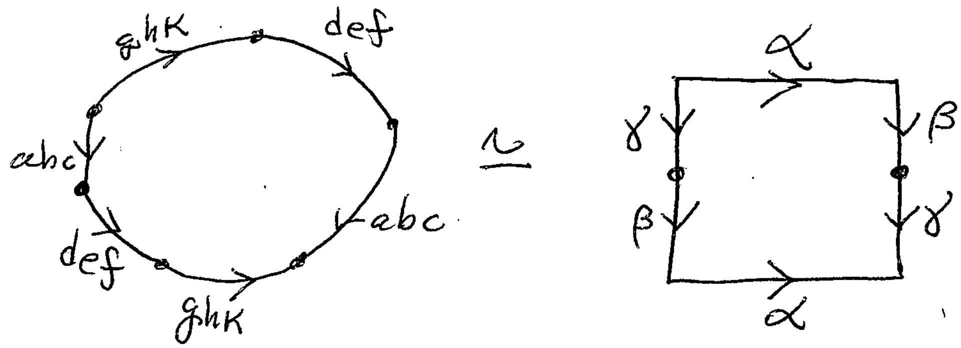
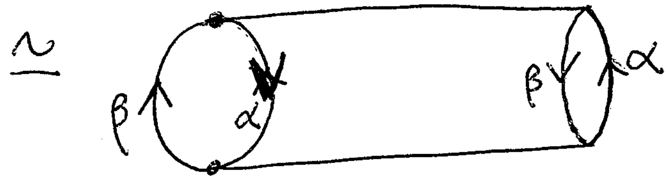
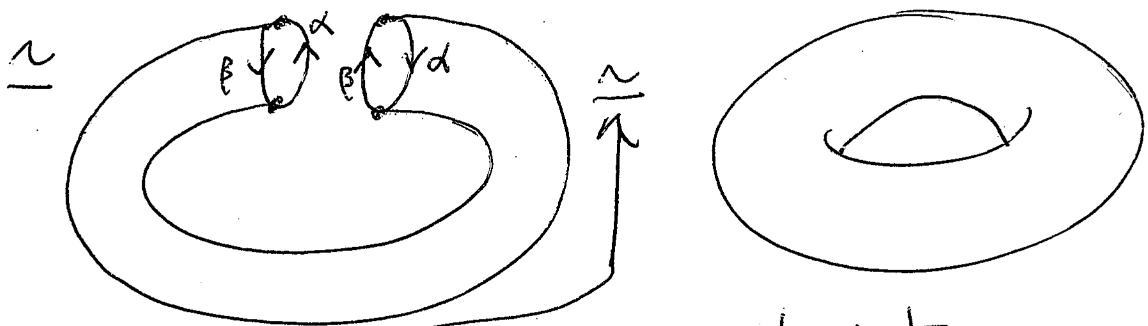
Graph Theory Problem Set – Solutions

1.



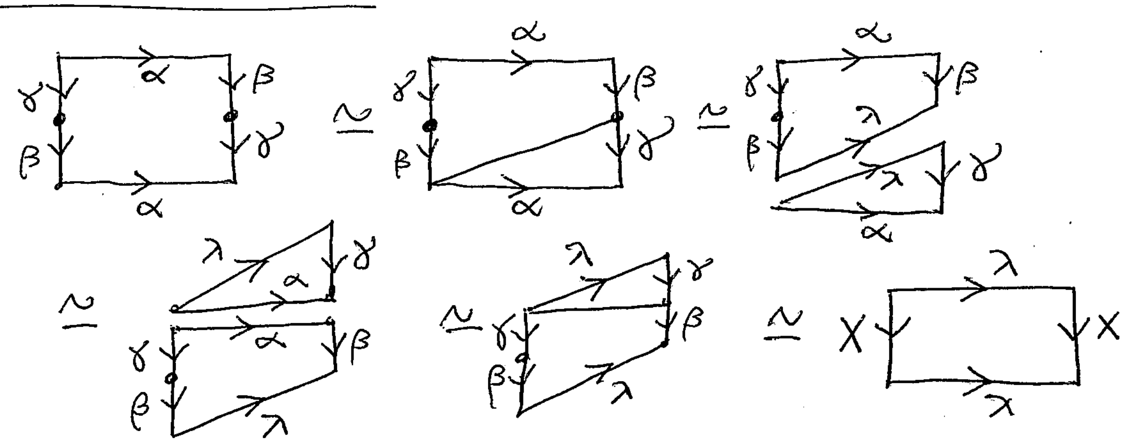


(join α to α)



