IMMERSIONS AND MOD-2 QUADRATIC FORMS

LOUIS H. KAUFFMAN AND THOMAS F. BANCHOFF

1. Introduction. We are going to consider an easily visualizable classification problem in topology and its close relationship with a corresponding classification problem in algebra. It is standard practice in topology to begin with the geometry and then mirror part of its structure in algebra. Usually it requires care to find a mirror which captures just enough geometry so that the corresponding algebra problems are both accessible and relevant.

It occasionally happens that this process illuminates both the algebra and the geometry. This is the case with our topic. Hence we obtain a particularly nice way to learn about two things simultaneously — immersions of surfaces and mod-2 quadratic forms.

It is difficult to talk about two things at once. We shall therefore begin with the topology, show how it leads to quadratic forms, and then discuss quadratic forms in more detail. By doing this, we obtain geometric proofs and interpretations for the basic algebraic identities which underlie the theory of mod-2 quadratic forms.

This paper is relatively self-contained except for a few facts about surfaces that are summarized in sections 3 and 4 and some facts about homology in section 6. In such cases, we have tried to make the facts geometrically plausible.

A word about quadratic forms: Consider a polynomial function $f(x, y) = ax^2 + bxy + cy^2$ with a, b, and c real numbers. The locus $f(x, y) =$ constant represents a conic section in the plane and it is a standard exercise to determine the geometric form of this conic by changing variables to eliminate the xy term in the expression. A modern approach is to consider the plane as the vector space R^2 of ordered pairs $v = (x, y)$ of real numbers, and to think of $f: R^2 \to R$ as a function on R^2. We then can define a bilinear form on R^2 by setting $\langle v, w\rangle = f(v + w) - f(v) - f(w)$. Conversely, given a bilinear form on R^2, we may obtain a quadratic form $q: R^2 \to R$ by setting $q(v) = \frac{1}{2}\langle v, v\rangle$. Similarly in any field where $1 + 1 \neq 0$, there is a one-to-one correspondence between bilinear forms and quadratic forms. But our geometric study will lead us to quadratic forms on a vector space V over the field Z_2 with two elements, where $1 + 1 = 0$. Although for any such form $q: V \to Z_2$ we get a bilinear form $\langle v, w\rangle = q(v + w) + q(v) + q(w)$, the quadratic form is no longer completely determined by the bilinear form. Thus the study of mod-2 quadratic forms involves a particular subtlety. We shall see corresponding phenomena mirrored in the study of immersions, and this correspondence is the main point of this paper.

The paper is organized as follows. Section 2 discusses immersions of circles into R^2 and the two sphere, S^2, and shows how consideration of the two sphere leads to mod-2 phenomena. Sections 3 and 4 discuss surfaces and their immersions. Letting $\mathscr{C}(M)$ denote the set of embedded curves on a surface M, we obtain, for each immersion $f: M \to S^2$, a function $N(f): \mathscr{C}(M) \to Z_2$. This function measures how curves on M are immersed into S^2. Section 4 introduces an invariant, $B(f)$, for immersions of surfaces by examining $N(f)$ on the boundary curves. We study immersions of punctured disks up to an equivalence relation called image homotopy. This section introduces some basic homotopies (handle sliding and permutation) which will be used later. In section 6 we show how $N(f): \mathscr{C}(M) \to Z_2$ leads to a quadratic form $q(f): \mathscr{H}(M) \to Z_2$ where $\mathscr{H}(M)$ is the mod-2 homology group of M. In section 7 we discuss certain homotopies of immersions and show how they correspond to isomorphisms of quadratic forms. It is then easy to explain the classification of mod-2 forms. Section 8 completes the classification of immersions of surfaces up to image homotopy.

A concise exposition of mod-2 quadratic forms may be found in [1, pp. 52–56]. For more information about topology and quadratic forms, the reader may enjoy looking at [3] and [5]. We remark that our results are related to the research article by Rourke and Sullivan [6].

2. Regular closed curves in the plane and the sphere. In order to approach the study of surfaces we have to understand closed curves. A closed curve may be thought of as the path traced out by a point moving continuously in the plane R^2, so that the point ends where it began. We may describe such a path α by writing $\alpha(\theta)$ to indicate the position of the point that corresponds to the angle θ on the circle S^1, so that $\alpha(0) = \alpha(2\pi)$. We will restrict ourselves to *regular curves* $\alpha : S^1 \to R^2$. These curves are also called *immersions* of the circle. They are defined by the condition that the velocity vector $\alpha'(\theta)$ varies continuously and is non-zero for all values of θ and $\alpha'(0) = \alpha'(2\pi)$. As a point traces out a regular curve, there is a well-defined continuously turning direction vector $\alpha'(\theta)/\|\alpha'(\theta)\|$ for each θ, and as θ goes from 0 to 2π, this unit vector goes a certain number of times around the unit circle in a counterclockwise direction. We define this integer to be the *degree* $D(\alpha)$ of the regular closed curve. Figure 1 indicates the degrees of a number of regular closed curves with arrows indicating which way each curve is to be traversed.

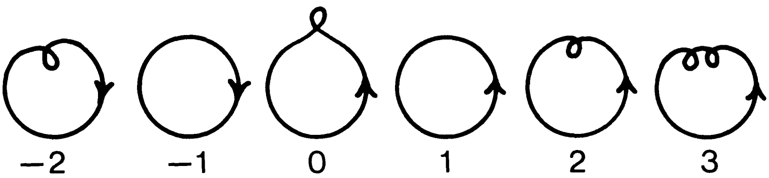

FIG. 1

The degree of a curve does not change if we deform the curve slightly so that the tangent directions move in a continuous manner. We call a one-parameter family $\alpha_t : S^1 \to R^2$, $0 \leqq t \leqq 1$, of regular curves a *regular homotopy* if the tangent vectors $\alpha'_t(\theta)$ to these curves vary continuously as θ and t change, and we say that α and $\bar{\alpha}$ are *regularly homotopic* if there is such a deformation beginning with $\alpha = \alpha_0$ and ending with $\bar{\alpha} = \alpha_1$. In this case we write $\alpha \simeq \bar{\alpha}$. The definition of regular homotopy is set up so that if α and $\bar{\alpha}$ are regularly homotopic, then $D(\alpha) = D(\bar{\alpha})$.

Our story really begins with a theorem of Hassler Whitney ([7], p. 279) which establishes the converse of this last statement. His main result shows that the integer $D(\alpha)$ completely characterizes the curves that are regularly homotopic to α.

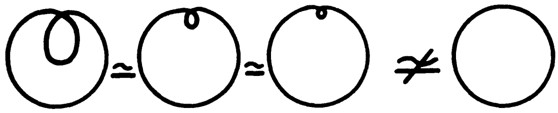

FIG. 2

THEOREM 2.1 (Whitney–Graustein). *If α and $\bar{\alpha}$ are two regular curves in R^2 with $D(\alpha) = D(\bar{\alpha})$ then α and $\bar{\alpha}$ are regularly homotopic.*

The condition $\alpha'(\theta) \neq 0$ for all θ indicates that for any θ_0, the curve is approximated by the tangent line, at least for a small interval about the value θ_0, so that as θ moves through this interval the regular curve α is one-to-one, with no double points. It is possible however for the whole regular curve to have double points. A double point is a point in R^2 that is the image of two distinct points on the circle. That is, the point p satisfies the condition: $p = \alpha(\theta_0) = \alpha(\theta_1)$ for some $0 \leqq \theta_0 \leqq \theta_1 < 2\pi$ and $\alpha^{-1}(p) = \{\theta_0, \theta_1\}$. Similarly, we may speak of points whose pre-image consists of a finite number of points on the circle (triple points, etc.). We call a double point a *normal crossing* if the tangent lines to the curve at the two points are different. We say that a regular curve is *normal* if all of its self-intersections are double points with normal crossings. Any regular curve may be deformed by a regular homotopy into a normal curve. During a regular homotopy, the number of normal crossings may change, but the parity will remain the same. (See Figure 3 for an example of a regular homotopy

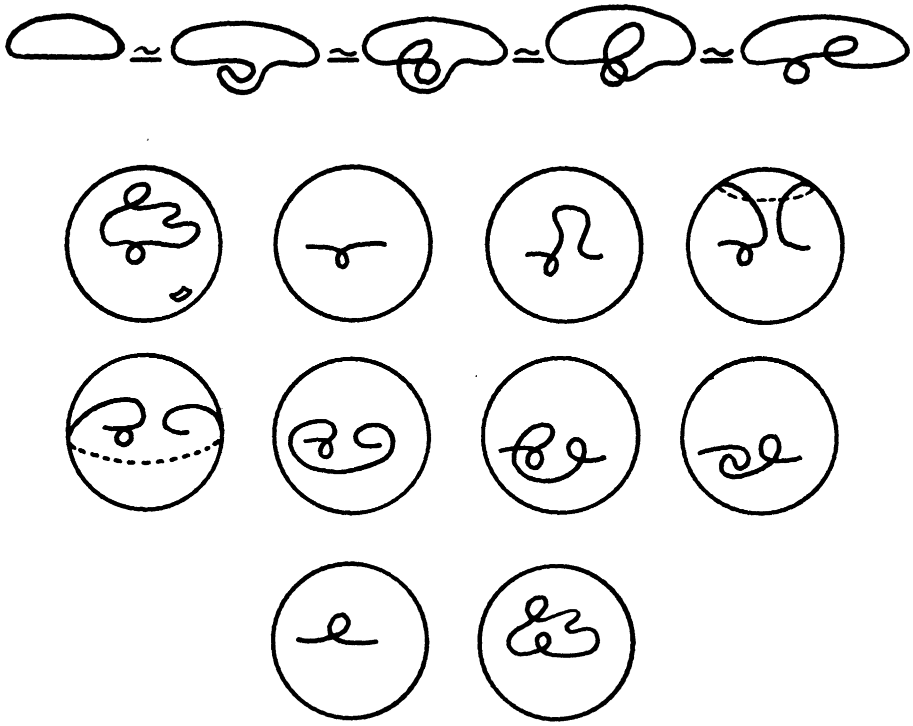

FIG. 3

which introduces two new normal crossings.) Thus if α and $\tilde{\alpha}$ are normal curves with $\alpha \simeq \tilde{\alpha}$, then the number of normal crossings of α and the number of normal crossings of $\tilde{\alpha}$ are either both even or both odd. (Regularity fails at the last instant in the parity-changing deformation illustrated in Figure 2.) If α is normal, we define the *crossing number* $N(\alpha)$ of α to be the number of normal crossings reduced modulo two.

