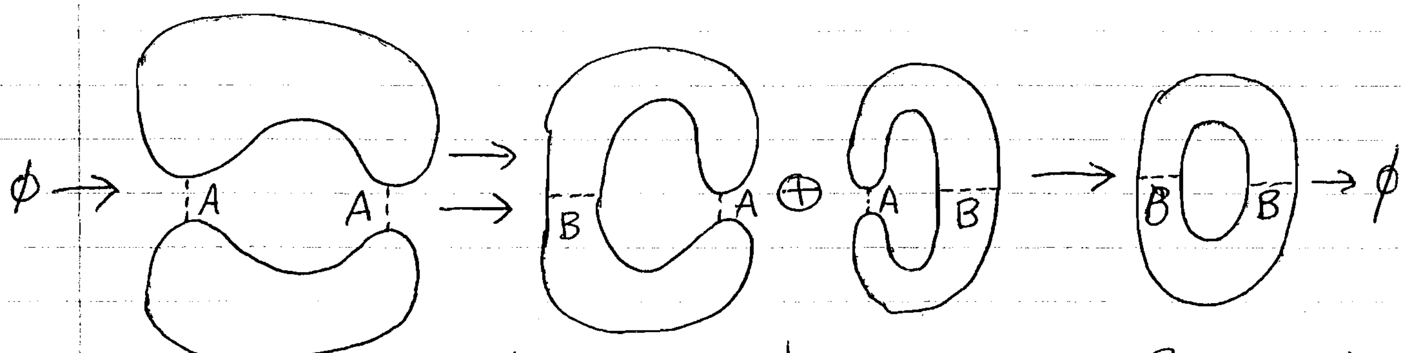


$= (1 - q(q+q^{-1}))\langle - \rangle$

$= -q^{2}\langle - \rangle$

Note

$gr(X) = q^{-1}$
$gr(1) = q$



$$\phi \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \to \phi$$

$$\phi \to V\otimes V \xrightarrow{m\otimes m} V \oplus V \xrightarrow{(-\Delta)\oplus\Delta} V\otimes V \to \phi$$

(This includes the signs.)

$$H^0 \cong Ker(m) = \langle X\otimes X,\ 1\otimes X \text{ [illegible] } X\otimes 1 \rangle$$

(continued on next page)

$$\boxed{\begin{aligned} gr(X) &= q^{-1} \\ gr(1) &= q \end{aligned}}$$

$$\phi \to V\otimes V \xrightarrow{m \oplus m} V \oplus V \xrightarrow{(-\Delta)\oplus\Delta} V\otimes V \longrightarrow \phi$$

$$H^0 \cong Ker(m) = \langle X\otimes X,\ 1\otimes X - X\otimes 1 \rangle$$

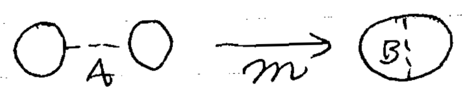
$$H^1 = \frac{Ker(-\Delta\oplus\Delta)}{Im(m\oplus m)} = \frac{\langle (v, v) \rangle}{\langle (m(\tilde{v}), m(\tilde{v})) \rangle} = 0$$

$$H^2 = \frac{V\otimes V}{Im(-\Delta\oplus\Delta)} = \frac{V\otimes V}{\langle \Delta v - \Delta w \rangle} = \frac{V\otimes V}{\langle \Delta(v-w) \rangle} = \frac{V\otimes V}{\langle \Delta(v) \rangle}$$

$$\Delta(a+bX) = a(1\otimes X + X\otimes 1) + b\, X\otimes X$$

$$\Rightarrow \quad H^2 \cong \langle 1\otimes 1,\ 1\otimes X \rangle$$

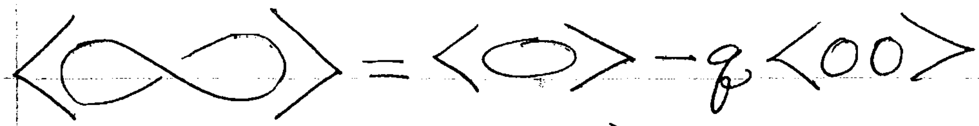
So
$$\left.\begin{aligned} H^0 &\leftrightarrow (q^{-2}+1) \\ H^1 &\leftrightarrow 0 \\ H^2 &\leftrightarrow (q^2+1) \end{aligned}\right\} \chi = q^0(q^{-2}+1) + q^2(q^2+1)$$
$$= q^{-2}+1+q^4+q^2$$
$$= \langle L \rangle \checkmark$$