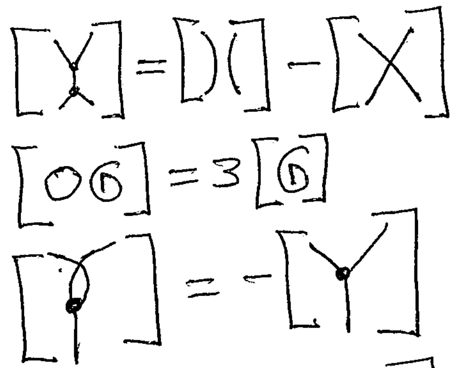
attach with a twist.

Second Solution

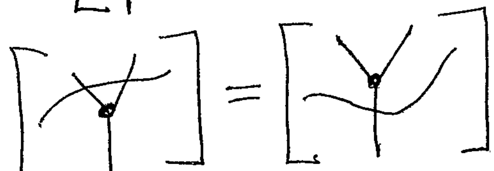


≅ Torus

2. Background

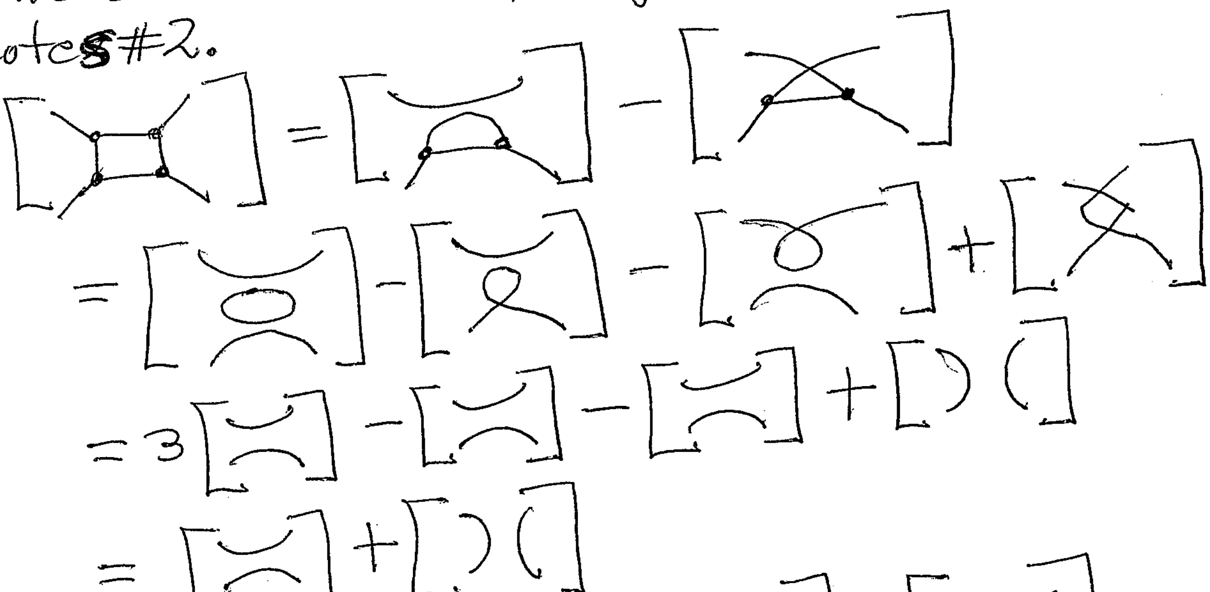


Also

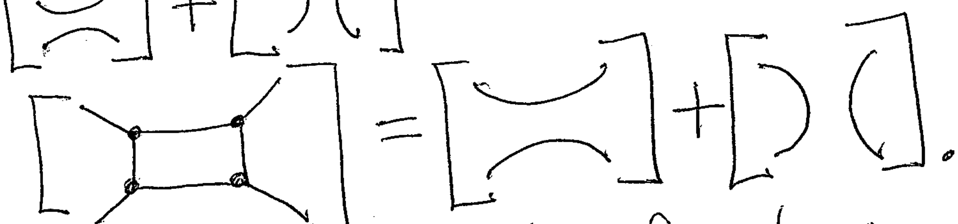


follows from the definition of

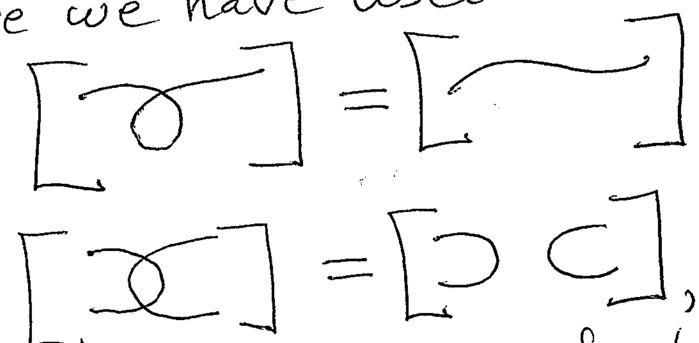
given in the class notes #2.



Thus

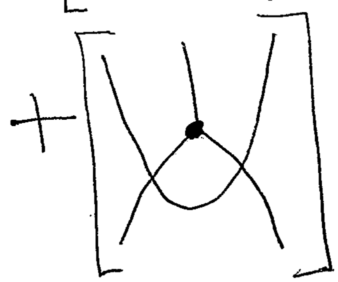


Here we have used the facts:

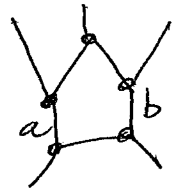


both consequences of the definition of the Penrose evaluation.

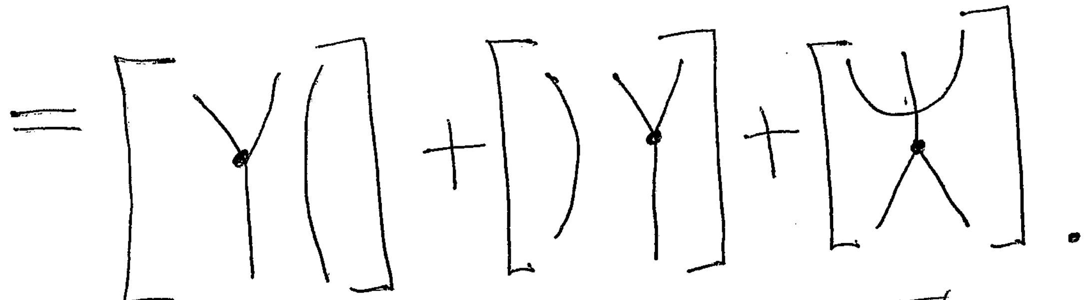
2. (continued)



(applying Penrose formula to a + b:



.)



This is the best we can do. There is no way to eliminate the crossing in the third term.