Whitney [7] established a relation between the degree of a normal curve and the number of normal crossings:

THEOREM 2.2. *If α is a normal curve on R^2, then $D(\alpha)$ and the number of normal crossings of α have opposite parity. That is, $D(\alpha)+1$ reduced modulo 2 equals $N(\alpha)$.*

We now wish to consider closed curves not in the plane but in S^2, the unit sphere in R^3. The notions of regular curve, regular homotopy, and normal curve carry over to the case where we map to S^2 rather than R^2, but now there is an additional kind of deformation available, in which we swing a loop over the back of the sphere. This is illustrated in Figure 3. The first part of the figure shows how to create or destroy two normal crossings. Note that when this occurs in the plane the two tiny loops so created contribute oppositely to the degree since the tangent vector turns in opposite directions on the two loops. In the plane it is not possible to change a $+1$ loop to a -1 loop by a regular homotopy. However, this can be done in S^2. The second part of Figure 3 shows how to switch such a loop on a curve without disturbing the rest of the curve. For this figure we ask the reader to imagine that he or she is looking down towards the surface of a transparent sphere. As the curve on the sphere is deformed, part of it swings over the back of the sphere (dotted lines). In the intermediate stages of the regular homotopy we have not drawn the entire curve. What is not drawn remains stationary. A deformation such as this obliterates the difference between curves 3 and 1 of Figure 1, or between curves 2 and 0.

Although the notion of degree as we have defined it does not apply to a normal curve on the sphere, we can still speak about the normal crossing number, $N(\alpha)$. The number $N(\alpha)$ is an invariant of the regular homotopy class of α. Whitney's result shows that the crossing number completely classifies normal curves on the sphere up to regular homotopy:

THEOREM 2.3. *For normal curves α and $\bar{\alpha}$ on S^2, $\alpha \simeq \bar{\alpha}$ if and only if $N(\alpha) = N(\bar{\alpha})$.*

3. Immersions of the annulus in the sphere. In this paper we want to use the properties of regular curves to study immersions of surfaces with boundary into S^2. Perhaps the simplest such surface from our point of view is the *annulus*, which we may think of as an interval of concentric circles in the plane, with radii varying from r_0 to r_1. By an *immersion* of the annulus into R^2 or S^2, we mean a mapping $\alpha(\theta, r)$, $0 \leqq \theta \leqq 2\pi$, $r_0 \leqq r \leqq r_1$, such that the partial derivative vectors $\partial\alpha/\partial\theta$, $\partial\alpha/\partial r$ are non-collinear vectors at each point which vary continuously as r and θ change. This definition guarantees that the annulus is mapped in a locally one-to-one way. Note that it follows that, for each fixed r, the curve $\alpha(\theta, r)$ is an immersion of the circle. For example, a small strip neighborhood of a regular curve in the plane is the image of an immersion of the annulus. If we would deform this center curve by a regular homotopy $\alpha_t(\theta, r)$, then we could obtain a similar one-parameter family $\alpha_t(\theta, r)$ of immersions of the entire annulus. We say that such a family of immersions is a *regular homotopy* of the annulus if for every r, the family $\alpha_t(\theta, r)$ is a regular homotopy of curves and if the partial derivative vectors $\partial\alpha_t/\partial\theta$ and $\partial\alpha_t/\partial r$ vary continuously as θ, r and t change.

As in the case of curves, we are interested in classifying regular homotopy classes of immersions of the annulus into the sphere, and we may use our results on curves to give a complete classification.

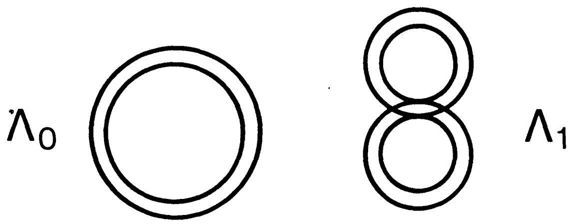

FIG. 4

THEOREM 3.1. *Any immersion of the annulus in S^2 is regularly homotopic to an immersion whose image is a strip neighborhood of either a circle or of a figure eight.* (See Figure 4.)

Proof. Any immersion of the annulus may be shrunk down to a strip neighborhood of the center curve. (This involves the tubular neighborhood theorem of differential topology. See Milnor, *Topology from the Differentiable Viewpoint*, The University Press of Virginia, (1965) p. 46.) We may then use the results of the previous section to find a regular homotopy of this curve to a circle or a figure eight carrying along the strip neighborhood of the curve.

There is one additional subtlety that makes the theory of immersions of the annulus different from that of curves and that has to do with *orientation*. The partial derivative vectors $\partial\alpha/\partial\theta$ and $\partial\alpha/\partial r$ are tangent vectors to the sphere that are assumed to be non-collinear, so their cross-product either points out of the ball or into the ball at every point of the annulus. In the first case we say that α preserves orientation and in the second we say that α reverses orientation. The property of preserving or reversing orientation does not change during a regular homotopy, so our classification theorem may be stated more precisely as follows:

THEOREM 3.2. *Two immersions of the annulus into the sphere are regularly homotopic if and only if their center curves are regularly homotopic and both immersions either preserve or reverse orientation.*

Henceforth we shall assume that all of our immersions of surfaces are orientation preserving.

4. Immersions of surfaces with boundary into the sphere. An *orientable surface with boundary* is obtained by removing a finite number of non-overlapping discs from an orientable surface such as a sphere, a torus, or more generally, a sphere with handles. An annulus, for example, may be described as a sphere with two discs removed. The disc itself is a sphere with a disc removed. A basic theorem in the study of orientable surfaces states that any such surface may be obtained by starting with a polygonal region in the plane and identifying certain pairs of boundary edges (see for example [4], Chapter 1). A torus may be described in this way as a square with opposite sides identified, and if we remove a quarter disc about each corner of the square before identifying the opposite sides, we obtain a torus with a disc removed, which we shall refer to as a *punctured torus* (see Figure 5).

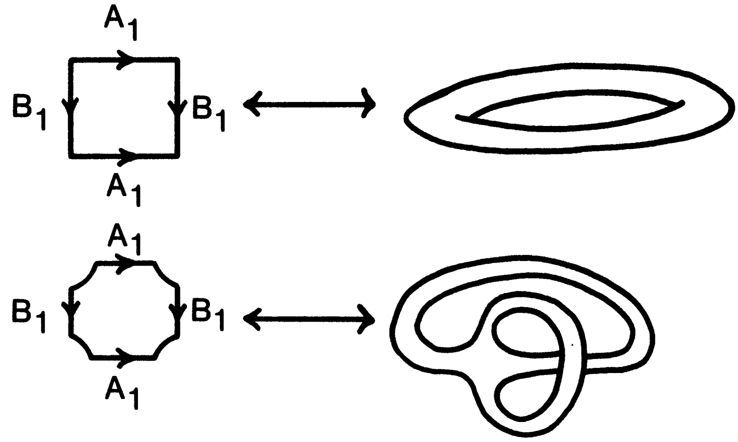

FIG. 5

Although it is not possible to find a locally one-to-one mapping of the entire torus into the plane, it is possible to produce several different immersions of the punctured torus into the plane. To do this we *attach bands* to the square without corners by finding immersions of half of an annulus $\alpha(\theta, r)$, $0 \leqq \theta \leqq \pi$, $r_0 \leqq r \leqq r_1$, so that the ends match up with the sides that are to be identified. In a region about any point of the punctured torus we may then find parameters so that the partial derivative vectors are non-collinear and so that they vary continuously as the parameters change. We call such a mapping an *immersion* of the punctured torus into the plane (or into S^2).