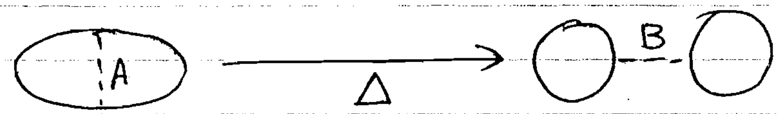
$$: \langle \infty \rangle = \langle \bigcirc\bigcirc \rangle - q\langle \bigcirc \rangle$$
$$= (q+q^{-1})(q+q^{-1}-q)$$
$$= 1+q^{-2}$$

$$V\otimes V \xrightarrow{m} V$$

$$Ker(m) = \langle X\otimes X,\ 1\otimes X - X\otimes 1 \rangle$$
$$Im(m) = V$$
$$\left.\begin{aligned} H^0 &: q^{-2}+1 \\ H^1 &: 0 \end{aligned}\right\} \chi = q^{-2}+1 \checkmark$$



$$= (q+q^{-1})(1 - q(q+q^{-1}))$$
$$= (q+q^{-1})(1 - q^2 - 1)$$
$$= -q^3 - q$$



$$V \longrightarrow V \otimes V$$

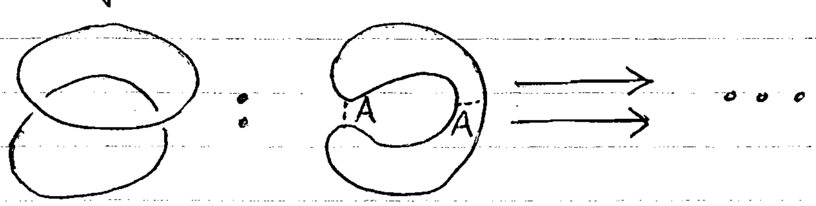
$\operatorname{Ker} \Delta = \{0\}$

$H^0 = 0, \; H^1 = V \otimes V / \operatorname{Im}(\Delta)$

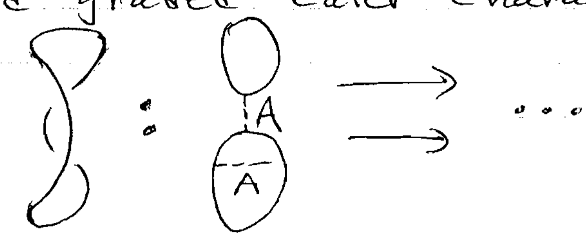
$$= \langle 1 \otimes 1, 1 \otimes X \rangle \leftrightarrow q^2 + 1$$

$$\chi = -q(q^2+1) = -q^3 - q \checkmark$$

<u>Exercise</u>: Work out the cohomology complex and graded Euler characteristic for



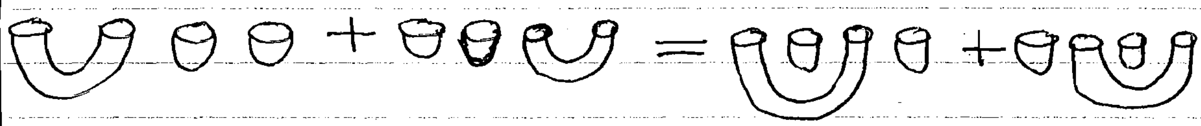
<u>Exercise</u>: Work out the cohomology complex and graded Euler characteristic for



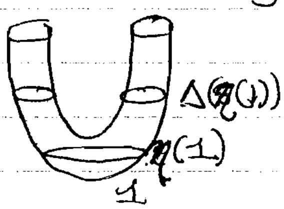
V. We will come back to more of these direct calculations, but now lets look at the whole matter more abstractly. Dror Bar Natan discovered an approach at the abstract but topological Frobenius algebra level, and we will work with this. His approach is based on a key property of our favorite Frobenius algebra $\mathcal{A} = \langle 1, X \rangle$:

$\boxed{\Delta(1) = 1 \otimes X + X \otimes 1,\ \Delta(X) = X \otimes X,\ X^2 = 0\ ;\ \varepsilon(1) = 0,\ \varepsilon(X) = 1}$ $\boxed{\eta(1) = 1}$.