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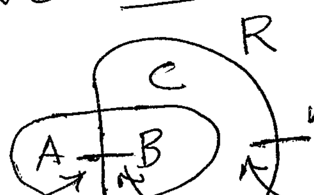
3. (a) In a planar endgame of Brussels Sprouts we have one marker in each region.

e.g.



single marker in outer region R

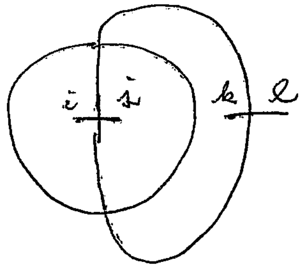
single marker in C

single marker in B

single marker in A

(If there were > 1 marker in any region, then one could make a move in that region, connecting two markers. Thus we would not have an endgame position.)

3. (continued)

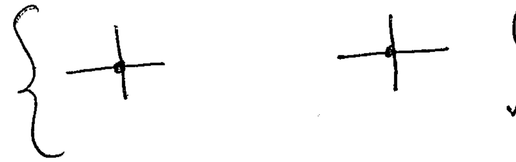


Note that in this endgame position there are <u>4 markers</u>.

i j k l

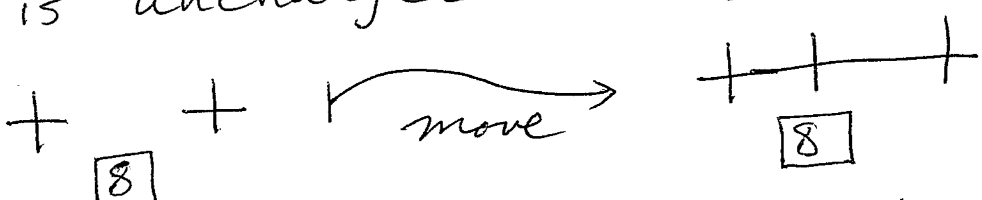
- At the beginning of a game with <u>n</u> starting nodes, there are <u>$4n$</u> markers.

e.g.