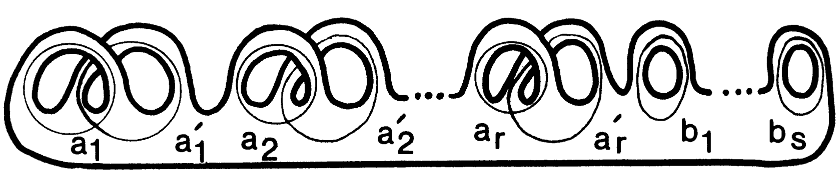

FIG. 6

We obtain the *center curve of a band* by taking the center curve of the half annulus and connecting its endpoints to the center of the square by a one-to-one arc to form an immersion of the circle. These remarks apply equally well to an arbitrary surface. In the general case the procedure described above leads to a surface with boundary curves as illustrated in Figure 6. In Figure 6 the collection of center curves to the bands is given by the set

$$\{a_1, a'_1, a_2, a'_2, \ldots, a_r, a'_r, b_1, \ldots, b_s\}.$$

A band is said to be *untwisted* if its center curve is regularly homotopic to an embedded curve (that is, a curve with no self-crossings) and to be *twisted* if it is regularly homotopic to a normal curve with just one crossing. In Figure 7 we indicate four immersions of the punctured torus: T_{00} with both bands untwisted, T_{01} and T_{10} with one band twisted and one band untwisted, and T_{11} with both bands twisted.

As in the case of the annulus, we define a *regular homotopy* of immersions of a surface M with boundary to be a one-parameter family of immersions $f_t : M \to S^2$ so that for some region about any point we may find a parametrization so that the partial derivative vectors change continuously as t varies.

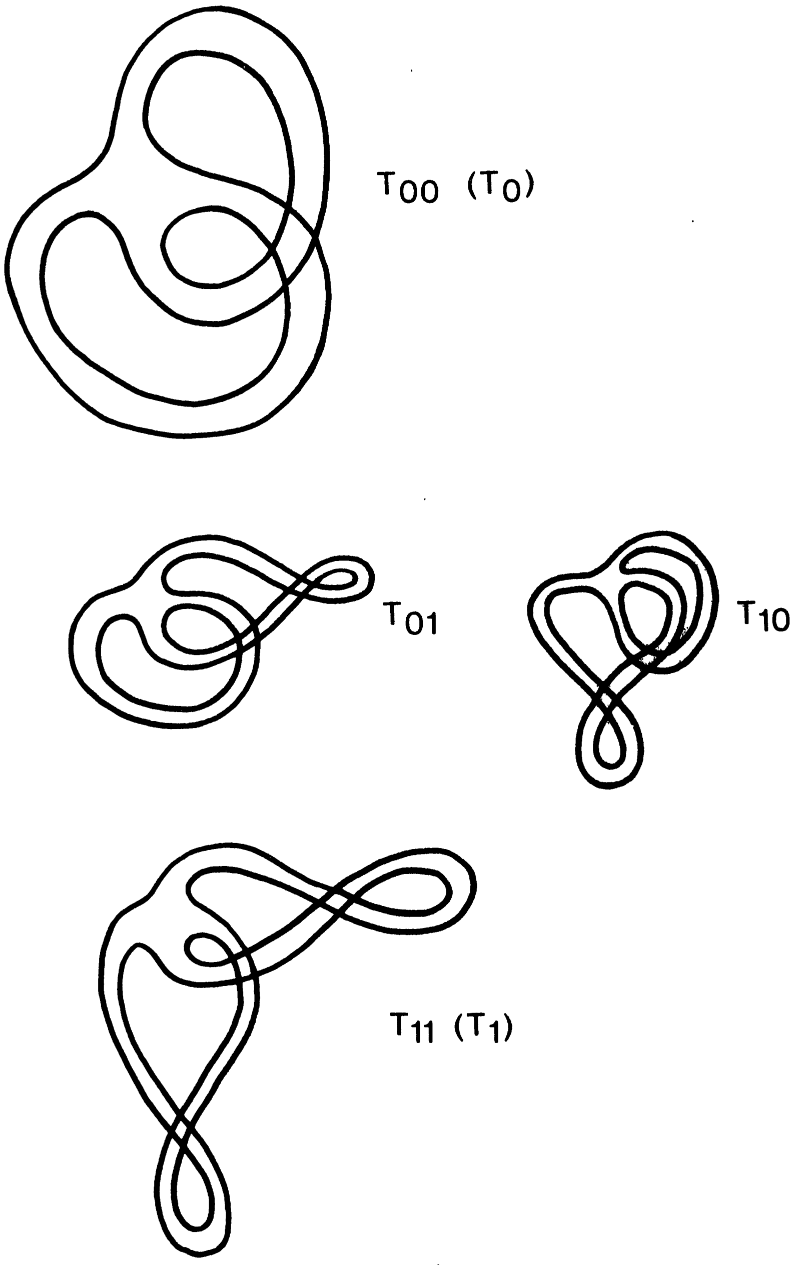

FIG. 7

The regular homotopy classification of surface immersions is not hard. Our next theorem is a generalization of Theorems 3.1 and 3.2. Since the proof is similar to the proofs of 3.1 and 3.2, we shall omit it.

THEOREM 4.1. *Let $f: M \to S^2$ be an orientation preserving immersion of a surface M with band curves $\{\alpha_1, \ldots, \alpha_k\}$. Another immersion, $g: M \to S^2$, is regularly homotopic to f if and only if $N(f \circ \alpha_i) = N(g \circ \alpha_i)$ for each $i = 1, 2, \ldots, k$.*

Thus an immersion is determined up to regular homotopy by the crossing numbers of its band curves. There are four regular homotopy classes of immersions of the punctured torus. An example of an immersion in each class is given in Figure 7. The images in the figure are denoted T_{00}, T_{01}, T_{10}, and T_{11}.

The immersions T_{01} and T_{10} differ very little in appearance. In fact, the distinction we are making between them depends upon more than their images. We are assuming that maps $f: M \to S^2$ and $g: M \to S^2$ are given so that $f(M) = T_{01}$, $g(M) = T_{10}$. Furthermore, if α and β are the band curves on M then we also assume that $N(f \circ \alpha) = 0$, $N(f \circ \beta) = 1$ while $N(g \circ \alpha) = 1$, $N(g \circ \beta) = 0$. By changing a map without changing its image one can produce non-regularly homotopic immersions with the same image. For example, let $h: M \to M$ be a homeomorphism (that is, a one-to-one, onto, continuous mapping with continuous inverse) of the punctured torus that switches the two bands. If h satisfies the same differentiability criteria that we imposed upon an immersion, then so will the composition $f \circ h$. Hence $f \circ h: M \to S^2$ is also an immersion. We assumed that h interchanged the bands and therefore $N((f \circ h) \circ \alpha) = 1$ and $N((f \circ h) \circ \beta) = 0$. Thus, while $T_{01} = f(M) = f \circ h(M)$, we see that f and $f \circ h$ are not regularly homotopic. In fact, by Theorem 4.1, $f \circ h$ is regularly homotopic to g. Thus, depending upon the maps representing them, the images T_{01} and T_{10} may represent distinct or equivalent immersions.

We wish to concentrate on the images of immersions. For this purpose it is useful to say that an orientation preserving homeomorphism $h: M \to M$ (M any orientable surface) is a *diffeomorphism* if h satisfies the local derivative conditions for an immersion. It then follows that if $f: M \to S^2$ is an immersion and $h: M \to M$ is a diffeomorphism, then $f \circ h: M \to S^2$ is also an immersion. Note that f and $f \circ h$ have identical images.

We are going to study an equivalence relation on immersions called *image homotopy*. Intuitively, two immersions are image homotopic if there is a regular homotopy between their images. For example, consider the bottom line of Figure 8; it illustrates an image homotopy between T_{00} and T_{01}. Thus image homotopy is weaker than regular homotopy. Since our rigorous definition of image homotopy is a bit technical, we defer it to the end of this section. The reader may wish to look forward into the rest of the paper before examining the exact concept of image homotopy.

In setting up algebraic invariants of image homotopy it will be useful to consider curves embedded in a given surface. Let $\mathscr{C}(M)$ denote the collection of curves $\alpha: S^1 \to M$ with α an embedding. Given an immersion $f: M \to S^2$ and $\alpha \in \mathscr{C}(M)$ we obtain a regular curve $f \circ \alpha: S^1 \to S^2$. Thus we may define

$$N(f): \mathscr{C}(M) \to Z_2 \quad \text{by} \quad N(f)(\alpha) = N(f \circ \alpha).$$

In the next section $N(f)$ will be applied to the boundary curves of M. We then turn to arbitrary curves and see how this leads to a connection between image homotopy and mod-2 quadratic forms.

Notation. Since T_{00}, T_{01}, and T_{10} are image homotopic, we shall use the notation T_0 for T_{00} and T_1 for T_{11} in all later sections.

Here is the promised definition:

DEFINITION 4.2. Two immersions f, $g: M \to S^2$ are image homotopic ($f \approx g$) if there is a diffeomorphism $h: M \to M$ so that $f \circ h$ is regularly homotopic to g.