The 4Tu Identity. (Algebraic Versions (A) and (B))

(A) 

So (A) says that



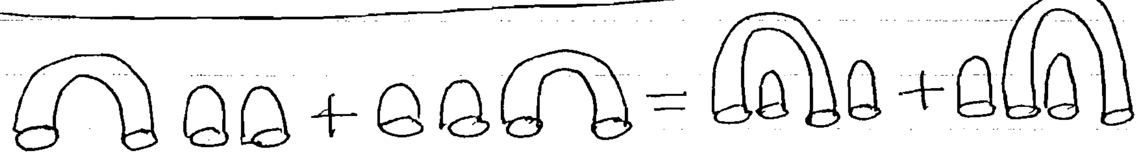
$$\Delta(1) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Delta(1) = \Delta_1(1) \otimes 1 \otimes \Delta_2(1) \otimes 1 + 1 \otimes \Delta_1(1) \otimes 1 \otimes \Delta_2(1)$$

Check this: $\Delta(1) = 1 \otimes X + X \otimes 1$

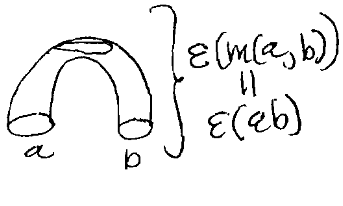
So $\Delta(1) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Delta(1) = 1 \otimes X \otimes 1 \otimes 1 + X \otimes 1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes X + 1 \otimes 1 \otimes X \otimes 1$

while $\Delta_1(1) \otimes 1 \otimes \Delta_2(1) \otimes 1 + 1 \otimes \Delta_1(1) \otimes 1 \otimes \Delta_2(1) = 1 \otimes 1 \otimes X \otimes 1 + X \otimes 1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1 \otimes 1$

and these are equal.

(B) 

So we need to verify that



$$\boxed{\begin{aligned}&\varepsilon(ab)\varepsilon(c)\varepsilon(d) + \varepsilon(a)\varepsilon(b)\varepsilon(cd)\\ &= \varepsilon(ac)\varepsilon(b)\varepsilon(d) + \varepsilon(a)\varepsilon(bd)\varepsilon(c)\end{aligned}}$$

For example: $\underset{a}{X}, \underset{b}{X^*}, \underset{c}{1}, \underset{d}{X}$