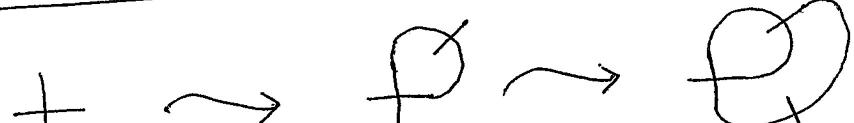
$n = 2$
markers $= 8$.
$\mathcal{M}$

- The number $\mathcal{M}$ of markers is unchanged when you move.



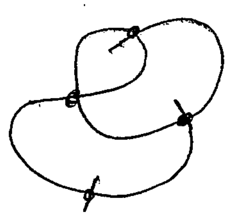
- Therefore, if a game starts with $\mathcal{M}$ markers, <u>any endgame position will have $\mathcal{M}$ regions</u>.

e.g.



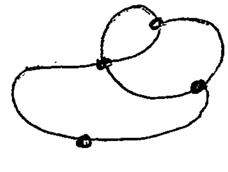
$\mathcal{M} = 4$

$\mathcal{M} = 4 = f$



<u>Endgame E</u>

Each endgame has a corresponding graph as shown.



<u>Graph of Endgame</u> $G(E)$

3. (continued)

Let G be the graph of an endgame E. Then

$$\begin{cases} f(G) = m = 4n \\ \qquad (n = \#\text{ of starting nodes}) \\ e(G) = 2m \quad \text{where } \underline{m = \#\text{ of moves in the game}} \\ v(G) = n + m \end{cases}$$

In a planar game, $v - e + f = 2$.

Thus $(n+m) - 2m + 4n = 2$

$$\Rightarrow 5n - m = 2$$

$$\Rightarrow \boxed{m = 5n - 2}$$

Thus we have proved that <u>for a planar game of Brussels Sprouts, every game (starting with n nodes) lasts for exactly $5n-2$ moves.</u>

(b) This solves (b) as well.

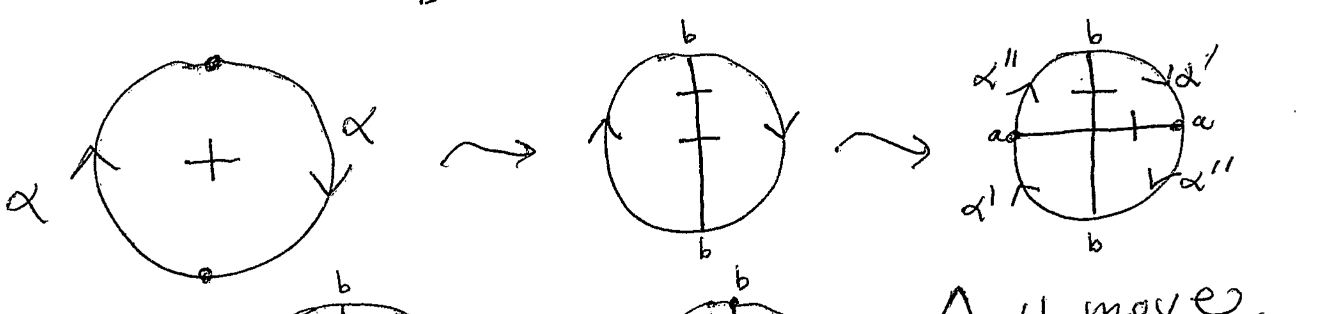
(c) Suppose you play on an orientable surface S_g of genus g + <u>suppose your endgame position has only disk like regions</u>. Then $v - e + f = 2 - 2g$ and (next page)

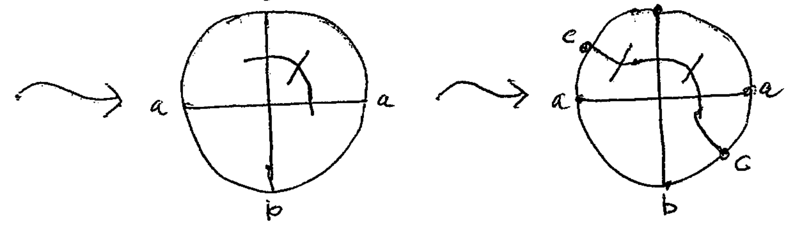
$$(n+m) - 2m + 4n = 2 - 2g$$
$$5n - m = 2 - 2g$$
$$\boxed{m = (5n-2) + 2g}$$

Thus playing on a surface can yield up to $2g$ more moves. If you simply play and get to an endgame (possibly with some non-disk regions), then you will find that $\boxed{5n-2 \leq m \leq (5n-2) + 2g}$.