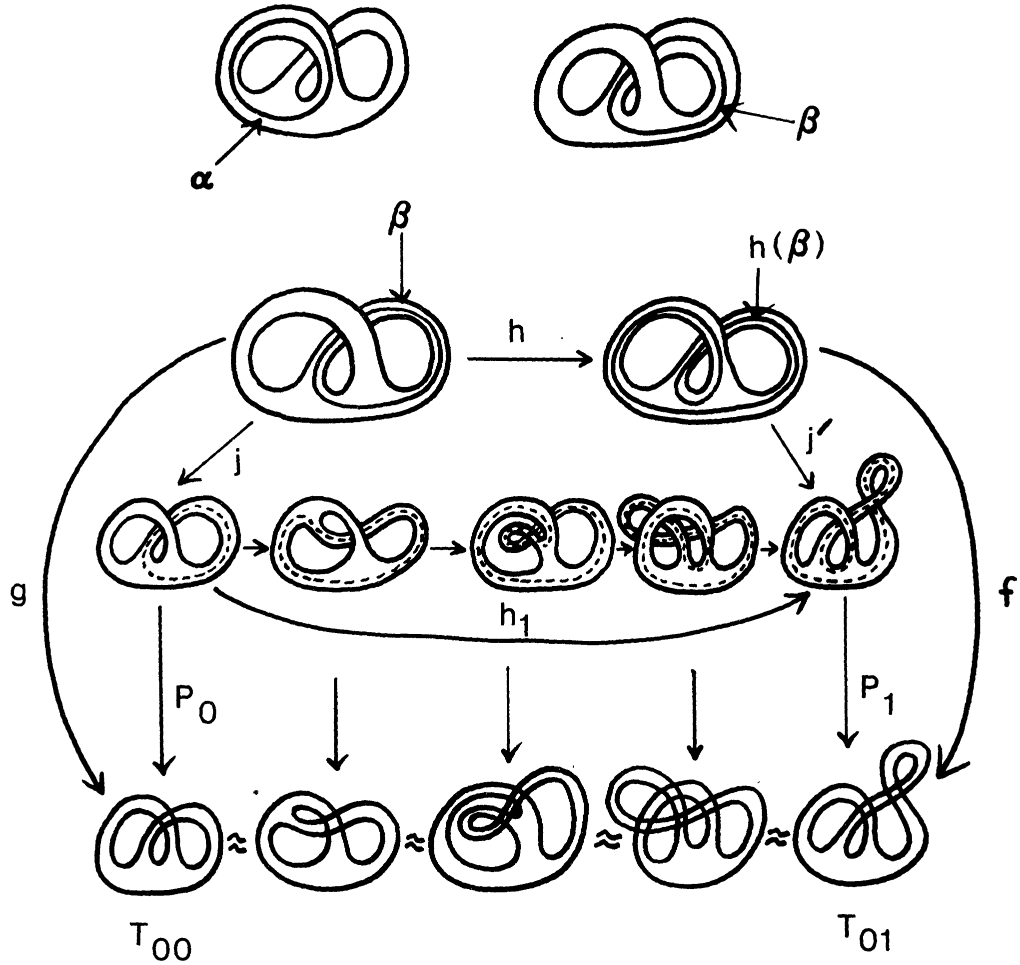

FIG. 8

While this definition may seem hard to visualize, in practice this is not the case. A diffeomorphism of a surface may often be viewed as the result of a deformation. View Figure 8. It illustrates (middle row) an embedding of the punctured torus in R^3 and a deformation through embeddings to a new punctured torus in R^3. More precisely, we are given a time-parameter family of diffeomorphisms $h_t : R^3 \to R^3$ so that $h_0 = \text{identity}$, and an embedding $j : M \to R^3$. Then $j(M)$ represents the embedded torus at time $t = 0$ and $h_1(j(M))$ represents the torus at time $t = 1$. There is an obvious map $j' : M \to R^3$ so that $j'(M) = h_1(j(M))$ and so that j' embeds the disk and maps each band to the corresponding embedded band. Define $h : M \to M$ by the formula $h(x) = (j'^{-1} \circ h_1 \circ j)(x)$. As the picture suggests, $h(\alpha) = \alpha$ while $h(\beta)$ is a sort of combination of α and β, where α and β are the band curves on M. The bottom part of the figure illustrates a sequence of projections $p_t : h_t(j(M)) \to R^2$ (or S^2). In each case, $p_t \circ h_t \circ j(M)$ is an immersed surface. Let $f = p_1 \circ j'$ and $g = p_0 \circ j$. Then f and g are immersions with $N(g)(\alpha) = N(g)(\beta) = 0$ while $N(f)(\alpha) = 0$, $N(f)(\beta) = 1$. However, by our construction we see that $N(f \circ h)(\alpha) = 0 = N(f \circ h)(\beta)$. Hence, by Theorem 4.1, $f \circ h$ is regularly homotopic to g.

We put this in a nutshell by saying that T_{00} and T_{01} are image homotopic. Intuitively, the image homotopy is pictured in the bottom line of Figure 8. Any such picture may be unfolded as we have done, to produce a diffeomorphism h. We shall use this convention from now on, drawing image homotopies as in the bottom of Figure 8 and leaving the unfolding to the reader.

5. The boundary invariant. Recall that our surfaces have boundary. Each boundary component of a surface M may be viewed as a curve on M. If C is a boundary component of M, then we may choose an embedding $c : S^1 \to M$ so that $c(S^1) = C$. In this way we regard each boundary curve as an element of $C(M)$. Given an immersion $f : M \to S^2$, we may compute the crossing number, $N(f)(C) = N(f \circ c)$, for each boundary component C. For a given component C, this number does not depend upon the choice of embedding $c : S^1 \to M$.

DEFINITION 5.1. The boundary invariant, $B(f) \in Z$, of an immersion $f : M \to S^2$ is the total number of boundary curves $C \subset M$ such that $N(f)(C) = 1$.

LEMMA 5.2. *If f and g are image homotopic immersions of a surface M into S^2, then $B(f) = B(g)$.*

Proof. First suppose $f = g \circ h$ where $h : M \to M$ is a diffeomorphism. Since h only permutes the boundary components, $B(f) = B(g)$. If f is regularly homotopic to g then $f|_C$ is regularly homotopic to $g|_C$ for each boundary curve C. Hence $N(f)(C) = N(g)(C)$ and therefore $B(f) = B(g)$. Since it is sufficient to check these two cases, we conclude that if f is image homotopic to g, then $B(f) = B(g)$.

LEMMA 5.3. *For any surface immersion $f : M \to S^2$, the boundary invariant is even. In particular, if M has only one boundary component, then $B(f) = 0$.*

Proof. Given an immersion $f : M \to S^2$ we may assume that each boundary curve is immersed with normal crossings, and that the immersions of distinct boundary curves cross each other normally. Let the total crossing number, $T(f) \in Z$, be the total number of crossings (self and mutual) among the immersed boundary curves. Since any two curves in S^2 intersect in an even number of points, it is clear that $B(f) \equiv T(f)$ (modulo 2).

We shall prove that $T(f) \equiv 0$ (modulo 2) by induction on the number of bands necessary to represent M. If there are no bands, then M is a disk and so there is a regular homotopy of the boundary curve to a small circle about a point. Such a circle has crossing number zero, and so the original crossing number was even. For a surface with one or more bands, we may eliminate a band by cutting across it as in Figure 9. The total crossing number is unaffected by this operation. Hence the lemma follows by induction.

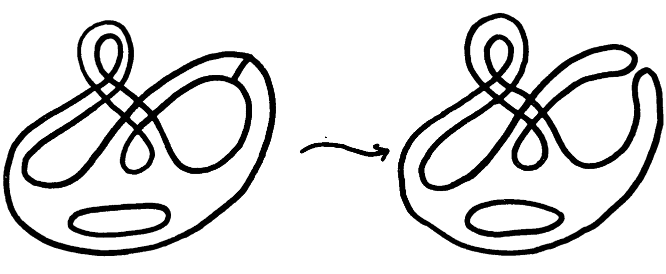

FIG. 9

The boundary invariant easily distinguishes certain immersions. For example, let X and Y denote the immersions in Figure 10. Then $B(X) = 2$ while $B(Y) = 4$. Hence X is not image homotopic to Y.