$\varepsilon(ab)\,\varepsilon(c)\,\varepsilon(d) = 0$

$\varepsilon(a)\,\varepsilon(b)\,\varepsilon(cd) = 1$

$\varepsilon(ac)\,\varepsilon(b)\,\varepsilon(d) = 1$

$\varepsilon(a)\,\varepsilon(bd)\,\varepsilon(c) = 0$

$\underset{a}{X}, \underset{b}{1}, \underset{c}{1}, \underset{d}{X}$

$\varepsilon(ab)\,\varepsilon(c)\,\varepsilon(d) = 0$

$\varepsilon(a)\,\varepsilon(b)\,\varepsilon(cd) = 0$

$\varepsilon(ac)\,\varepsilon(b)\,\varepsilon(d) = 0$

$\varepsilon(a)\,\varepsilon(bd)\,\varepsilon(c) = 0$

$\underset{a}{1}, \underset{b}{X}, \underset{c}{X}, \underset{d}{X}$

$\varepsilon(ab)\,\varepsilon(c)\,\varepsilon(d) = 1$

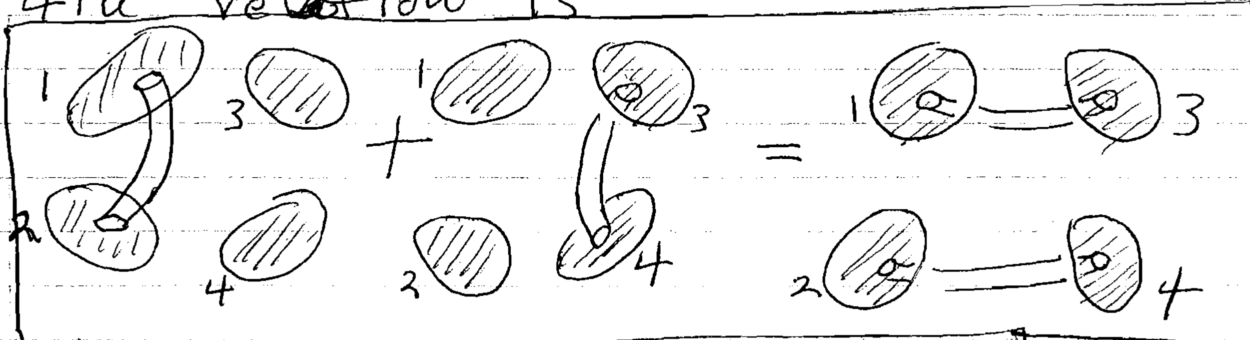
$\varepsilon(a)\,\varepsilon(b)\,\varepsilon(cd) = 0$

$\varepsilon(ac)\,\varepsilon(b)\,\varepsilon(d) = 1$

$\varepsilon(a)\,\varepsilon(bd)\,\varepsilon(c) = 0$

Etc...

At the level of surfaces, the 4Tu relation is

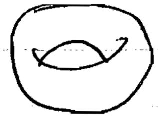


$$\boxed{S_{12} + S_{34} = S_{13} + S_{24}}$$

With 4-bits of nearby surface, you are allowed to rearrange the tubing as shown above!

We already have (for Q) that

S :  $= 0$

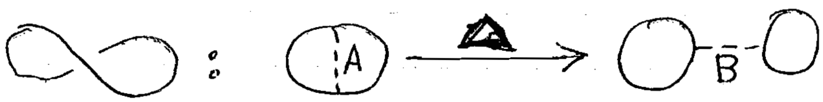
T :  $= 2$.

It will turn out that

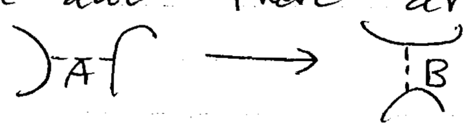
$$H^*\left[\frac{\text{Abstract Complexes}}{\langle S, T, 4Tu\rangle}\right]$$

is invariant under the Reidemeister moves.

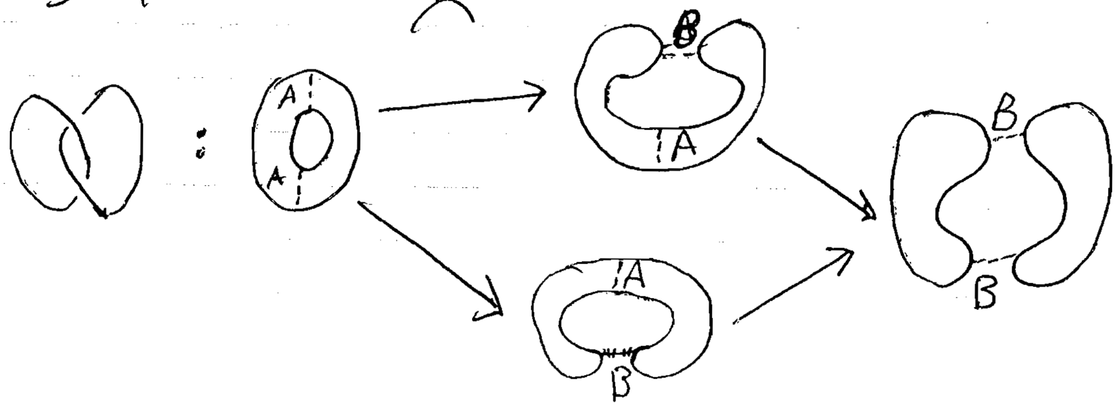
First we have to discuss the matter of abstract complexes. Given a knot or link diagram we have a little <u>category</u> :



