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Hopf algebras and invariants of 3-manifolds

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Abstract

This paper studies invariants of 3-manifolds derived from finite dimensional Hopf algebras. The invariants are based on right integrals for the Hopf algebras. In fact, it is shown that the defining property of the right integral is an algebraic translation of a necessary condition for invariance under handle slides in the Kirby calculus. The resulting class of invariants is distinct from the class of Witten–Reshetikhin–Turaev invariants.

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1. Introduction

The purpose of this paper is to indicate a method of defining invariants of 3-manifolds intrinsically in terms of right integrals on certain Hopf algebras. We call such an invariant a *Hennings invariant* [6], as Hennings was the first person to point out that invariants could be defined in this way. The work reported in this paper appears more fully in joint work of the author and David Radford [11].

The present paper contains a significant innovation that goes beyond our previous work [11]. We had previously verified the invariance of our functional for 3-manifolds by using the restricted Kirby move, a special case of handling sliding. In fact, the argument is most transparent when done in general! Here, we shall see that the notions of right integral and invariance under handle sliding are actually translations of each other between algebraic and geometrical categories. These matters are explained in Section 3.

Hennings invariants were originally defined using oriented links. It is not necessary to use invariants that are dependent on link orientation to define 3-manifold invariants via surgery and Kirby calculus. For that reason the invariants discussed in this paper are formulated for unoriented links. This results in a simplification and conceptual clarification of the relationship of Hopf algebras and link invariants. The practical benefit is a simplified algorithmic structure for the calculation or reasoning

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about the invariants. *Further reference to invariants of* 3*-manifolds in this paper will, unless otherwise specified, be to this version of the Hennings invariant for unoriented links.*

We show in [11] that invariants defined in terms of right integrals, as considered in this paper, are distinct from the invariants of Reshetikhin and Turaev [23, 24]. We show that our invariant is non-trivial for the quantum group $U_q(\mathrm{sl}_2)'$ when q is an fourth root of unity. The Reshetikhin–Turaev invariant is trivial at this quantum group and root of unity. The non-triviality of our invariant is exhibited by showing that it distinguishes all the Lens spaces $L(n, 1)$ from one another. This proves that there is non-trivial topological information in the non-semisimplicity of $U_q(\mathrm{sl}_2)'$.

The paper is organized as follows. Section 1 recalls Hopf algebras, quasi-triangular Hopf algebras and ribbon Hopf algebras. Section 2 discusses the conceptual setting of the invariant. This involves a summation over labellings of the link diagram by elements of the Hopf algebra. We work in a category that allows immersed diagrams so that the special grouplike element in the Hopf algebra and the ribbon element in the Hopf algebra both have diagrammatic interpretations. A trace function on the Hopf algebra that is invariant under the antipode is shown to yield a link invariant. In Section 3 we show that traces of the kind discussed in Section 2 are constructed from right integrals in many cases and that under suitable conditions these traces yield invariants of the 3-manifolds obtained by surgery on the links. Section 4 sketches the application to $U_q(\mathrm{sl}_2)'$.

1. Algebra

Recall that a *Hopf algebra A* [25] is a bialgebra over a commutative ring k that has associative multiplication, coassociative comultiplication and is equipped with a counit, a unit and an antipode. The ring k is usually taken to be a field.

A is an algebra with multiplication $m: A \otimes A \to A$. The associative law for m is expressed by the equation $m(m \otimes 1) = m(1 \otimes m)$ where 1 denotes the identity map on A.

A is a bialgebra with coproduct $\Delta: A \to A \otimes A$. The coproduct is a map of algebras. Δ is coassociative. Coassociativity of Δ is expressed by the equation $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ where 1 denotes the identity map on A.

The unit is a mapping from k to A taking 1 in k to 1 in A, and thereby defining an action of k on A. It will be convenient to just identify the units in k and in A, and to ignore the name of the map that gives the unit.

The counit is an algebra mapping from A to k denoted by $E: A \to k$. The following formulas for the counit dualize the structure inherent in the unit: $(E \otimes 1)\Delta = 1 = (1 \otimes E)\Delta$. Here the 1 denotes the identity map on A.

It is convenient to write formally $\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \in A \otimes A$ to indicate the decomposition of the coproduct of x into a use of first and second factors in the two-fold tensor product of A with itself. We shall further adopt the summation convention that $\sum x_{(1)} \otimes x_{(2)}$ can be abbreviated to just $x_{(1)} \otimes x_{(2)}$. Thus we shall write $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

The antipode is a mapping $s: A \to A$ satisfying the equations $m(1 \otimes s)\Delta(x) = E(x)1$, and $m(s \otimes 1)\Delta(x) = E(x)1$, and $m(s \otimes 1)\Delta(x) = E(x)1$ where 1 on the right-hand side of these equations denotes the unit of k as identified with the unit of A. It is a consequence of this definition that $s(xy) = s(y)s(x)$ for all x and y in A.

A quasitriangular Hopf algebra A [4] is a Hopf algebra with an element $\rho \in A \otimes A$ satisfying the following equations:

(1) $\rho\Delta = \Delta'\rho$ where Δ' is the composition of Δ with the map on $A \otimes A$ that switches the two factors.

(2) $\rho_{12}\rho_{13} = (1 \otimes \Delta)\rho$, $\rho_{13}\rho_{23} = (\Delta \otimes 1)\rho$.

These conditions imply that ρ has an inverse, and that

$$\rho^{-1} = (1 \otimes s^{-1})\rho = (s \otimes 1)\rho.$$

It follows easily from the axioms of the quasitriangular Hopf algebra that ρ satisfies the Yang–Baxter equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}.$$

A less obvious fact about quasi-triangular Hopf algebras is that there exists an element u such that u is invertible and $s^2(x) = uxu^{-1}$ for all x in A. In fact, we may take $u = \sum s(e')e$ where $\rho = \sum e \otimes e'$.

An element G in a Hopf algebra is said to be *grouplike* if $\Delta(G) = G \otimes G$ and $E(G) = 1$ (from which it follows that G is invertible and $s(G) = G^{-1}$). A quasi-triangular Hopf algebra is said to be a *ribbon Hopf algebra* [10, 23] if there exists a grouplike element G such that (with u as in the previous paragraph) $v = G^{-1}u$ is in the centre of A and $s(u) = G^{-1}uG^{-1}$. We call G a *special* grouplike element of A.

