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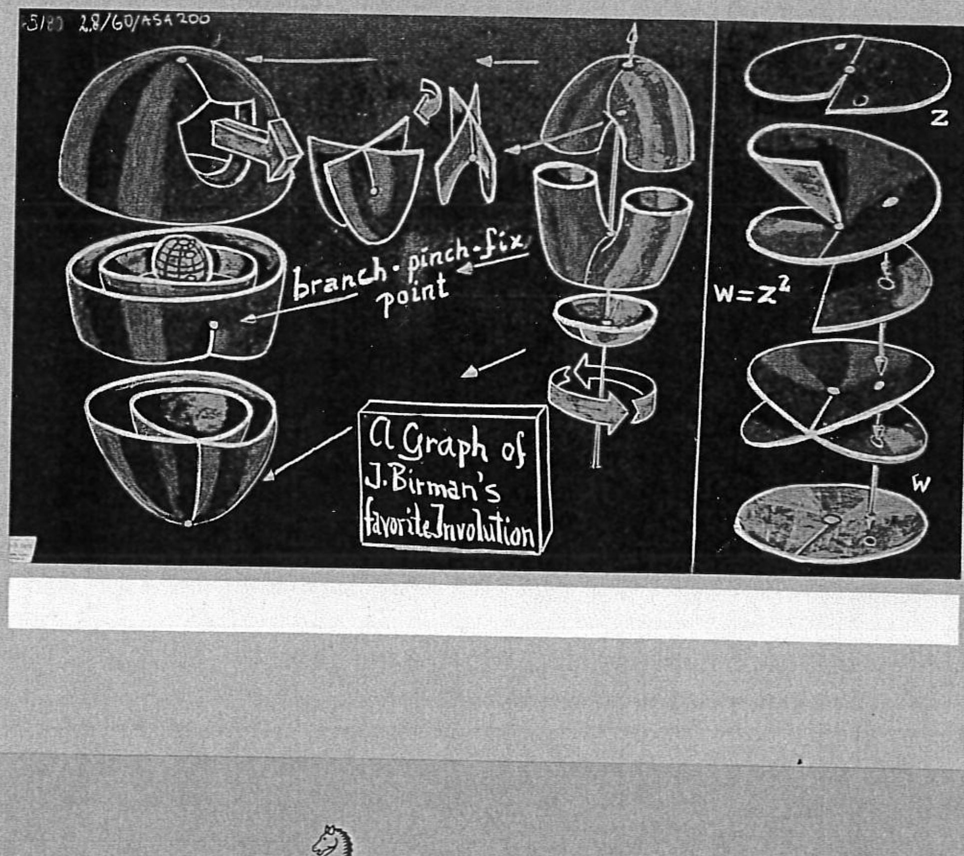
A Topological Picturebook



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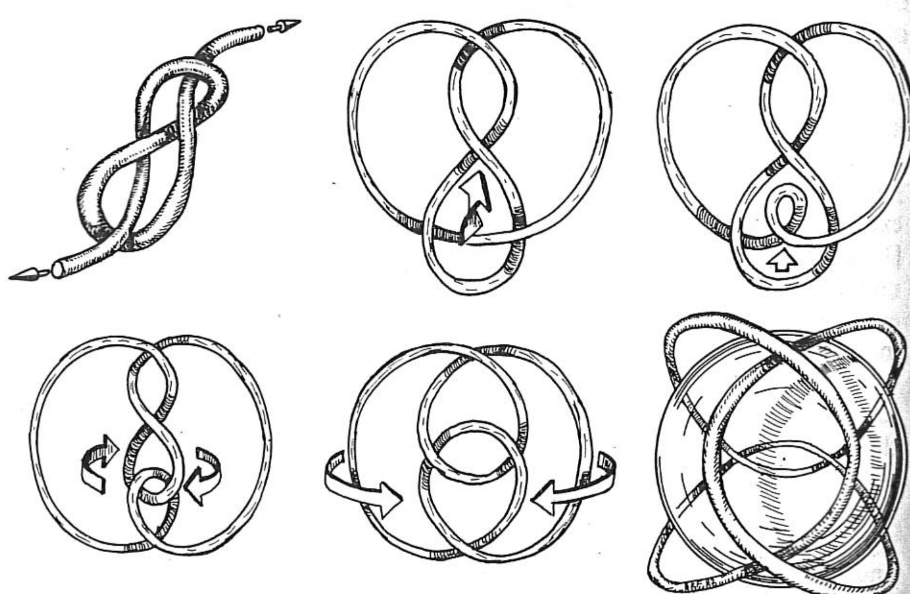
THE FIGURE EIGHT KNOT

My final picture story is about visualizing how a knot complement fibers over the circle. This exercise of the imagination consists of filling the void of space, closed up by a single point at infinity to form the 3-sphere, with a continuous succession of surfaces spanning the knot. That is, through each point not on the curve, there will pass a unique copy of a surface whose boundary is the knot.

The reason for wanting to do this comes from the project of imposing a non-Euclidean (hyperbolic) geometry on most 3-dimensional manifolds, as explained in the survey articles by Bill Thurston [1982] and John Milnor [1982]. The complement of the figure-8 knot plays a major role in Thurston's unpublished textbook. For my story about this knot, I used two manuscript editions [1977,1982] of the text. It is important to realize that the examples in this chapter belong to an area of topology that already has a highly developed and effective graphical shorthand. This style consists of "schematic" diagrams which are quite terse and frequently incomprehensible to the novice. Recognizable pictures of familiar shapes, artistically arranged in space as dictated by the abstract diagrams, are sometimes welcome. That is the purpose of my efforts here.