In fact, immersions of punctured disks are classified by the boundary invariant:

PROPOSITION 5.4. *Let M be a punctured disk with k holes, represented as a disk with k attached bands. Two immersions f, $g : M \to S^2$ are image homotopic if and only if $B(f) = B(g)$.*

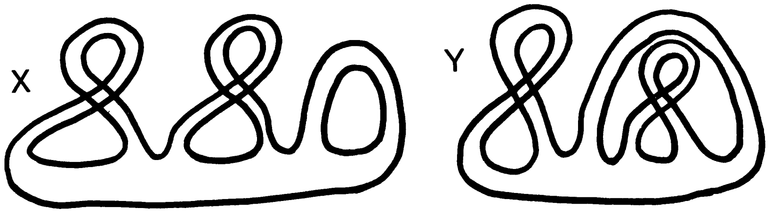

FIG. 10

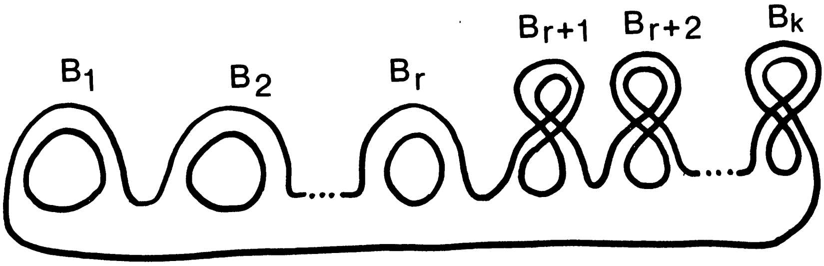

FIG. 11

Proof. We may assume (by using a regular homotopy) that f embeds the disk part of M in standard fashion and that the bands are immersed disjointly, with either one twist or no twist. If M is represented as in Figure 11 with bands $B_1, B_2, \ldots, B_k$, then we may assume that $B_1, \ldots, B_r$ have no twist, while $B_{r+1}, \ldots, B_k$ each have a single twist. This is accomplished by using a sequence of permutations as in Figure 12A. There are $(k+1)$ boundary curves $C_0, C_1, \ldots, C_k$. The curve C_0 is the outer boundary. Hence $B(f) = (k-r) + N(C_0)$. Since $N(C_0) = 0$ or 1, this presentation is not quite canonical. However, if $k - r \equiv 0 \pmod 2$ then $N(C_0) = 0$, and if $k - r \equiv 1 \pmod 2$ then $N(C_0) = 1$ (since $B(f)$ is even). If k is odd and $B(f) = k+1$, there is no ambiguity. Otherwise, there are two immersions of this form corresponding to each value of $B(f)$, one with $N(C_0) = 0$ and one with $N(C_0) = 1$. They are image homotopic via the "handle-sliding" operation illustrated in Figure 12 for $k = 2$ and $k = 4$.

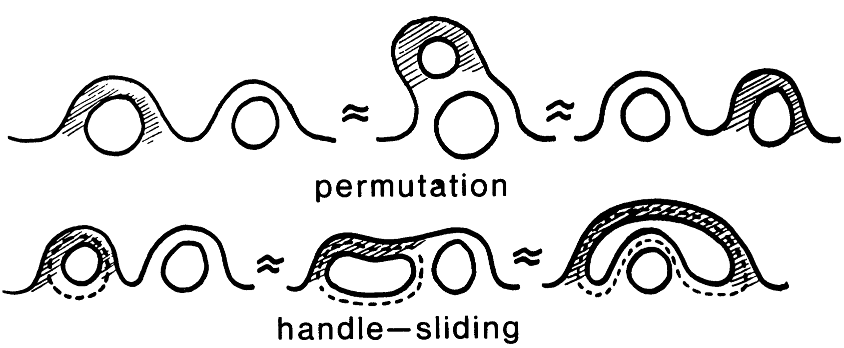

FIG. 12A

In the general case we slide the right hand end of B_1 across all other bands, cancel some twists using the mod-2 character of immersions in S^2, and proceed as in Figure 12. This completes the proof.

6. Curves, homology, and quadratic forms. If M is an arbitrary surface and $f: M \to S^2$ an immersion, then the boundary invariant is insufficient to determine the image homotopy class of f. We

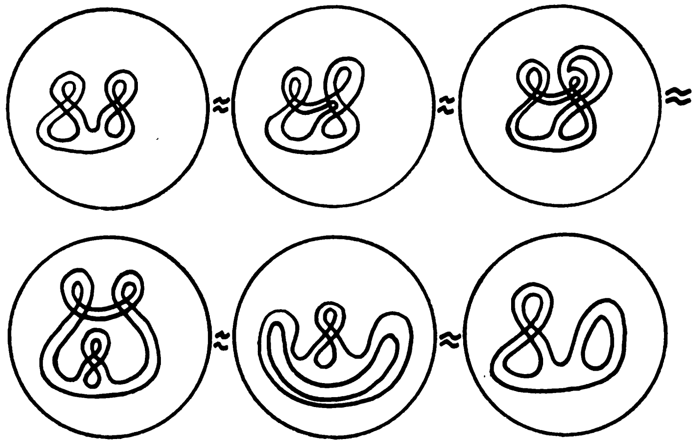

FIG. 12B

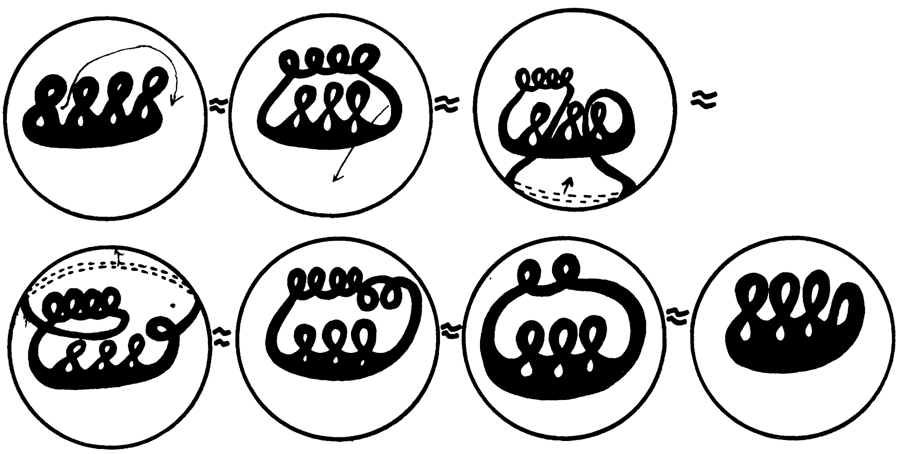

FIG. 12C

shall see that $N(f): \mathscr{C}(M) \to Z_2$ contains all of the extra information that is needed. However, $\mathscr{C}(M)$ is a very large collection of curves; some simplification is called for.

Suppose $\alpha, \beta \in \mathscr{C}(M)$. It may happen that α is regularly homotopic to β. That is, there may be a map $F: S^1 \times [0,1] \to M$ such that each F_t, $0 \leqq t \leqq 1$, is an immersion and $F_0 = \alpha$, $F_1 = \beta$. Under these circumstances $f \circ \alpha$ and $f \circ \beta$ are regularly homotopic immersions of S^1 to S^2. (The regular homotopy is $f \circ F: S^1 \times [0,1] \to S^2$.) Thus $N(f)(\alpha) = N(f)(\beta)$.

DEFINITION 6.1. Let $\bar{\mathscr{C}}(M)$ denote the collection of regular homotopy classes of elements of $\mathscr{C}(M)$. Given $\alpha \in \mathscr{C}(M)$, let $\bar{\alpha}$ denote its regular homotopy class in $\bar{\mathscr{C}}(M)$. Then we may define $\bar{N}(f): \bar{\mathscr{C}}(M) \to Z_2$ by $\bar{N}(f)(\bar{\alpha}) = N(f)(\alpha)$. The remarks above assure us that this definition makes sense.

Now $\bar{\mathscr{C}}(M)$ has extra structure which can be exploited. Given α and $\beta \in \mathscr{C}(M)$, we can deform each of them by a regular homotopy so that $\alpha(S^1)$ and $\beta(S^1)$ intersect normally. Assuming that α and β intersect normally, let $\alpha \cdot \beta \in Z_2$ denote the mod-2 residue class of their number of mutual intersections. This intersection number in Z_2 depends only on the regular homotopy classes of the curves. We obtain a pairing $\cdot: \bar{\mathscr{C}}(M) \times \bar{\mathscr{C}}(M) \to Z_2$ by setting $\bar{\alpha} \cdot \bar{\beta} = \alpha \cdot \beta$ where α and β are chosen to intersect normally.

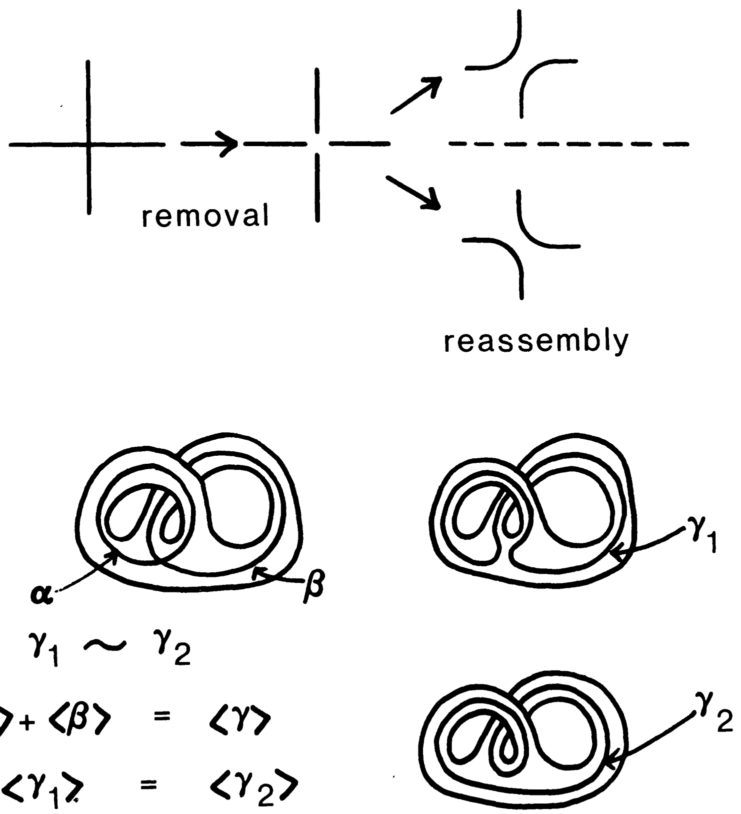

FIG. 13

It is also possible to add elements of $\mathscr{C}(M)$. This involves removing intersection points. Suppose two arcs intersect normally as in Figure 13. There are two ways to remove the intersection point and reassemble the remainder into two non-intersecting arcs (see Figure 13). Given curves $\alpha, \beta \in \mathscr{C}(M)$ intersecting normally, we may systematically remove all the intersection points and then reassemble to obtain a connected closed curve with no self-intersections. There are many ways to do this. Call the set of curves obtained in this way $\alpha \oplus \beta$.