Since $v = G^{-1}u$ is central, $vx = xv$ for all x in A. Therefore $G^{-1}ux = xG^{-1}u$, whence $s^2(x) = uxu^{-1} = GxG^{-1}$. Thus $s^2(x) = GxG^{-1}$ for all x in A. Similarly, $s(v) = s(G^{-1}u) = s(u)s(G^{-1}) = G^{-1}uG^{-1}G = G^{-1}u = v$. Thus the square of the antipode is represented as conjugation by the special grouplike element in a ribbon Hopf algebra, and the central element $v = G^{-1}u$ is invariant under the antipode.

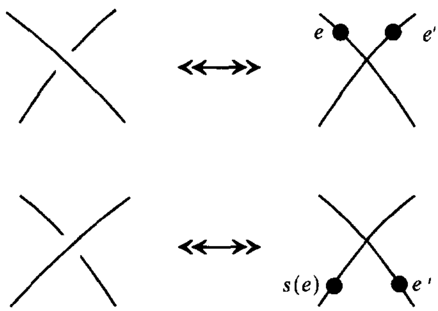
2. Diagrammatic geometry and the trace

A function $\mathrm{tr}: A \to k$ from the Hopf algebra to the base ring k is said to be a *trace* if

$$\mathrm{tr}(xy) = \mathrm{tr}(yx), \quad \mathrm{tr}(s(x)) = \mathrm{tr}(x)$$

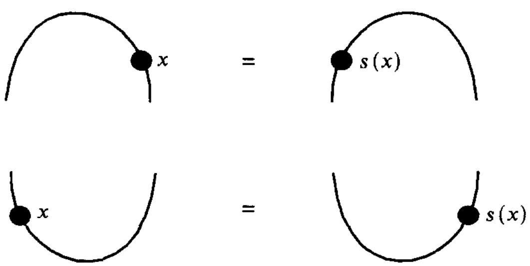
for all x and $y \in A$. In this section we describe how a trace function on a ribbon Hopf algebra yields an invariant, $\mathbf{TR}(K)$, of regular isotopy of knots and links [7, 8].

The link diagram is arranged with respect to a vertical direction so that the crossings form the two types indicated below, and so that other than the crossings the only critical points of the height function are maxima and minima. Each crossing is decorated with elements of the Hopf algebra as shown below. (Here $\rho = \sum e \otimes e'$ is the Yang–Baxter element in $A \otimes A$, and s denotes the antipode.)



It is implicit in this formalism that there is a summation over all the pairs e, e' for each Yang–Baxter element.

Hopf algebra elements may be moved across maxima or minima at the expense of application of the antipode. That is, if a Hopf algebra element is moved across a maximum or minimum, then it is replaced by the application of the antipode to that element if the motion is anti-clockwise. If the motion is clockwise, then the inverse of the antipode is applied to the element. See the diagram below.



The link diagram is subject to deformations that generate regular isotopy [9]. Since the diagram is presented with respect to a choice of vertical direction (discriminating the maxima, minima and crossing types), regular isotopy is generated by a set of moves that include the cancellation of adjacent pairs of maxima and minima and the switching of an arc across a maximum or minimum. The full set of moves is shown in Fig. 1. We have labelled these moves as follows:

= . cancellation of maxima and minima,

II. cancellation of opposite crossings,

III. braiding,

IV. switching,

IV$'$. twist of crossings.

IV$'$ is equivalent to IV in the presence of the cancellation of maxima and minima. These moves generate regular isotopy for diagrams arranged with respect to a vertical direction.

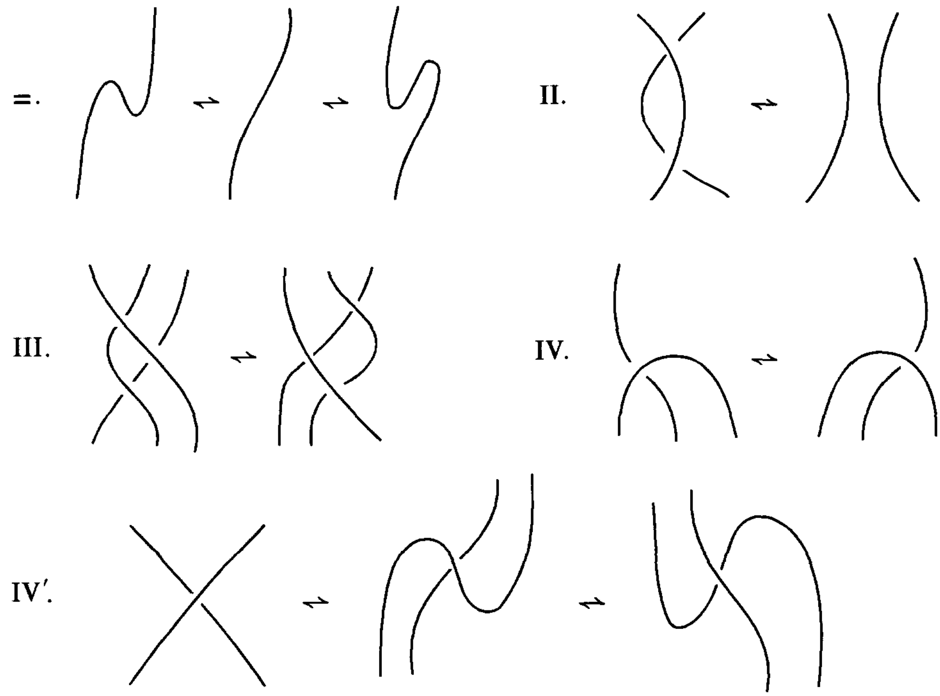
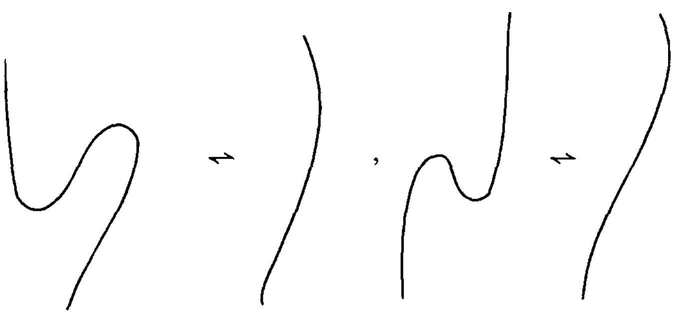