In Thurston's grand scheme, it is the topology of a manifold that limits and frequently determines its possible geometries. For simplicity's sake, the text that goes with my pictures is topological. Here is a brief note about the group theory and geometry which I have left out. The way to get a geometrical manifold is to take a polyhedral chunk from a geometrical space and identify its faces pairwise with each other. In order that the geometry matches up properly across the faces and around the edges, it is best to let a group of isometries on the ambient space dictate the identification. In the story about the Penrose tribar the polytope was a cube in Euclidean space, and the isometries were rotations and translations. In 1912, Gieseking identified the faces of an ideal tetrahedron in hyperbolic space as dictated by a group of orientation reversing isometries, see Magnus [1974, p.153ff]. By finding a suitable representation of the fundamental group of the figure-8 knot complement, Riley [1975] found a hyperbolic manifold homeomorphic to the knot complement. Troels Jϕrgensen [1977] used the way the knot complement fibers over the cirlce to demonstrate its hyperbolic structure. Thurston showed how it is also the (orientable) double cover of Gieseking's manifold.

The way from the knot complement to the gluing diagram on a hexahedron and the way back from a diagram to the complement is not part of the fibering story. I have included it here because it is such a rewarding subject for descriptive topology. This is also true of the digression on the Hopf fibration at the end. Hopf's decomposition of the 3-sphere into a bundle of 1-dimensional fibers (circles that link) that arrange themselves exactly like the points on a 2-sphere lead to Seifert's [1933] theory of (singular) fibrations of 3-manifolds over surfaces. The theory provides a complete classification of the 3-manifolds with the geometry of the 3-sphere.



PROJECTIONS OF THE KNOT

Figure 1

As Tait observed [1911], the figure-8 knot, by sailors "used only to prevent ropes from unreeving; it forms a large knob" has been part of "topology" ever since Listing [1847] coined this name for the mathematical theory of position. Tait called it the "four-knot" because it is the only simple closed

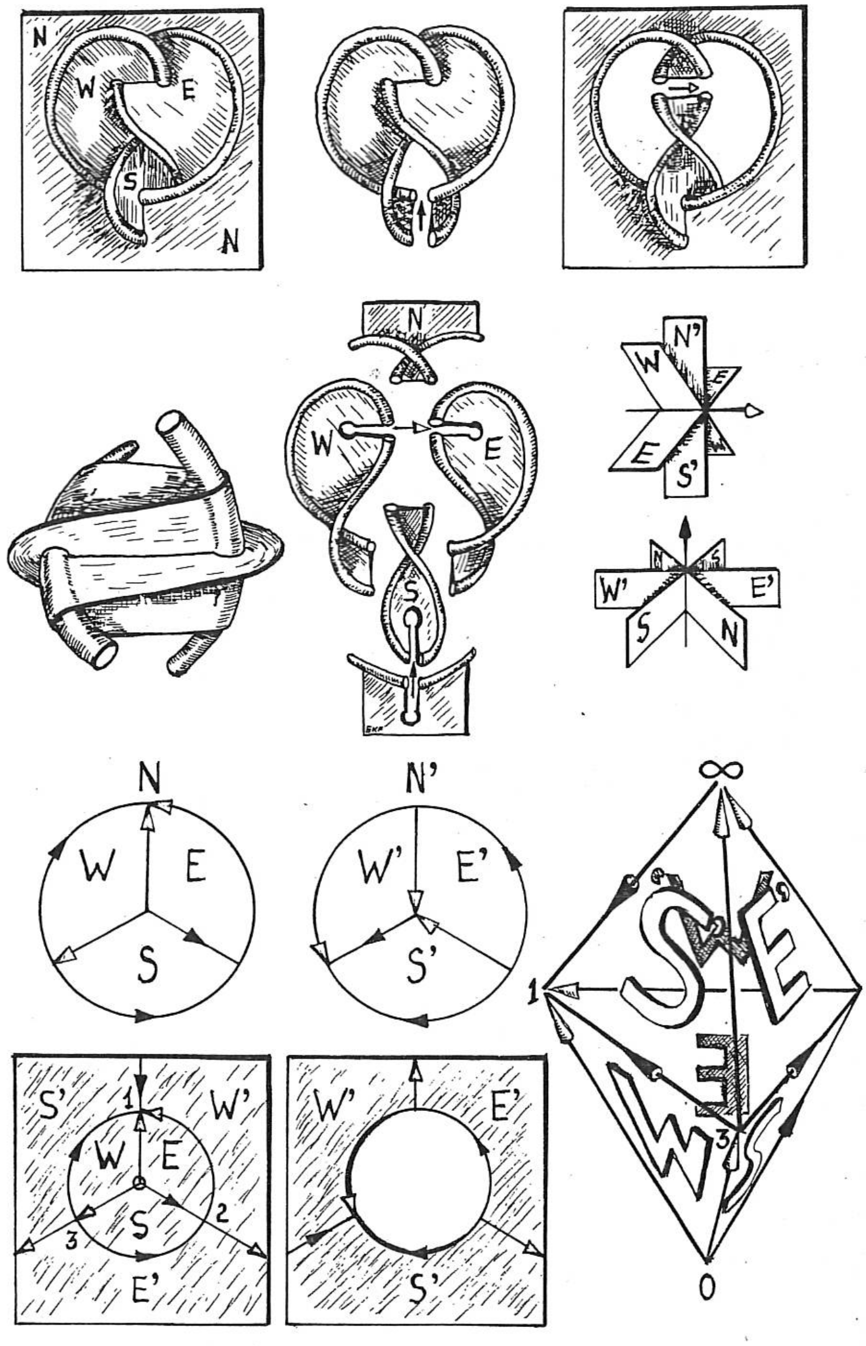


Figure 2 HEXAHEDRAL COMPLEMENT