Thus we have a procedure for addition:

(1) Choose $\alpha, \beta \in \mathscr{C}(M)$.

(2) Find $\alpha', \beta' \in \mathscr{C}(M)$ so that α' and β' intersect normally, and $\alpha' \approx \alpha$, $\beta' \approx \beta$.

(3) Remove all intersection points of α' and β'.

(4) Reassemble to form an embedded curve γ.

(5) Then $\gamma \in \alpha \oplus \beta$.

There is an important relationship between this addition process and $N(f)$.

LEMMA 6.2. *Let $f: M \to S^2$ be an immersion. Let $\alpha, \beta \in \mathscr{C}(M)$ be two normally intersecting curves. Then, for any choice of γ in $\alpha \oplus \beta$,*

$$N(f)(\gamma) = N(f)(\alpha) + N(f)(\beta) + \alpha \cdot \beta.$$

Proof. We may assume that $\alpha(S^1) \cap \beta(S^1) \neq \varnothing$ since this can be arranged by a regular homotopy. By the same reasoning, we may assume that $f \circ \alpha(S^1)$ and $f \circ \beta(S^1)$ have normal self and mutual intersections. Now $N(f)(\gamma)$ equals the total mod-2 number of intersections of $f \circ \alpha(S^1)$ and $f \circ \beta(S^1)$ (self and mutual) minus those mutual intersections which already occur on M. Since the total mutual intersection of two curves on S^2 is even, we conclude that

$$N(f)(\gamma) = N(f)(\alpha) + N(f)(\beta) + \alpha \cdot \beta.$$

This lemma suggests that it would be advantageous to place an equivalence relation on $\mathscr{C}(M)$ so that both sums and the mod-2 intersection number are well-defined on the equivalence classes. Such an equivalence relation must include regular homotopy; it should also make all elements of the set $\gamma_1 \oplus \gamma_2$ equivalent when γ_1 and γ_2 are normally intersecting curves on the surface. Thus we make the following definition:

DEFINITION 6.3. We say that two curves $\alpha, \beta \in \mathscr{C}(M)$ are *homologous* ($\alpha \sim \beta$) if one may be obtained from the other by a finite sequence of elementary homologies. An elementary homology may be of two types:

(1) $\alpha \sim \beta$ if α and β are regularly homotopic on M ($\bar{\alpha} = \bar{\beta} \in \bar{\mathscr{C}}(M)$).

(2) $\alpha \sim \beta$ if there are curves $\gamma_1, \gamma_2 \in \mathscr{C}(M)$, intersecting normally, so that α and β are each members of $\gamma_1 \oplus \gamma_2$.

Let $\mathscr{H}(M)$ denote the set of homology classes of curves in $\mathscr{C}(M)$. If $\alpha \in \mathscr{C}(M)$, let $\langle\alpha\rangle \in \mathscr{H}(M)$ be its homology class.

Given $\langle\alpha\rangle, \langle\beta\rangle \in \mathscr{H}(M)$, we may assume that the representatives α and β have non-empty, normal intersection. (Since the surface is connected, it is always possible to change two curves by a regular homotopy so that they have non-empty intersection.) We therefore define the sum by the equation $\langle\alpha\rangle + \langle\beta\rangle = \langle\gamma\rangle$ for any γ in $\alpha \oplus \beta$. The intersection pairing is defined by the equation $\langle\alpha\rangle \cdot \langle\beta\rangle = \alpha \cdot \beta$. In fact $\mathscr{H}(M)$ becomes a group. The identity element, 0, is represented by any small curve about a point on M. Note that it follows from Lemma 6.2 that $N(f)$ is well-defined on $\mathscr{H}(M)$ and that

$$N(f)(\langle\alpha\rangle + \langle\beta\rangle) = N(f)(\langle\alpha\rangle) + N(f)(\langle\beta\rangle) + \langle\alpha\rangle \cdot \langle\beta\rangle.$$

(We set $N(f)(\langle\alpha\rangle) = N(f \circ \alpha)$.)

It is not hard to verify the next lemma:

LEMMA 6.4. *If M is any surface, then $\mathscr{H}(M)$ is an abelian group such that every element has order* 2. *Thus we may regard $\mathscr{H}(M)$ as a Z_2-vector space. If M has normal form as in Figure 6 then $\mathscr{H}(M)$ has Z_2-basis $\mathscr{B} = \{a_1, a'_1, \cdots, a_r, a'_r, b_1, \cdots, b_s\}$ where each of these classes represents a curve which traverses a single band on M.*

The intersection pairing $\cdot : \mathscr{H}(M) \times \mathscr{H}(M) \to Z_2$ is bilinear and symmetric, and $x \cdot x = 0$ for all $x \in \mathscr{H}(M)$. In the basis $\mathscr{B}$,

$$\begin{aligned} &a_i \cdot a'_i = 1 \\ &a_i \cdot a_j = a_i \cdot a'_j = 0, \qquad i \neq j \\ &a_i \cdot b_j = a'_i \cdot b_j = 0, \qquad \text{any } i, j. \end{aligned}$$

(Compare with Figure 6.)

In fact, $\mathscr{H}(M) \simeq H_1(M; Z_2)$, the usual first homology group of M with Z_2 coefficients. (See [2], pp. 92–94.)

We are now in a position to reformulate Lemma 5.2.

DEFINITION 6.5. Let V be a finite dimensional Z_2 vector space and $\cdot : V \times V \to Z_2$ a symmetric, bilinear form such that $x \cdot x = 0$ for all $x \in V$. We say that a function $q : V \to Z_2$ is a *mod-2 quadratic form* associated with the pairing $\cdot$ if

$$q(x+y) = q(x) + q(y) + x \cdot y$$

for all $x, y \in V$.

COROLLARY 6.6. *Let $f: M \to S^2$ be an immersion. Define $q(f): \mathcal{H}(M) \to Z_2$ by $q(f)\langle\alpha\rangle = N(f)(\alpha)$. Then $q(f)$ is a mod-2 quadratic form associated with the intersection pairing on $\mathcal{H}(M)$.*

Note that, by 6.4, we can actually compute $q(f)$ by finding the mod-2 degrees of each band-curve on M. How does $q(f)$ behave under an image homotopy of f? Obviously it remains unchanged under regular homotopy of f. The next lemma shows what happens when the image homotopy involves a diffeomorphism:

LEMMA 6.7. *If $h: M \to M$ is a diffeomorphism, then h induces a vector space isomorphism $h_*: \mathcal{H}(M) \to \mathcal{H}(M)$ and $(h_* a)\cdot(h_* b) = a \cdot b$ for all $a, b \in \mathcal{H}(M)$.*

Proof. Define $h_*\langle\alpha\rangle = \langle h \circ \alpha\rangle$. The rest is easy to check.

Now suppose that $g = f \circ h$ where h is a diffeomorphism of M. Then $q(g)\langle\alpha\rangle = N(f \circ h \circ \alpha) = q(f)\langle h \circ \alpha\rangle = q(f) \circ h_*\langle\alpha\rangle$. Thus $q(g) = q(f) \circ h_*$.

DEFINITION 6.8. Let $q, q': V \to Z_2$ be two quadratic forms. One says that q is isomorphic to q' ($q \simeq q'$) if there is a vector space isomorphism $T: V \to V$ such that $q' = q \circ T$.

COROLLARY 6.9. *If f and $g: M \to S^2$ are image homotopic immersions, then the quadratic forms $q(f)$ and $q(g)$ are isomorphic.*

We can now prove that the punctured torus immersions T_0 and T_1 are not image homotopic. Let $\phi_0 = q(T_0)$ and $\phi_1 = q(T_1)$. Then $V = \mathcal{H}(M) \simeq Z_2 \oplus Z_2$ with basis $\{a, b\}$ so that $\phi_0(a) = \phi_0(b) = 0$, $\phi_1(a) = \phi_1(b) = 1$. Hence $\phi_0(a + b) = a \cdot b = 1$ and $\phi_1(a + b) = 1 + 1 + 1 = 1$. Therefore ϕ_1 takes a majority of elements of V to 1 while ϕ_0 takes a majority of elements of V to 0. If $\phi_1 = \phi_0 \circ f$ where $f: V \to V$ is an isomorphism, then ϕ_1 would also take a majority to 0. Therefore ϕ_0 and ϕ_1 are not isomorphic, and hence T_0 and T_1 are not image homotopic.

On the other hand, our pictures of homotopies give rise to specific isomorphisms. Consider the handle-sliding operation of Figure 12. If b denotes the curve (dotted) on the band being moved, while a is the curve on the stationary band, then our picture shows that $a \mapsto a$ while $b \mapsto a + b$ under the image homotopy.