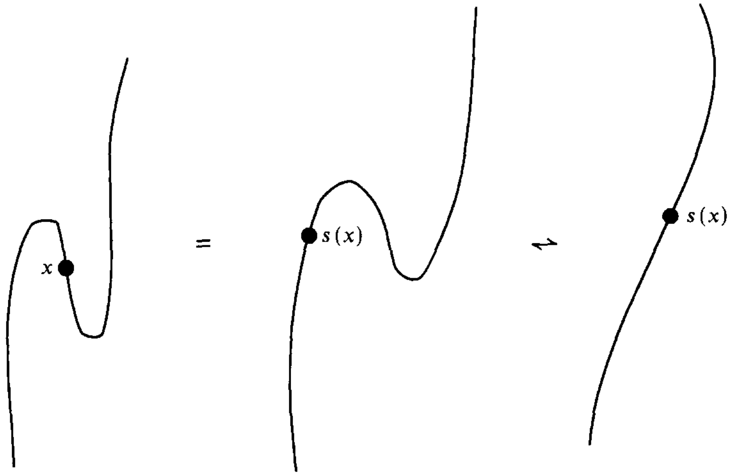
Fig. 1

Remark. The symbol $\simeq$ is used to denote the replacement of one figure by an equivalent figure. We shall sometimes use an equals sign (=) to perform the same purpose. The symbol $\twoheadleftarrow\!\twoheadrightarrow$ or $\rightleftharpoons$ will be used to indicate a correspondence. For example, a link diagram corresponds to the diagram obtained from it by decoration with elements of the Hopf algebra.

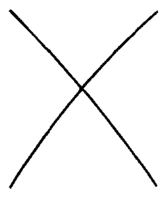
An invariant of regular isotopy must remain unchanged by the moves shown in Fig. 1. The simplest move is the cancellation of a pair consisting of a maximum and a minimum.



This paper cancellation gives a reformulation of the slide rule for the antipode: The antipode is accomplished by "composition with a maximum and a minimum".

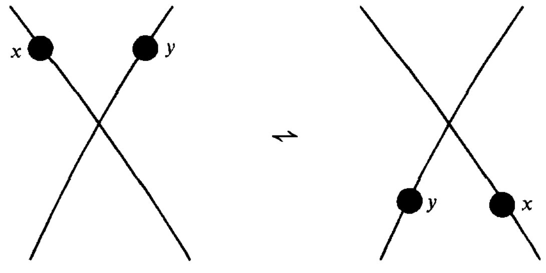


Note also that once the crossings of a link diagram have been labelled with elements of the Hopf algebra, the resulting diagram is depicted as a labelled immersion of a curve or curves in the plane. This is quite natural since the translation from algebraic braiding element to knot-theoretic braiding element is accomplished via the composition with a transposition, and the simplest diagrammatic representation of a transposition is the crossing of two arcs in the plane.



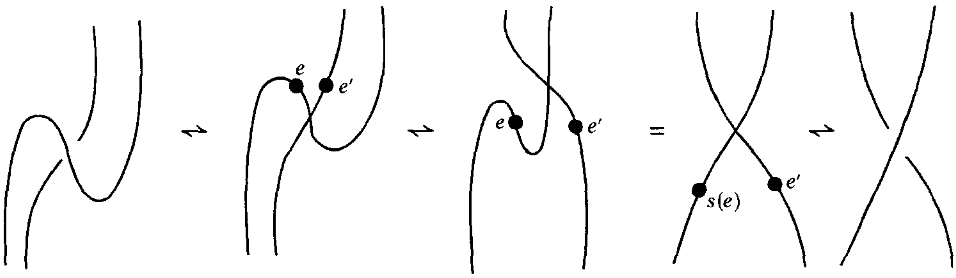
These immersions can be deformed up to regular homotopy that respects the given vertical direction. In other words, one can perform the projected forms of the moves of Fig. 1. If algebra is present on the lines then the following extra move is added (sliding an external line past an algebra element).

V. slide rule

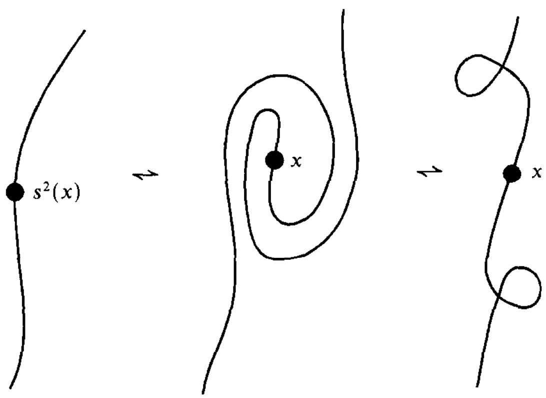


Since algebra elements are configured with respect to the vertical direction, we do not allow the cancellation of a maximum and a minimum that have an algebra element between them. This allows the representation of the antipode as described above.

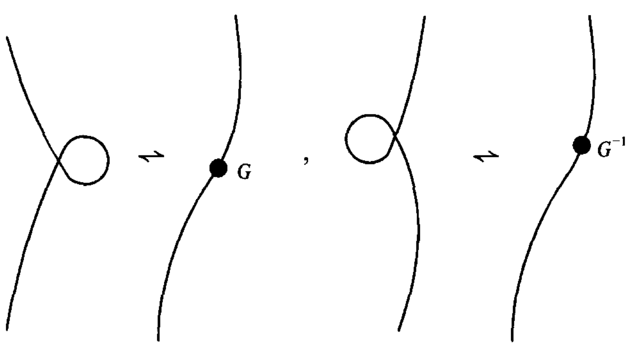
It is now easy to check the twist relation (IV′) for crossings:



With these conventions, the square of the antipode is equivalently diagrammed as a "composition with two curls" as shown below:

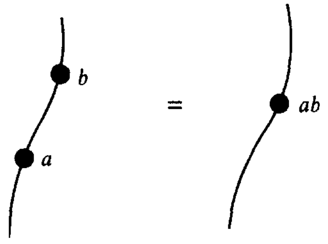


These curls are identified with the special grouplike elements G and G^{-1} in the Hopf algebra.

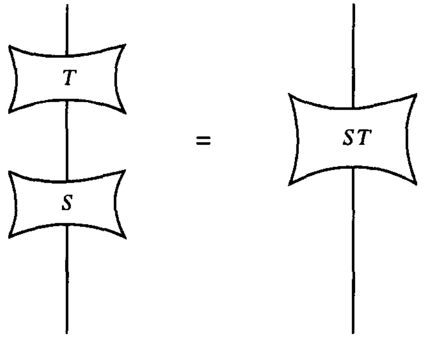


Thus the diagram for the square of the antipode represents directly the formula $s^2(x) = GxG^{-1}$.