If you are on the projective plane and the endgame has only disk-like regions, then $v - e + f = 1$ (see notes on Euler characteristic).

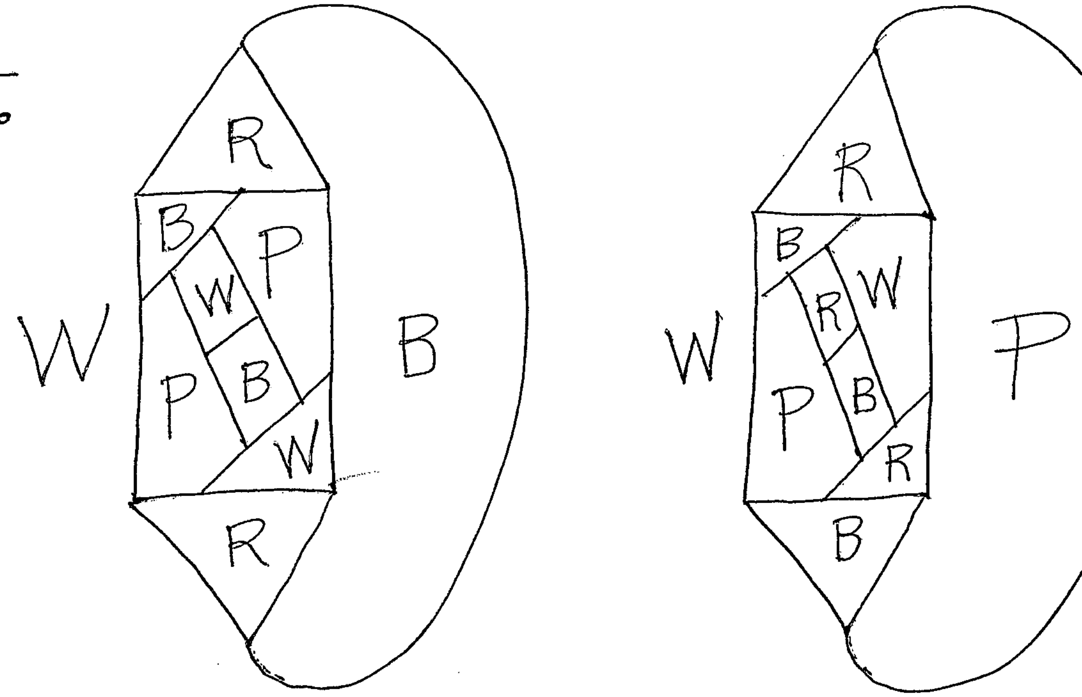
$$\therefore \quad 5n - m = 1$$
$$\boxed{m = 5n - 1}$$ is possible on $\mathbb{P}^2$.





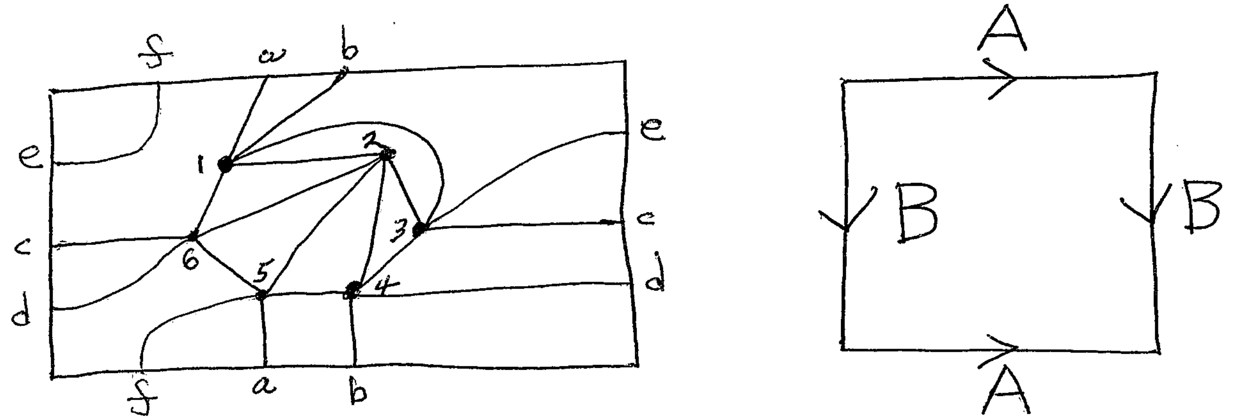
A 4 move game with $n=1$, on $\mathbb{P}^2$.