For example, Figure 8 shows an image homotopy of T_0 to T_{01}. We know that $q(T_0) = \phi_0$; let $\phi = q(T_{01})$. Then the isomorphism $h_*: V \to V$ is given by the equations $h_*(a) = a$ and $h_*(b) = a + b$.

The association of immersions and quadratic forms is a double-edged tool. It will let us determine the structure of both immersions and quadratic forms.

7. Basic homotopies and isomorphisms. Immersions may be combined. The beginning of Figure 14 illustrates the result of joining two copies of T_0 together by an untwisted band. This is called the connected sum. Given any two immersions A and B, their connected sum, $A \# B$, is formed in the same way. If one of the surfaces has more than one boundary component, then it is possible to form different immersions by joining along different boundary components. For example, Figure 10 illustrates two ways of taking the connected sum of Λ_1, Λ_1, and Λ_0. Nevertheless, the symbol $A \# B$ will denote any one of the possible connected sums of A and B.

The algebraic analogue of connected sum is the direct sum of quadratic forms.

DEFINITION 7.1. Let $q: V \to Z_2$ and $q': V' \to Z_2$ be mod-2 quadratic forms. Define the direct sum, $q \oplus q': V \oplus V' \to Z_2$ by the formula $q \oplus q'(r, r') = q(r) + q'(r')$ for $r \in V$ and $r' \in V'$. It is easy to see that $q \oplus q'$ is a quadratic form.

If A and B are immersions of surfaces M and N, then $A \# B$ is an immersion of $M \# N$ and $\mathcal{H}(M \# N) \simeq \mathcal{H}(M) \oplus \mathcal{H}(N)$. It is clear that $q(A \# B) \simeq q(A) \oplus q(B)$.

For example, let Λ_0 and Λ_1 be the untwisted and single-twist annulus immersions. Let η_0 and η_1 be the mod-2 quadratic forms on $V = Z_2$ defined by setting $\eta_0(1) = 0$ and $\eta_1(1) = 1$. Then $q(\Lambda_0) = \eta_0$ and

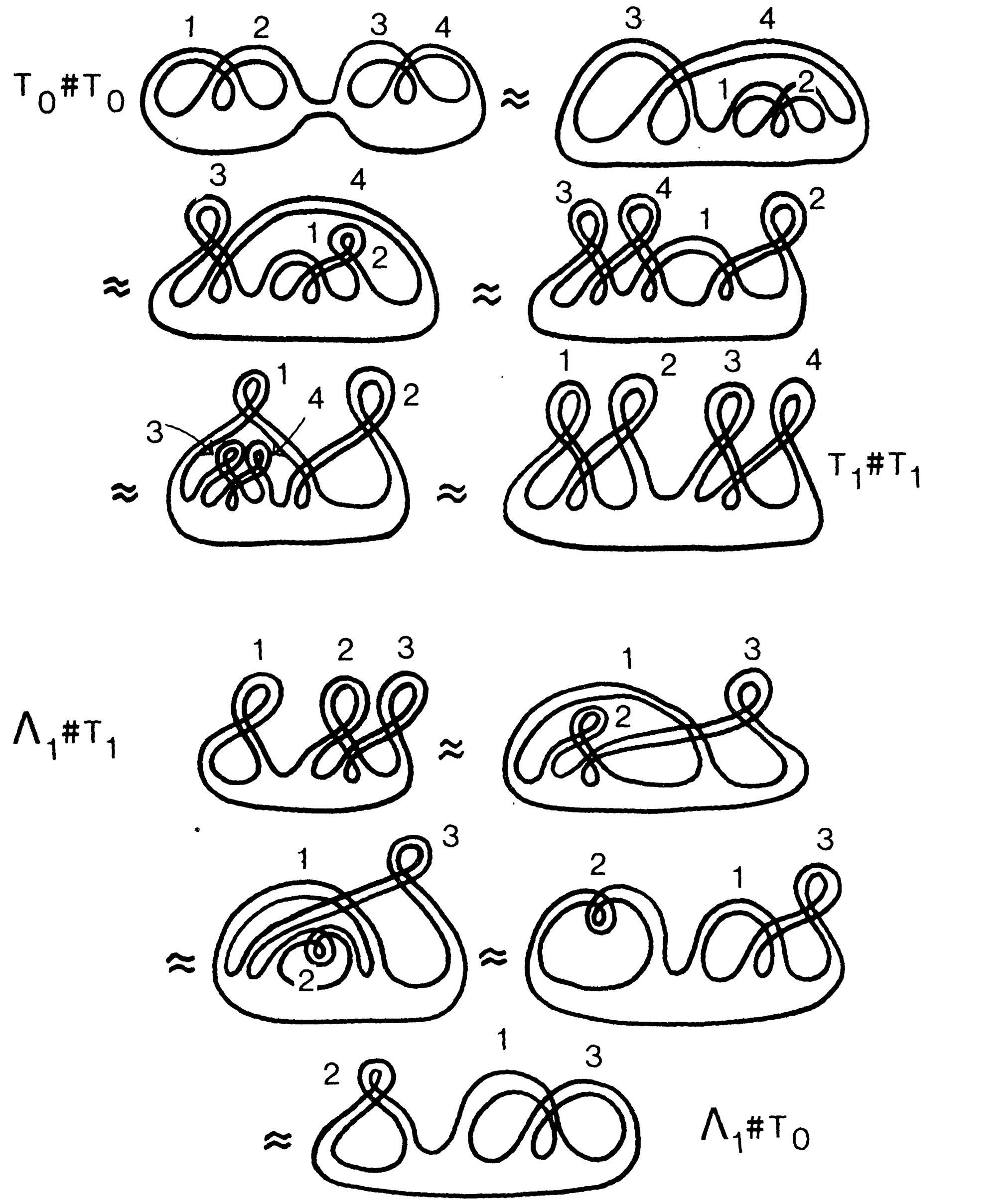

FIG. 14

$q(\Lambda_1) = \eta_1$. Figure 12B may be interpreted as $\Lambda_0 \# \Lambda_1 \approx \Lambda_1 \# \Lambda_1$. Hence $\eta_0 \oplus \eta_1 \simeq \eta_1 \oplus \eta_1$. Since direct sum of forms is well-defined, commutative, and associative, this implies that $\eta_1 \oplus \eta_1 \oplus \eta_1 \simeq \eta_0 \oplus \eta_0 \oplus \eta_1$. This type of reduction occurs for forms but not for immersions (the boundary invariant gets in the way).

Certain basic homotopies lead to fundamental isomorphisms of quadratic forms:

THEOREM 7.2. (a) $T_0 \# T_0 \approx T_1 \# T_1$; (b) $\Lambda_1 \# T_1 \approx \Lambda_1 \# T_0$; (c) $\Lambda_1 \# \Lambda_1 \approx \Lambda_0 \# \Lambda_1$.

Hence, (a′) $\phi_0 \oplus \phi_0 \simeq \phi_1 \oplus \phi_1$, (*recall that* $\phi_0 = q(T_0)$, $\phi_1 = q(T_1)$); (b′) $\eta_1 \oplus \phi_1 \simeq \eta_1 \oplus \phi_0$; (c′) $\eta_1 \oplus \eta_1 \simeq \eta_0 \oplus \eta_1$.

Proof. The proof involves handle sliding as illustrated in Figure 14. For (a) the bands are labelled 1, 2, 3, 4. The first step involves sliding the $1-2$ group over 3. Then 3 is slid around 1, 2 and 4; and 2 is slid around 1. Note that in sliding 3 past the $1-2$ group one slides 3 along 1, then around 2, then around 1, and finally across 2. This adds two twists to band 3. Sliding around 4 adds another twist. Thus in the second step, band 3 acquires 3 twists. Since twists cancel in pairs on S^2 (by the global swing-around of Figure 3) we have illustrated band 3 with the single resulting twist. This mod-2 twist arithmetic goes on throughout most of the rest of the deformation.

In the third step , 4 slides over 2, and acquires a twist. Then 1 slides over 3 acquiring a twist. Finally the $3-4$ group slides out across 2 and we have $T_1 \# T_1$.

For (b) the procedure is similar. Slide 1 across 2. Slide the left end of 2 around 1. Slide 2 out. Slide 3 around 1.

We have discussed (c) in the remarks prior to the theorem. This completes the proof.

The algebra isomorphisms (a′), (b′), (c′) deserve algebraic proofs. It is not hard to ferret out proofs from our homotopies by following a homology basis throughout the deformations. For example, let $\psi = q(T_0 \# T_0)$ and $\psi' = q(T_1 \# T_1)$. Let V have basis a_1, a_2, a_3, a_4 where a_i is the curve corresponding to the band labelled i. Then $\psi(a_1) = \psi(a_2) = \psi(a_3) = \psi(a_4) = 0$ while $\psi'(a_1) = \cdots = \psi'(a_4) = 1$. Here are the transformations corresponding to each step in the deformation for part (a) (refer to Figure 14).

(1) identity

(2) $a_1 \mapsto a_1$
$a_2 \mapsto a_1 + a_2$
$a_3 \mapsto a_3 + a_4$
$a_4 \mapsto a_4$

(3) $a_i \mapsto a_i \qquad i = 1, 2, 3$
$a_4 \mapsto a_2 + a_4$

(4) $a_1 \mapsto a_1 + a_3$
$a_i \mapsto a_i, \qquad i = 2, 3, 4$

(5) identity.