where the objects are the states of the diagram and there are morphisms

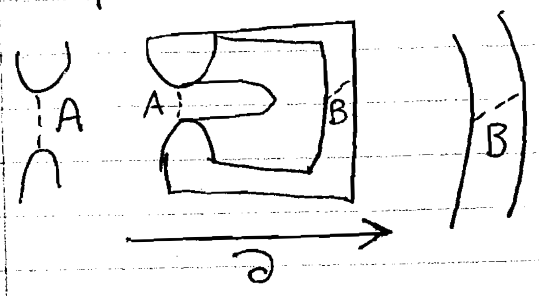


<u>Thus</u>



We have been associating chain complexes to these <u>state - categories</u>

by using (Frobenius Picture) surfaces as morphisms between states as in



and taking a functor from this category to the additive ($\oplus$) tensor category ($\otimes$) generated by our Frobenius algebra $\mathcal{A} = V$. We made chain complexes by taking alternating sum of the maps going out of any given state object. OK.

But why not put in additivity formally and make the abstract patterns of the chain complex without looking at the algebra happening inside?! We will do exactly that.

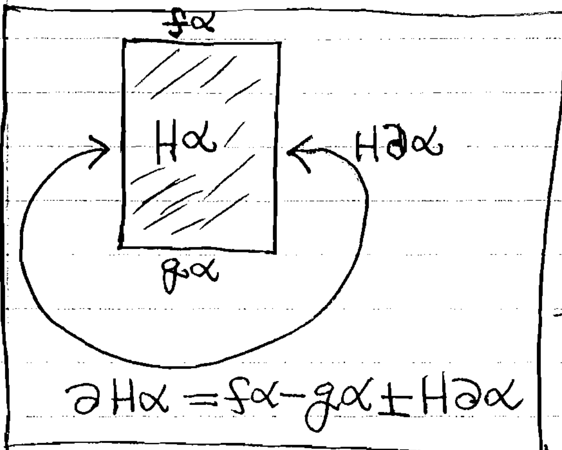
Some aspects of homological algebra can be described "from the outside" and we shall make use of them. For example, a chain mapping

$$\begin{tikzcd} \cdots \arrow[r] & C^n \arrow[r, "d"] \arrow[d, "f^n"] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ \cdots \arrow[r] & \tilde{C}^n \arrow[r, "\tilde{d}"] & \tilde{C}^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

just means that you have maps f^n and the diagrams commute. We understand this without looking at internal structure.

Suppose $f, g: C \longrightarrow \tilde{C}$. We say f and g are homotopic ($f \sim g$) if $\exists$ a function $H: C^{*+1} \longrightarrow C^{*}$ such that