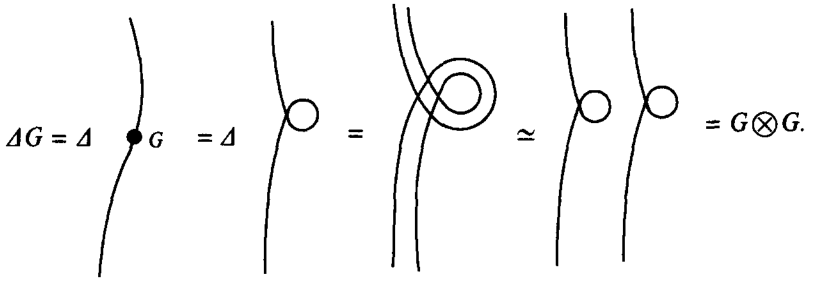
Along a vertical line, algebra elements combine by multiplication.



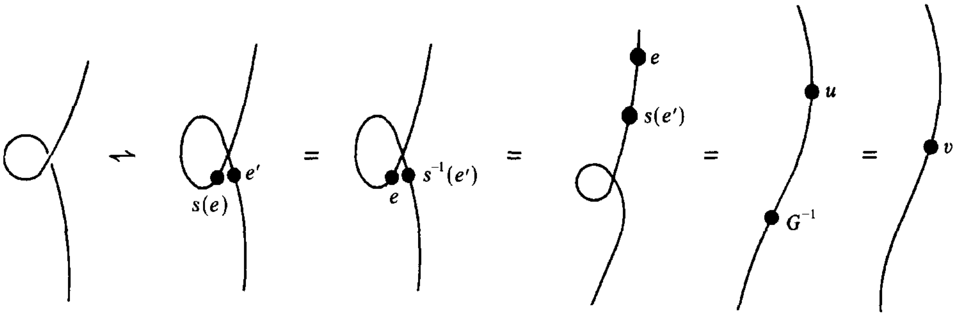
The product in the Hopf algebra corresponds to the multiplication of single strand tangles. (A single strand tangle is a bit of link diagram with two free ends arranged with respect to the vertical so that one end is down and the other end is up. Tangles are multiplied by attaching the down end of one tangle to the top end of the other.)



The coproduct $\Delta : A \to A \otimes A$ in the Hopf algebra corresponds to a mapping on tangles $\Delta : T^{(1)} \to T^{(2)}$ from single strand tangles to double strand tangles obtained by forming the parallel (two strand) cable of the given tangle. The tangles in question can be immersions. For example, we see that the formula $\Delta(G) = G \otimes G$ corresponds to the regular isotopy shown below.

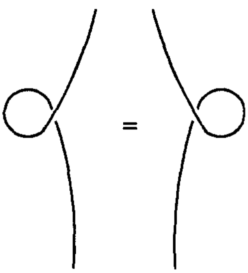


In this way knots on a line can be resolved into algebra elements. For example the twist shown below is equivalent to the ribbon element v. Note how the factorization of v into a product of G^{-1} and $u = \sum s(e')e$ is related to the slide convention for the antipode (In the diagrammatic calculation shown below we use the fact that $(s \otimes s)\rho = \rho$.)



$$v = G^{-1}u$$

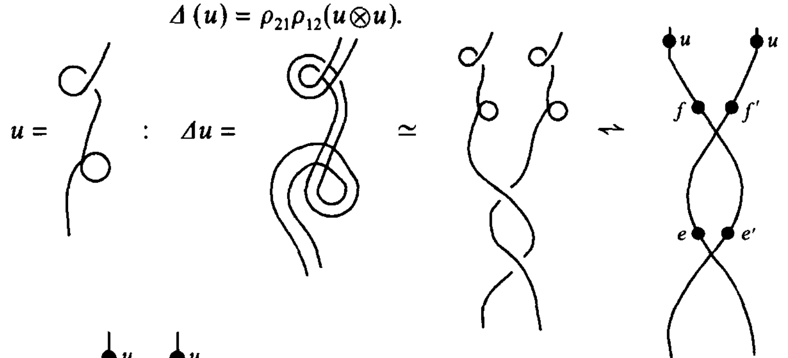
Note that $s(v) = v$ corresponds to the identification shown below.

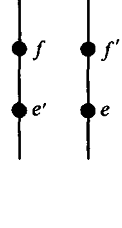


When this identification is added to regular isotopy, the twists catalogue only the framing, and the equivalence relation on the link diagrams is equivalent to ambient isotopy of framed links. We call this equivalence relation on link diagrams *ribbon equivalence*.

Finally, returning to the diagrammatic coproduct we see the interpretation of the following formula of Drinfeld

$$\Delta(u) = \rho_{21}\rho_{12}(u \otimes u).$$



$\Rightarrow \Delta u =$  $\rightharpoondown \; \rho_{21}\rho_{12}(u \otimes u).$

In general, if T is a single strand tangle, and $F(T)$ is the corresponding element in the Hopf algebra A that is determined by our correspondence, then $F(\Delta(T)) = \Delta(F(T))$ where the first Δ is the diagrammatic coproduct and the second Δ is the algebraic coproduct. This fact follows from the axioms for a quasi-triangular Hopf algebra in conjunction with our diagrammatic conventions.

Definition and computation of TR(K). Suppose that $\mathrm{tr}: A \to k$ is a trace function. That is, tr is a linear function satisfying

1. $\mathrm{tr}(xy) = \mathrm{tr}(yx)$,
2. $\mathrm{tr}(s(x)) = \mathrm{tr}(x)$.

To define the trace TR(K) for a knot diagram K, slide all of the algebra into one vertical portion of the diagram. Amalgamate this algebraic expression according to the rule for multiplying algebra elements on the diagram, as we have done above. Call this localized algebra element w. It is a sum of products, and can be formally represented as a product where it is understood that there is a sum over all pairs of the type e, e'.

Let d be the Whitney degree of the flat diagram for K that is obtained by traversing K upward from the vertical portion where the algebra has been concentrated. The Whitney degree is the total turn of the tangent vector to the curve as one traverses it in the given direction. For example:


$d = 1$


$d = -1$.