These are obtained by repeatedly using the handle sliding basis-change discussed at the end of the last section. Note that sliding across a torus-group (like 1–2) induces the identity transformation since one must slide past each band in the group twice.

The mapping $h_*: V \to V$ is the composite of these five maps. We find

$$h_*(a_1) = a_1 + a_3$$
$$h_*(a_2) = a_1 + a_2 + a_3$$
$$h_*(a_3) = a_2 + a_3 + a_4$$
$$h_*(a_4) = a_2 + a_4.$$

It is easy to check that $\psi' \circ h_* = \psi$. This gives an algebraic proof that $\phi_0 \oplus \phi_0 \simeq \phi_1 \oplus \phi_1$.

The isomorphisms of Theorem 7.2 are the key to the classification of mod-2 quadratic forms! Here is the algebra in its own right:

Let $q: V \to Z_2$ be an arbitrary mod-2 quadratic form with associated pairing $\cdot: V \times V \to Z_2$. When $q = q(f)$ for an immersion $f: M \to S^2$ we know that V has a basis $\mathcal{B}$ as in 5.4. In fact one can always find a basis with these intersection properties for any symmetric bilinear pairing on V such that $x \cdot x = 0$ for all $x \in V$. The proof (algebraic proof) is a standard exercise in linear algebra. It then follows, just as in the geometric case, that any form $q: V \to Z_2$ is a direct sum involving the forms ϕ_0, ϕ_1, η_0 and η_1. Such a direct sum reduces to one of the following types by using (a′), (b′), and (c′).

(i) $\phi_1 \oplus l\phi_0 \oplus (s+1)\eta_0$

(ii) $(l+1)\phi_0 \oplus (s+1)\eta_0$

(iii) $(l+1)\phi_0 \oplus s\eta_0 \oplus \eta_1$.

Here $l\phi = \phi \oplus \phi \oplus \cdots \oplus \phi$ (l-factors).

These types are distinct. To see this, first characterize the subspace of V which supports the η_0 and η_1 factors. This is the *radical* of V, denoted Rad V. It is the subspace

$$\text{Rad } V = \{x \in V \mid x \cdot y = 0, \quad \text{for all } y \text{ in } V\}.$$

Thus, forms of type (i) or (ii) have $q \mid \text{Rad } V \equiv 0$. If this is the case, then we obtain $\bar{q} : V/\text{Rad } V \to Z_2$ and $\bar{q}$ is *non-degenerate.* That is, the matrix of the bilinear form for $\bar{q}$ is non-singular. The form $\bar{q}$ is isomorphic to $\phi_1 \oplus l\phi_0$ or to $(l+1)\phi_0$. In the first case, a majority of the elements of $V/\text{Rad } V$ go to 1, while in the second case a majority go to 0. We may therefore define the *Arf invariant* $c(q) = 1$ or 0 according to this majority vote by $\bar{q}$.

In case (iii) $q \mid \text{Rad } V \not\equiv 0$. But forms of this type are characterized by $\dim(V)$ and $\dim(\text{Rad } V)$.

This completes the classification of mod-2 quadratic forms. In the next section we apply our results and classify immersions.

REMARK. It is useful at this point to survey the correspondence that we have obtained. If V is a vector space over a field F and $\langle \ , \ \rangle$: $V \times V \to F$ is a symmetric bilinear form then, if $1 + 1 \neq 0$ in F, we may define a quadratic form $q : V \to F$ by $q(x) = \frac{1}{2}\langle x, x \rangle$. Note that we now have

$$q(x+y) = \tfrac{1}{2}\langle x+y, x+y \rangle = \tfrac{1}{2}(\langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle) = q(x) + q(y) + \langle x, y \rangle.$$

Thus q is a quadratic form associated with $\langle \ , \ \rangle$. Conversely, if q is quadratic form then we obtain a corresponding bilinear form. This correspondence breaks down when $1 + 1 = 0$ and there may be many quadratic forms associated with a given bilinear form. Just so in our geometry we take one surface (and its homology intersection form) and we find many immersions of this surface into S^2 (and their quadratic forms). For a given surface, each quadratic form is associated with one given bilinear form (the intersection form). Since image homotopy of immersions implies isomorphism of the corresponding mod-2 quadratic forms, it has been possible to give a geometric version of the theory of mod-2 forms.

8. Classification of immersions. We are now prepared to complete the classification of immersed surfaces.

THEOREM 8.1. *Let f and g be orientation preserving immersions of a surface with boundary M into the two-sphere, S^2. Then f is image homotopic to g if and only if the boundary invariants of f and g agree and the quadratic forms are isomorphic. That is,*

$$\text{(a)} \quad B(f) = B(g) \quad \textit{and} \quad \text{(b)} \quad q(f) \simeq q(g).$$

REMARK. If $B(f) = B(g) = 0$, then $q(f)$ and $q(g)$ have Arf invariants (see section 7) and we may replace (b) by

$$\text{(b}'\text{)} \quad c(q(f)) = c(q(g)).$$

If $B(f) = B(g) \neq 0$, then the quadratic forms are of type (iii) and hence are classified by the dimensions of their radicals.

Proof. As in the proof of 5.3, we may assume that M is in normal form as in Figure 6 (a disk with attached bands), that f embeds the disk, and that the bands corresponding to different pairs $\{a_i, a_i'\}$ or singlets $\{b_j\}$ do not intersect in the image. In other words, f is a connected sum involving T_0, T_1, Λ_0 and Λ_1 with all the copies of T_0 and T_1 appearing on a single boundary component. Hence, by using 7.2 we may write, f is image homotopic to one of the following forms:

(i) $f \approx T_1 \# kT_0 \# l\Lambda_0$

(ii) $f \approx kT_0 \# l\Lambda_0$

(iii) $f \approx kT_0 \# l\Lambda_0 \# s\Lambda_1$, $s \neq 0$.

Here the connected sum is the specific one arising from the normal form for M. Immersions of type (i) and (ii) have $B(f) = 0$ and are distinguished by the quadratic form, as we have seen. An immersion of type (iii) is clearly distinguished by $B(f)$ (which tells how many Λ_1's appear) and $q(f)$ (which tells how many Λ_0's and Λ_1's appear). This completes the proof.

Note the close parallel with the classification of quadratic forms. Boundary difficulties, as in section 5, prevent the correspondence from being perfect. This was to be expected since there are no distinguished elements in V for an arbitrary form $q: V \to Z_2$, while $\mathcal{H}(M)$ contains the homology classes of the boundary curves.

If M has a single boundary component then the quadratic form has no radical. Such forms are sums of ϕ_0 and ϕ_1. Theorem 8.1 implies that isomorphism classes of non-degenerate (Rad = 0) forms are in 1–1 correspondence with image homotopy classes of immersions with a single boundary component.

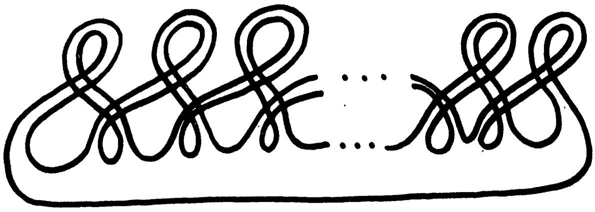

FIG. 15

Here is an exercise. Let N_k be the immersion pictured in Figure 15. It has k-bands. Reduce it to normal form by handle-sliding. What does this say about the corresponding quadratic forms as a function of $k = 1, 2, 3, 4, \ldots$?

Partially supported by NSF grants: MPS73–08855–A02 and GP–38582(X1), MPS73–08732–A02.

References

1. W. Browder, Surgery on Simply Connected Manifolds, Springer-Verlag, New York, 1972.
2. M. Greenberg, Lectures on Algebraic Topology, Benjamin, New York, 1967.
3. F. Hirzebruch, W. D. Neumann, and S. S. Koh, Differentiable Manifolds and Quadratic Forms, Marcel Dekker, New York, 1971.
4. W. Massey, Algebraic Topology: An Introduction, Harcourt Brace and World, New York, 1967.
5. J. Milnor, D. Husemoller, Symmetric Bilinear Forms, Springer-Verlag, New York, 1973.
6. C. P. Rourke and D. P. Sullivan, On the Kervaire obstruction, Ann. of Math., 94 (1971) 397–413.
7. H. Whitney, On regular closed curves in the plane, Comp. Math., 4 (1937) 276–284.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109.
DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912.

THE FIFTH U.S.A. MATHEMATICAL OLYMPIAD

S. L. GREITZER

The Fifth U.S.A. Mathematical Olympiad was held on May 4, 1976. As before, it consisted of five power questions requiring mathematical ingenuity as well as competence and knowledge of subject matter. The problems will be found at the end of this article.

As in previous years, most students were selected on the basis of their performance on the Annual High School Mathematics Examination. Several students from Michigan and Wisconsin were invited to participate. These states do not, as a rule, participate in the Annual High School Mathematics Examination, but have their own contests. In all, 96 students were asked to participate, and 94 finally took part. One student did not reply to the invitation, and one student withdrew suddenly.