$$Hd + dH = f - g.$$

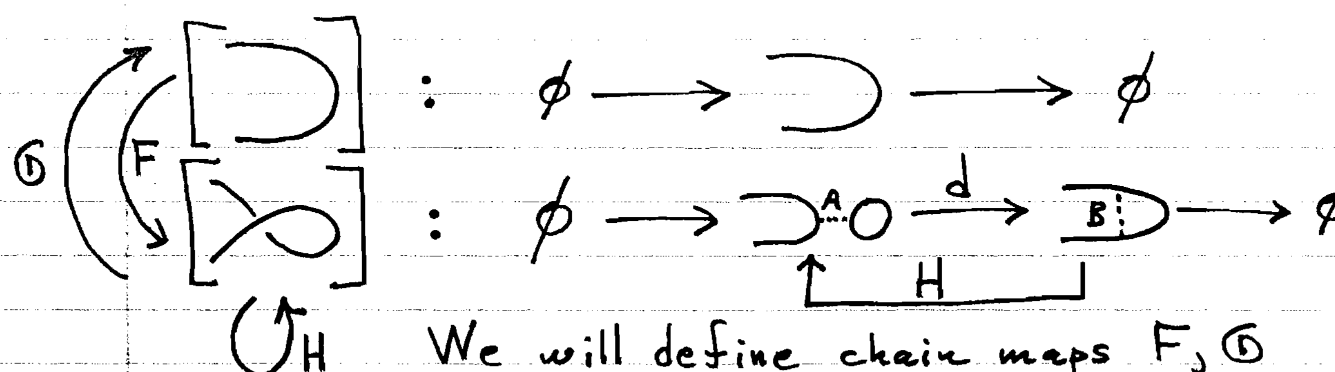


Boundary Illustration of a homotopy.

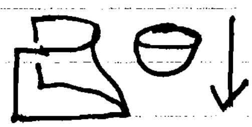
Proposition. $f, g: C \longrightarrow \tilde{C}$, $f \sim g$
$\Rightarrow f^{*} = g^{*}: H^{*}C \longrightarrow H^{*}\tilde{C}$

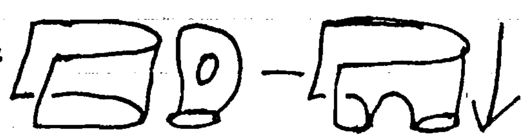
Proof. ~~If~~ $d\alpha = 0$ then
$H d\alpha + dH\alpha = f\alpha - g\alpha$
$\Rightarrow d(H\alpha) = f\alpha - g\alpha$
$\Rightarrow f\alpha$ and $g\alpha$ represent the same H^{*} class in $\tilde{C}$. //

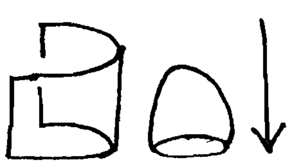
Now consider

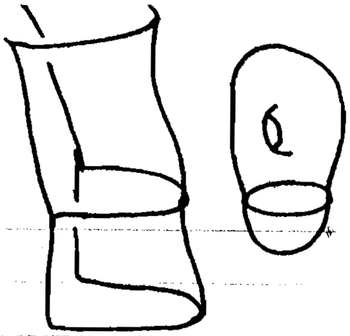
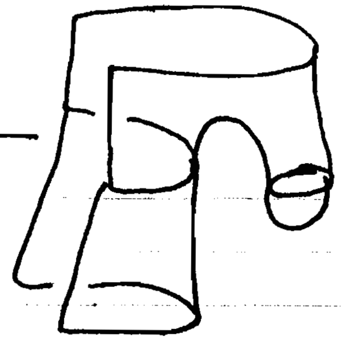


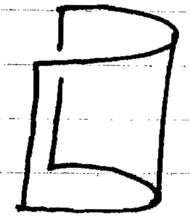
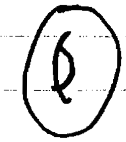
We will define chain maps F, G and a homotopy H.

G is specified by $G^{0} =$ 

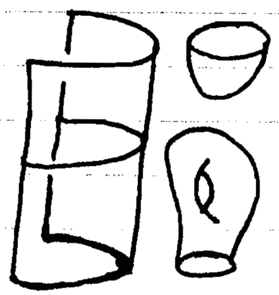
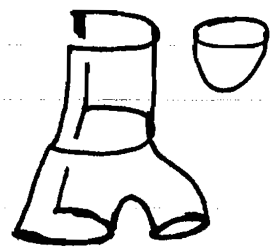
F is specified by $F^{0} =$ 

H is specified by 

GF :  — 

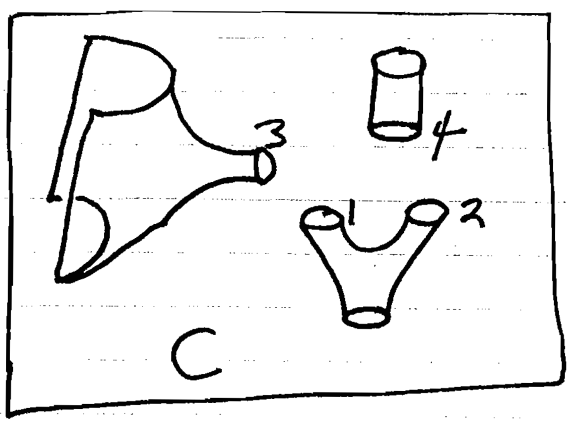
=  since  = 2.