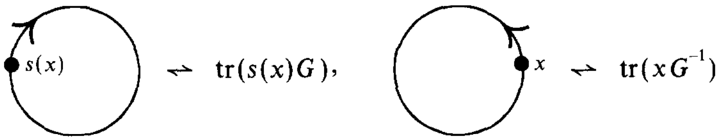
Define TR(K) by the formula $\mathrm{TR}(K) = \mathrm{tr}(wG^d)$. Note that w is itself a summation over all the pairs x, x' corresponding to Yang–Baxter elements on the diagram. TR(K) *defines a regular isotopy invariant of unoriented knots.* (The proof is primarily a matter of checking that TR(K) is independent of the place where we concentrate the algebra. This reduces to checking the independence in the case where the concentration is moved around a maximum or a minimum. See Example 2 below; and for a complete proof see [8, Theorem 5.1]).

In order to define an invariant of unoriented links, concentrate the algebra for each component of the link, and define

$$\mathrm{TR}(K) = \mathrm{tr}(w_1 G^{d_1})\,\mathrm{tr}(w_2 G^{d_2})\,\mathrm{tr}(w_3 G^{d_3}) \cdots \mathrm{tr}(w_n G^{d_n}),$$

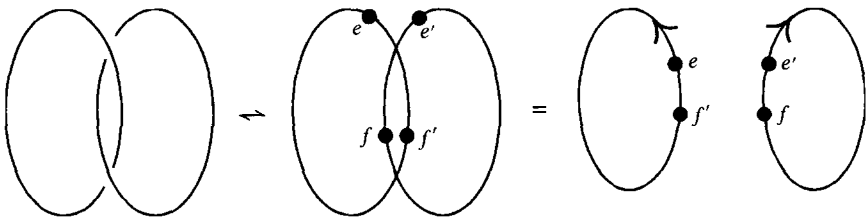
where the labels $1, 2, \ldots, n$ refer to the components of the link, and the implicit summation is the sum over all the pairs x, x' in these words. The elements $w_1, \ldots, w_n$ are the algebra concentrations for each link component, and the degrees $d_1, \ldots, d_n$ are the Whitney degrees of the components of the link.

Example 1. This example points out how the TR(K) is invariant under algebra slides:



$$\mathrm{tr}(s(x)G) = \mathrm{tr}(s(s(x)G)) = \mathrm{tr}(G^{-1}s^2(x)) = \mathrm{tr}(G^{-1}GxG^{-1}) = \mathrm{tr}(xG^{-1}).$$

Example 2. Here is the form of calculation for a link.



$$\mathrm{TR}(L) = \sum \mathrm{tr}(f'eG^{-1})\mathrm{tr}(fe'G).$$

If $\rho = \sum_{i=1}^{n} x_i \otimes y_i$, then $\mathrm{TR}(L) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{tr}(y_j x_i G^{-1})\mathrm{tr}(x_j y_i G)$.

This is how the regular isotopy invariant of the link would look as a specific sum of traces of algebra elements.

3. Invariants of 3-manifolds

The structure we have built so far can be used to construct invariants of 3-manifolds presented in terms of surgery on framed links. We sketch here our technique that simplifies an approach to 3-manifold invariants of Mark Hennings [6]. In fact the approach sketched here also simplifies our own previous method as explained in [11].

Recall that an element λ of the dual algebra A^* is said to be a *right integral* if $\lambda(x)1 = m(\lambda \otimes 1)(\Delta(x))$ for all x in A. For a unimodular [15, 19] finite dimensional ribbon Hopf algebra A there is a right integral λ satisfying the following properties for all x and y in A:

(0) λ is unique up to scalar multiplication when k is a field.

(1) $\lambda(xy) = \lambda(s^2(y)x)$.

(2) $\lambda(gx) = \lambda(s(x))$ where $g = G^2$, G the special grouplike element for the ribbon element $v = G^{-1}u$.

Given the existence of this λ, define a functional $\mathrm{tr}: A \to k$ by the formula $\mathrm{tr}(x) = \lambda(Gx)$. Note that, since $s^2(G) = G$, we have that $\lambda(Gx) = \lambda(s^2(G)x) = \lambda(xG)$. Thus $\mathrm{tr}(x) = \lambda(Gx) = \lambda(xG)$.

Theorem. *With* tr *defined as above,*

(1) $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ *for all* x, y *in* A.

(2) *$tr(s(x)) = \mathrm{tr}(x)$ for all x in A.*
(3) *$[m(\mathrm{tr}\otimes 1)(\Delta(u^{-1}))]u = \lambda(v^{-1})v$ where $v = G^{-1}u$ is the ribbon element.*

Proof. The proof is a direct consequence of the properties (1) and (2) of λ. Thus $\mathrm{tr}(xy) = \lambda(Gxy) = \lambda(s^2(y)Gx) = \lambda(GyG^{-1}Gx) = \lambda(Gyx) = \mathrm{tr}(yx)$, and $\mathrm{tr}(s(x)) = \lambda(Gs(x)) = \lambda(gG^{-1}s(x)) = \lambda(s(G^{-1}s(x))) = \lambda(s^2(x)s(G^{-1})) = \lambda(s^2(x)G) = \lambda(GxG^{-1}G) = \lambda(Gx) = \mathrm{tr}(x)$. Finally, $[m(\mathrm{tr}\otimes 1)(\Delta(u^{-1}))]u = G^{-1}[m(\lambda.G\otimes G)(\Delta(u^{-1}))]u = [m(\lambda\otimes 1)(\Delta(Gu^{-1}))]G^{-1}u = \lambda(Gu^{-1})G^{-1}u = \lambda(v^{-1})v$. This completes the proof. □

The upshot of this theorem is that for a unimodular finite dimensional Hopf algebra there is a natural trace defined via the existent right integral. Remarkably, this trace is just designed to behave well with respect to handle sliding. Handle sliding is the basic transformation on framed links that leaves the corresponding 3-manifold obtained by amed surgery unchanged. See [13]. This means that a suitably normalized version of this trace on framed links gives an invariant of 3-manifolds. For a link K, we let $\mathrm{TR}(K)$ denote the functional on links, as described in the previous section, defined via tr (above).