4.



These two colorings have opposite parity.

5. (a)



This diagram represents $K_6 \subset$ Torus.

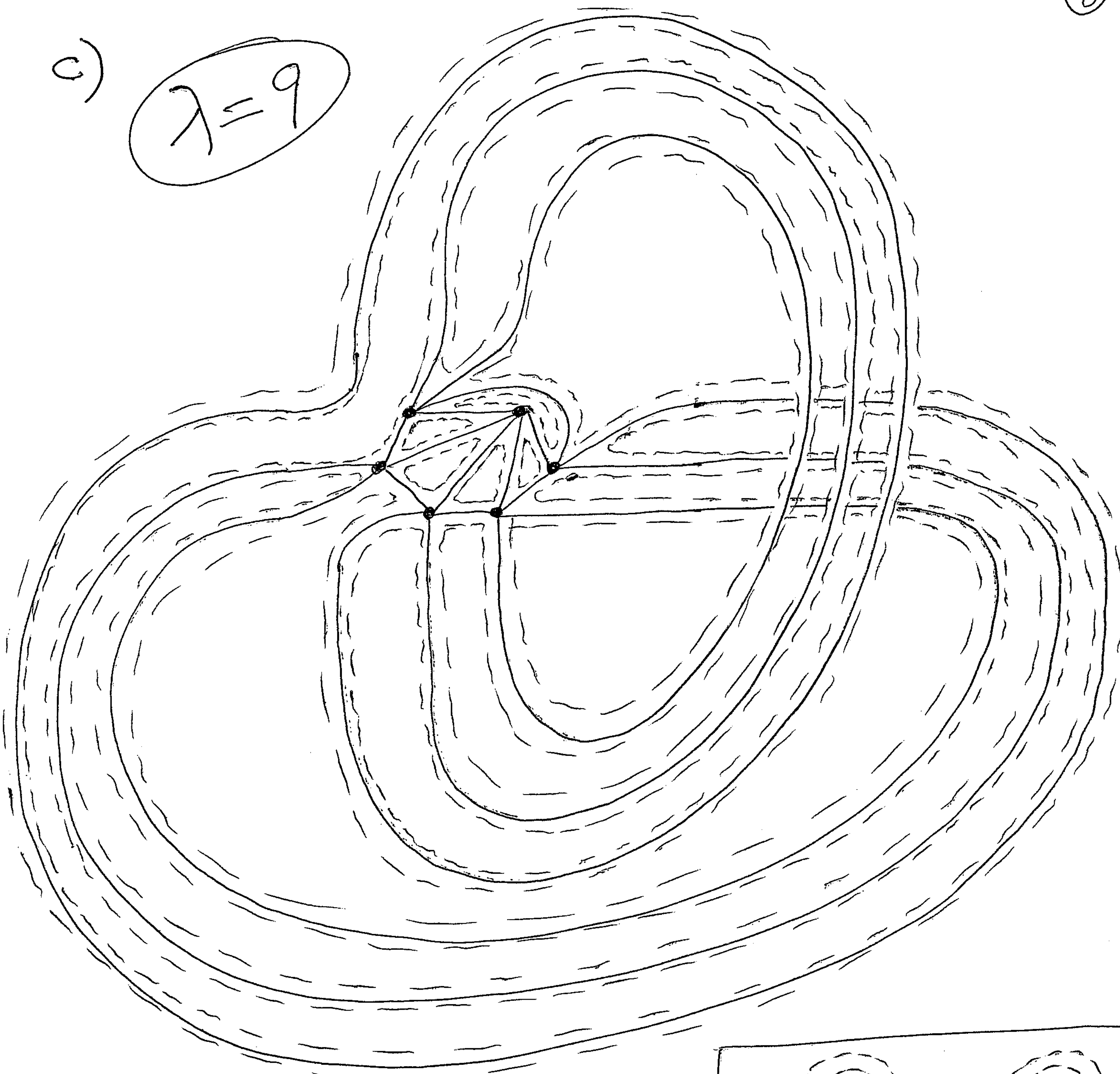
(b) $v - e + \lambda = 2 - 2g$

$g = 1$, $v = 6$, $e = 6 \cdot 5/2 = 15$

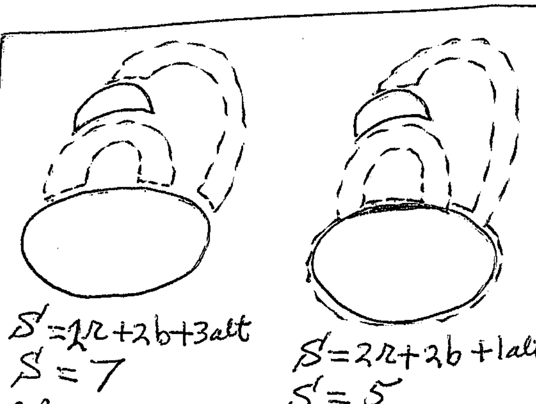
$6 - 15 + \lambda = 0$

$\Rightarrow \underline{\underline{\lambda = 9}}$

c) $\lambda = 9$



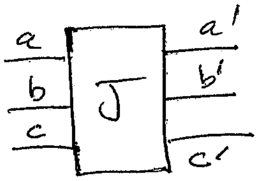
6. Solution omitted here. Will discuss in the problem session or come by in the fall.