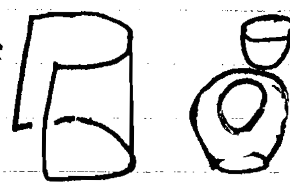
$\therefore$ GF = $\underline{1}$.

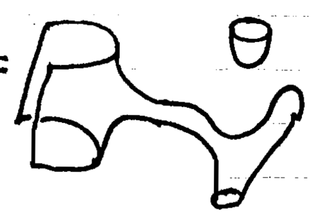
FG :  — 

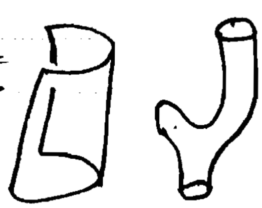
$= C_{12} - C_{13} = C_{24} - C_{34} = I - C_{34}$

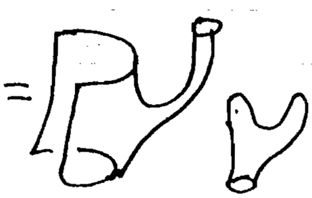
<u>See below</u>:

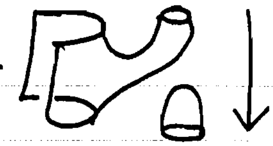


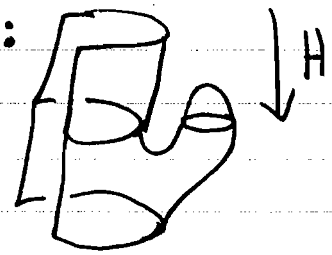
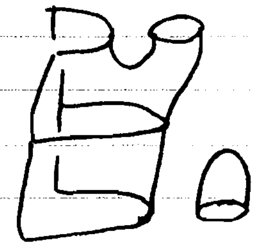
$C_{12} =$ 

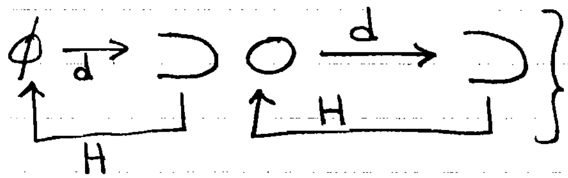
$C_{13} =$ 

$C_{24} =$ 

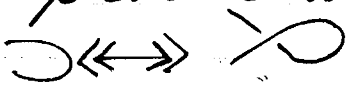
$C_{34} =$ 

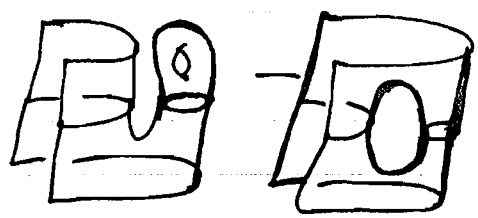
Thus $F^\circ G^\circ = I - C_{34} = I -$ 

dH:  Hd:  $= C_{34}$

 $\Rightarrow F^\circ G^\circ - I = -Hd$

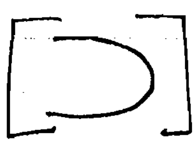
Thus $FG \sim I$.

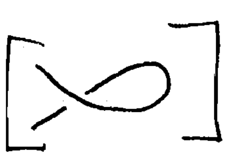
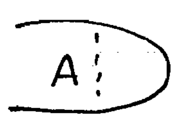
This proves invariance under .

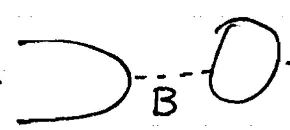
Remark: $dF^\circ =$  $= \phi$

This shows that F is a chain map.

Next we look at

 : $\phi \longrightarrow$  $\longrightarrow \phi$

 : $\phi \longrightarrow$ 

$\xrightarrow{d}$ 

$\longrightarrow \phi$