A proper normalization of $\mathrm{TR}(K)$ gives an invariant of the 3-manifold obtained by framed surgery on K. More precisely, (assuming that $\lambda(v)$ and $\lambda(v^{-1})$ are non-zero) let

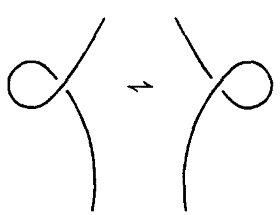
$$\mathrm{INV}(K) = \{[\lambda(v)\lambda(v^{-1})]^{-c(K)/2}[\lambda(v)/\lambda(v^{-1})]^{-\sigma(K)/2}\}\mathrm{TR}(K)$$

where $c(K)$ denotes the number of components of K, and $s(K)$ denotes the signature of the matrix of linking numbers of the components of K (with framing numbers on the diagonal), then $\mathrm{INV}(K)$ is an invariant of the 3-manifold obtained by doing framed surgery on K in the blackboard framing. This is our reconstruction of Hennings invariant [6] in an intrinsically unoriented context.

3.1. Handle sliding

The rest of this section is devoted to a discussion of the relationship between handle sliding and the trace function that we have defined on the Hopf algebra in terms of the right integral λ. This provides the reasoning behind the construction of the 3-manifold invariant. We begin with a quick review of the Kirby calculus [13].

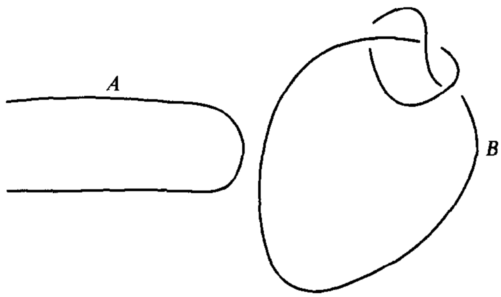
In Section 2 we discussed *ribbon equivalence* of link diagrams. Recall that this is the equivalence relation generated by regular isotopy (Reidemeister moves II and III) plus the equivalence of curls shown below (preserving writhe but changing the Whitney degree).



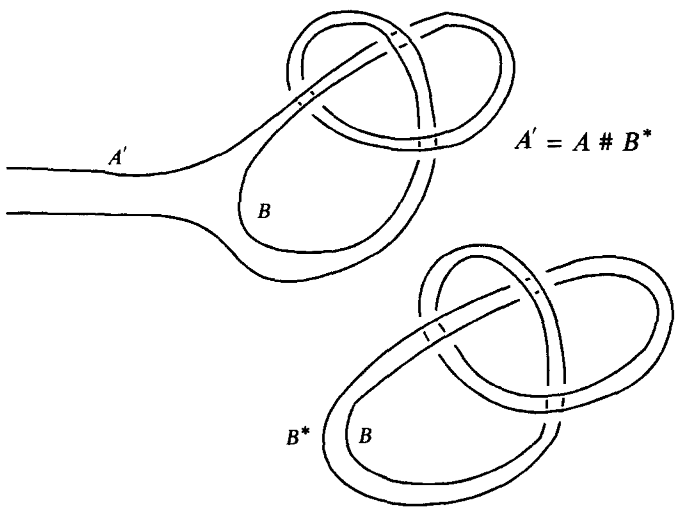
In these terms, Kirby calculus adds two more operations on link diagrams. These are

1. handle sliding,
2. blowing up and blowing down.

In handle sliding, we are given two link components A and B with A and B sharing parallel segments as shown below.

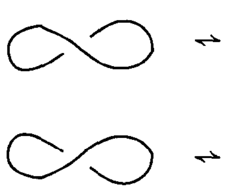


Then A is replaced by $A' = A \# B^*$ where B^* is a parallel copy of B and $\#$ denotes the connected sum of A and B^* along the parallel segment. See the illustration below.

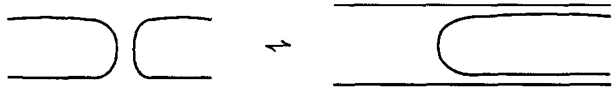


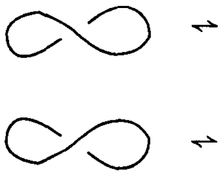
Note that B^* is one of the components of $\Delta(B)$ where we have defined $\Delta(K)$ to be the parallel cabling of K for any single component object K.

Blowing up and blowing down constitute the addition or removal from the link diagram of isolated singly twisted components as shown below.



Thus we can summarize the Kirby moves symbolically by the diagrams shown below:

1.  ,

2. 

Each diagram in a ribbon equivalence class gives a specific 3-manifold upon surgery on the corresponding link. (The writhe of the diagram determines the framing for the surgery). Kirby's theorem states that two such 3-manifolds are homeomorphic if and only if the corresponding links can be transformed into one another by a combination of ribbon equivalence, handle sliding and blowing up and down.

In order for an invariant of ribbon equivalence of links to be an invariant of 3-manifolds it is sufficient that it be unchanged under handle sliding and that it behave in a controlled way under blowing up and down. We now look at both of these issues for the trace function defined in this section.

In the case of handle sliding, we can assume that the algebra on the lines is already concentrated as illustrated below for the position just before the slide. In this illustration we have a component that we wish to slide with its algebra concentrated as the symbol y, and we have a closed loop with algebra concentrated as x. Thus the invariant will compute the closed loop part as $\mathrm{tr}(xG^{-1}) = \lambda(xG^{-1}G) = \lambda(x)$. We can indicate the computation for the whole situation by the formula $\mathrm{tr}(xG^{-1})y = \lambda(x)y$ where the y will come under the aegis of yet another trace function dependent upon the rest of the diagram.

After the slide, we see that, by functionality, x has been replaced by $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. As we illustrate in the diagrams below, this leads to the evaluation of the new closed loop as $\lambda(x_{(1)})$ ad the handle slide has multiplied y by the algebra $x_{(2)}$ so that the new trace evaluates to the formula

$$\sum \lambda(x_{(1)})x_{(2)}y.$$

Hence, for handle sliding invariance, it is sufficient to have the equality

$$\lambda(x)y = \sum \lambda(x_{(1)})x_{(2)}y.$$

Since λ is a right integral, we know that

$$\lambda(x)1 = \sum \lambda(x_{(1)})x_{(2)}.$$

This implies the desired equality, and hence the invariance of our trace under handle sliding! (see Fig. 2).