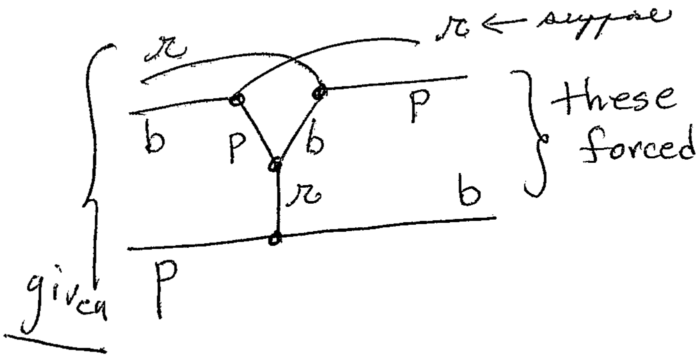
$S' = 2r + 2b + 3\,\text{alt}$
$S = 7$
$\pi \equiv 1$

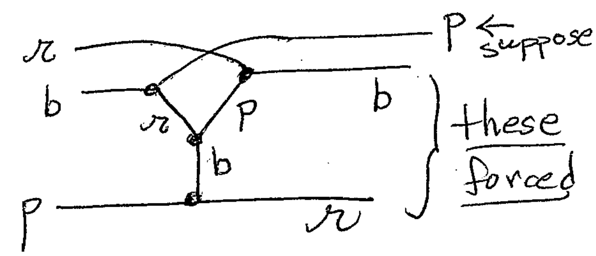
$S' = 2r + 2b + 1\,\text{alt}$
$S' = 5$
$\pi \equiv \underline{1}$

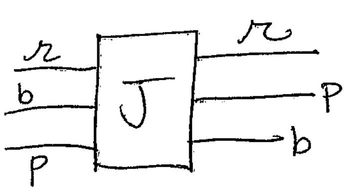
7. Consider possible colorings of J.



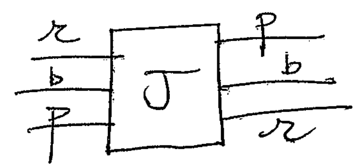
Case 1. a, b, c all distinct.

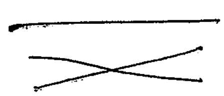




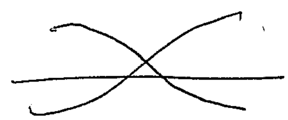


or



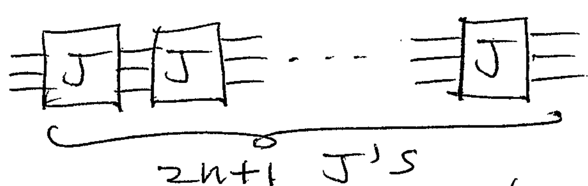


corres permutation of colors.