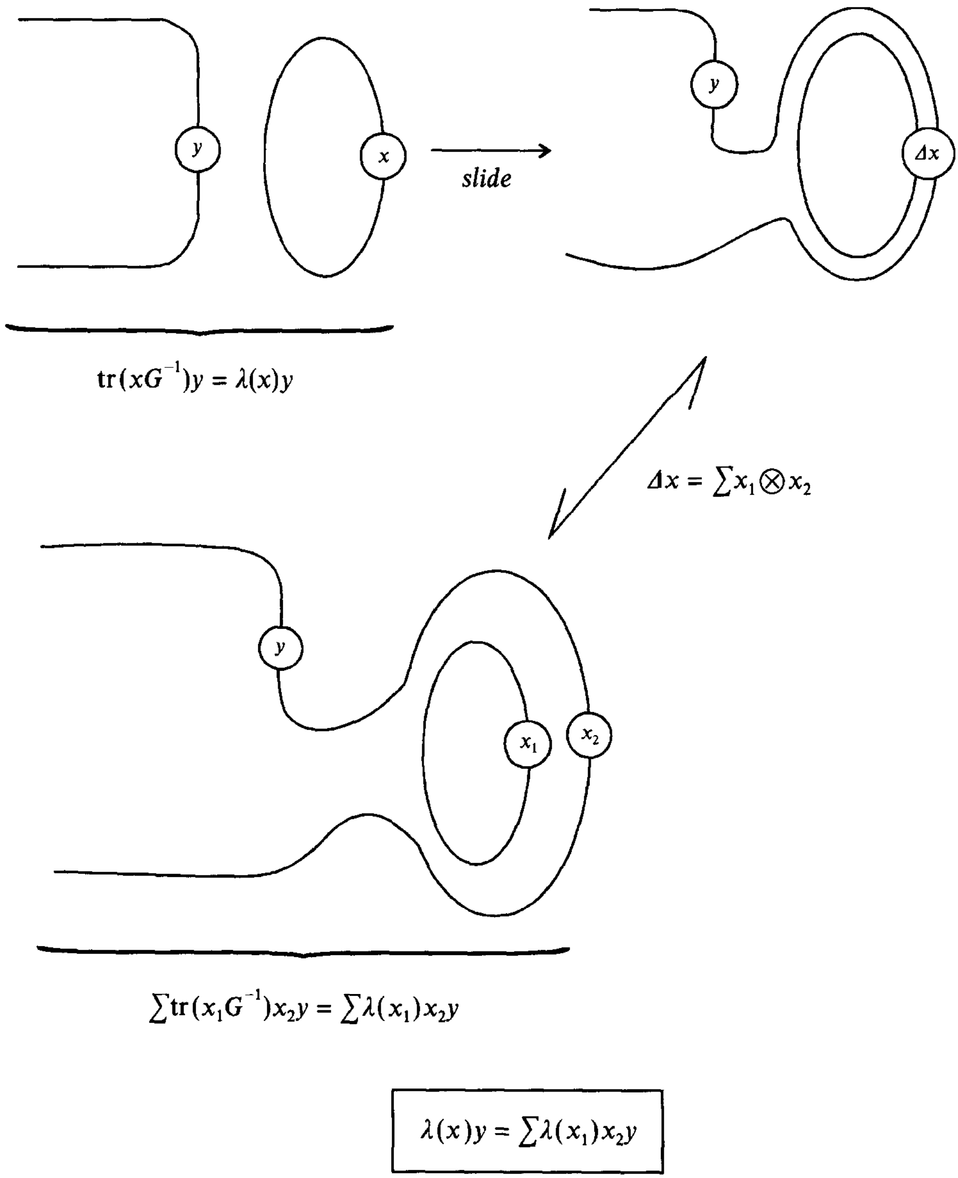


Fig. 2

For blowing up and blowing down, it is easy to see that the two 1-framed loops evaluate as $\lambda(v)$ and $\lambda(v^{-1})$ respectively. The formula

$$\mathrm{INV}(K) = \{[\lambda(v)\lambda(v^{-1})]^{-c(K)/2}[\lambda(v)/\lambda(v^{-1})]^{-\sigma(K)/2}\}\,\mathrm{TR}(K)$$

then gives the correctly normalized invariant of 3-manifolds.

4. $U_q(sl_2)'$

The purpose of this section is to set up part of the general calculations for $U_q(\mathrm{sl}_2)'$, and to sketch the calculation of the special case of the evaluation of the right integral on powers of the ribbon element v in the case $n = 8$. This will give us the result that the invariant INV(K) is distinct from the Witten–Reshetikhin–Turaev invariant at this root of unity. Complete details are found in [11].

Recall the algebraic structure of $U_q(\mathrm{sl}_2)'$.

Let t be a primitive nth root of unity, $q = t^2$, $m = \mathrm{order}(t^4)$. Assume $m \neq 1$ (that is $n \neq 1, 2, 4$). The algebra has generators and relations as given below.

$$ae = qea,\ af = q^{-1}fa,\ a^n = 1,\ e^m = 0 = f^m,$$

$$[e,f] = ef - fe = (a^2 - a^{-2})/(q - q^{-1}).$$

The Yang–Baxter element is given by the formula below [14, 20].

$$R = \sum_{v=0}^{m-1} \sum_{i,u \in Z/nZ} [(t^{-uv-i(u-v)-v}(q - q^{-1})^v)/(n(v)_q!)] f^v a^i \otimes e^v a^{-u}.$$

The coproduct is described by the formulas

$$\Delta a = a \otimes a,$$

$$\Delta x = x \otimes a^{-1} + a \otimes x, \quad x = e, f.$$

The counit is determined by the formulas

$$E(e) = E(f) = 0 \text{ and } E(a) = 1.$$

It follows from the definition of the antipode s that for $x = e$ or f, $0 = E(x)1 = m(s \otimes 1)\Delta(x) = s(x)a^{-1} + s(a)x = s(x)a^{-1} + a^{-1}x$. ($s(a) = a^{-1}$ since $\Delta(a) = a \otimes a$.).

This means $s(x) = -a^{-1}xa$, whence

$$s(e) = -q^{-1}e \quad \text{and} \quad s(f) = -qf.$$

The special grouplike element is $G = a^{-2}$. The special element u such that $s^2(x) = uxu^{-1}$ for all x, is given by the formula $u = \sum s(R^{(2)})R^{(1)}$. The next lemma gives a specific formula for u.

Lemma 1. $u = \sum_{v=0}^{m-1} \sum_{i,j \in Z/nZ} [(t^{j(i-v)-i^2-3v}(q^{-1} - q)^v)/(n(v)_q!)] a^j e^v f^v$.

Proof. See [11]. □

4.1. Change of basis

We now make the following change of basis. Replace e by $-(q-q^{-1})e$. Then

$$ae = qea,\ af = q^{-1}fa,\ a^n = 1,\ e^m = 0 = f^m,\ [f, e] = a^2 - a^{-2}.$$

Note that in the basis the formula for u becomes

$$u = \sum_{v=0}^{m-1} \sum_{i,j \in Z/nZ} [(t^{j(i-v)-i^2-3v})/(n(v)_q!)] a^j e^v f^v.$$

4.2. Right integral

A right integral λ for $A = U_q(\mathrm{sl}_2)'$ is described as follows. Consider the linear basis for A given by the set $\{a^i e^j f^k \mid 0 \le i \le n, 0 \le j, k < m\}$. Then $\lambda(w)$ for $w \in A$ is the coefficient of $a^{2(m-1)}e^{m-1}f^{m-1}$ in a writing of w in this basis. We can write $\lambda = \overline{a^{2(m-1)}e^{m-1}f^{m-1}}$ where the bar over the expression denotes the characteristic function of this element of the algebra A. That this formula gives the right integral can be verified by direct calculation [21].

4.3. Orthogonal idempotents

Let $\Lambda_i = (1/n)\sum_{j \in Z/nZ} t^{ij}a^j$. Then $\Lambda_i \Lambda_j = \Lambda_i \delta_{ij}$ where δ_{ij} is the Kronecker delta and $1 = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_{n-1}$. Thus $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}\}$ form a set of orthogonal idempotents for the group algebra $k[G]$ where $G = (a) = Z/nZ$.

From the relation

$$\sum_{j \in Z/nZ} t^{ik} = \begin{cases} n & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} \quad \text{for } k \in Z/nZ,$$

we have the following lemma.

Lemma 2. $a = \sum_{j \in Z/nZ} t^{-i}\Lambda_i$.

Proof. See [11]. $\square$

Hence

$$u = \sum_{v=0}^{m-1} \sum_{i \in Z/nZ} \left(\sum_{j \in Z/nZ} [(t^{-i^2-3v})/((v)_q!)]\,(t^{j(i-v)}a^j/n] \right) e^v f^v$$

$$= \sum_{v=0}^{m-1} \left(\sum_{i \in Z/nZ} (t^{-i^2-3v}/(v)_q!)\Lambda_{i-v} \right) e^v f^v$$

Lemma 3. $u = c(\sum_{v=0}^{m-1} [(t^{-3v-v^2})/(v)_q!]\,a^{2v}e^v f^v)$ where $c = \sum_{i \in Z/nZ} t^{-i^2}\Lambda_i$.

Proof. See [11]. □

4.4. The special case $n = 8$

Let $n = 8$. Then $m = 2$, $q = \sqrt{-1}$ and the algebraic relations for $U_q(\mathrm{sl}(2))'$ are

$$t^8 = 1,\ q = t^2,\ ae = qea,\ af = q^{-1}fa,\ a^8 = 1,\ e^2 = 0 = f^2,\ [f, e] = a^2 - a^{-2}.$$

Note that by the previous calculation,

$$u = c(1 + t^{-4}a^2ef) = c(1 - a^2ef)$$

with c given as in Lemma 3. Recall that $\lambda = \overline{a^{2(m-1)}e^{m-1}f^{m-1}}$ is a right integral for $U_q(\mathrm{sl}_2)'$. Thus, when $n = 8$, the right integral is $\lambda = \overline{a^2ef}$.

Lemma 4. *Let $X = -a^2ef$. Then $u = c(1 + X)$ and*

$$X^2 = (a^4 - 1)X = -2(\textstyle\sum_{i \text{ odd}} \Lambda_i)X.$$

Proof. See [11]. □

The special grouplike element in this case is $G = a^{-2}$. Thus the ribbon element is $v = G^{-1}u = a^2u$. Thus $v = a^2c(1 + X)$. To evaluate $\lambda(v^k)$, let $H = (a)$ be the cyclic group generated by a. Note that $v^k = c_0 + c_1X$, where $c_i \in k[H]$.

Lemma 5. *Writing* $c_1 = \sum_{i \in Z/8Z} \alpha_i \Lambda_i$, with $\alpha_i \in k$, $\lambda(v^k) = (-1/8)\sum_{i \in Z/8Z} \alpha_i$.

Proof. See [11]. □

Lemma 6. Let $n = 8$ *and let λ be the right integral and v be the ribbon element for $U_q(\mathrm{sl}(2))'$ as described above. Then $\lambda(v^k) = -k/2$.*

Proof. See [11]. □

Corollary. *The value of the* 3*-manifold invariant* $\mathrm{INV}(L(k1))$ *for $n = 8$ is given by the formula* $\mathrm{INV}(L(k, 1)) = \sqrt{-1}\, k$ *for $k \neq 0$.*

Proof. The surgery datum for $L(k, 1)$ is an unknotted loop with k curls. Hence the unnormalized invariant is given by the formula $\mathrm{TR}(v^kG^{-1}) = \lambda(Gv^kG^{-1}) = \lambda(v^kG^{-1}G) = \lambda(v^k) = -k/2$. The normalized invariant is given by the formula

$$\mathrm{INV}L(k, 1)) = [\lambda(v)\lambda(v^{-1})]^{-c(K)/2}[\lambda(v)/\lambda(v^{-1})]^{-\sigma(K)/2}\mathrm{TR}(K).$$

Here $c(K) = 1$ and $\sigma(K) = 1$ if $k > 0$, $\sigma(K) = -1$ if $k < 0$ since the link has one component, and the linking matrix is (k). We know that $\lambda(v) = -1/2$ and $\lambda(v^{-1}) = 1/2$. Therefore

$$\begin{aligned} \mathrm{INV}(L(k,1)) &= [(1/2)(-1/2)]^{-1/2}/(-1/2)]^{\pm 1}(-k/2) \\ &= (-2^2)^{1/2}(-1)(-k/2) = (-1)^{1/2}k. \end{aligned}$$

This completes the proof. □

Remark. This finishes our verification that the invariant INV is definitely different from the WRT invariant in the case $n = 8$, where WRT is trivial. During the preparation of our paper [11] it came to our attention that similar results have ben independently obtained by Tomotada Ohtsuki [18]. He finds that invariants defined for $U_q(\mathrm{sl}_2)'$ in a manner equivalent to ours necessarily vanish for 3-manifolds that are not rational homology spheres, and he performs calculations similar to ours for Lens spaces.

It should also be mentioned that a formalism similar to ours (without the use of right integrals) appears in the paper [22] by Reshetikhin, and that new approaches to these ideas can be found in the papers [12, 17].

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