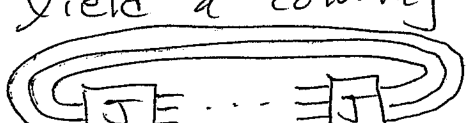


corres permutation of colors.

These are both odd permutations. Any odd product of odd permutations is odd. ∴

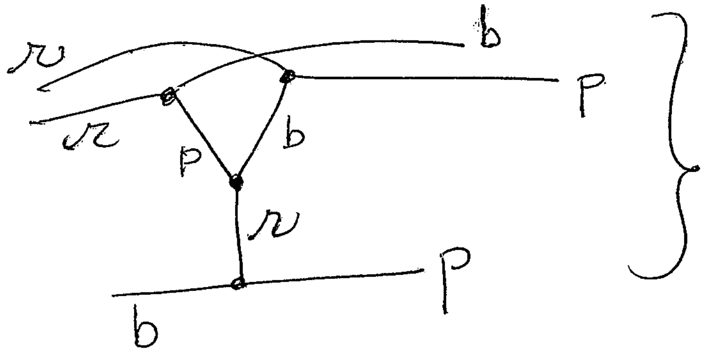


will produce an odd permutation of distinct input colors & ∴ cannot yield a coloring of



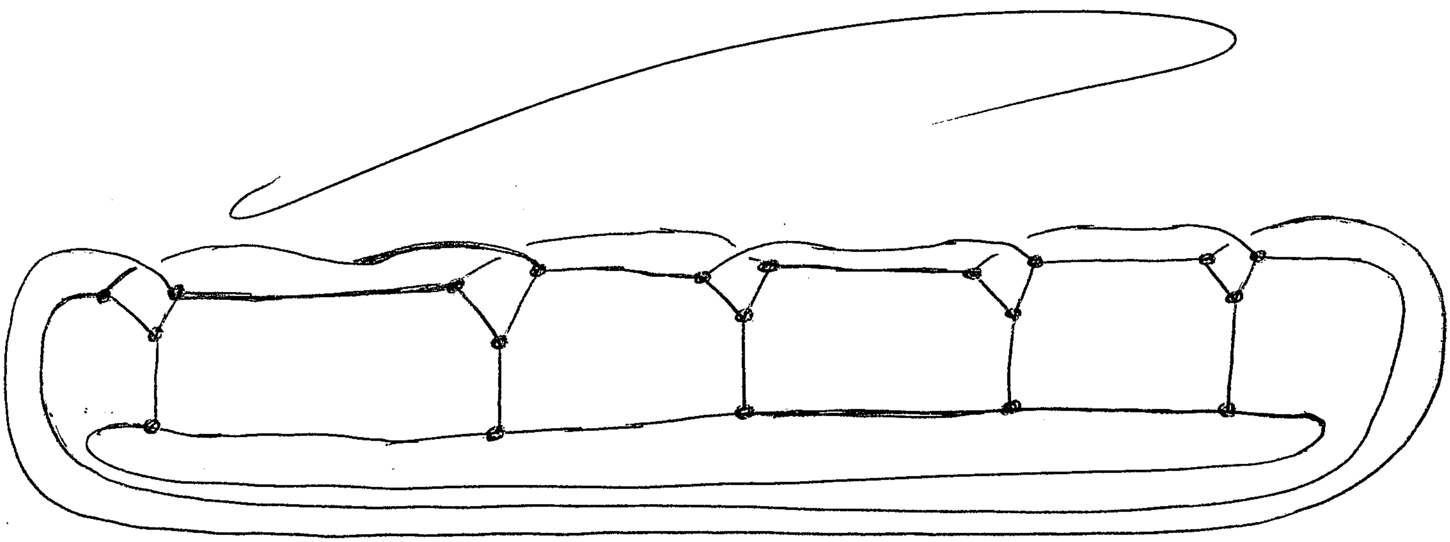
with distinct input colors.

$\boxed{\text{Case 2.}}$ Allow some inputs to be the same.



for example

You will find in all cases, the color missed on input appears on output. Thus an odd composition of J's can not have same colors as inputs and outputs. This completes the proof.



$J^{(5)}$