From Electricity to Trees

It is worthwhile recalling how the calculation of conductance occurs in electrical network theory, based on Ohm's and Kirchoff's laws. Ohm's law states that the electrical potential (voltage drop) between two nodes in a network is equal to the products of the current between these two nodes and the resistance between the two points. This law is expressed as $E = IR$, where E denotes the potential, I is the current, and R is the resistance. Kirchoff's law states that the total sum of currents into and out of any node is zero.

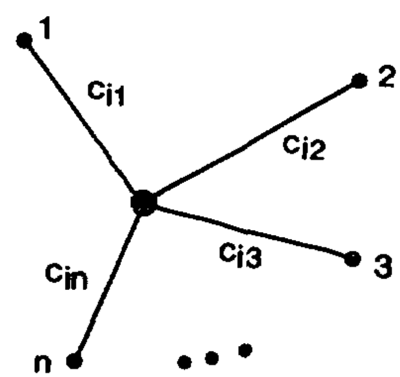

FIGURE A.1

These laws give the local rules for combining conductance. Conductance C is defined to be the reciprocal of resistance, viz., $C = 1/R$. Thus $E = I/C$ and $C = I/E$. Note that one also uses the principle that the voltage drop or potential along a given path in the network is equal to the sum of the voltage drops from node to node along the path.

With these ideas in mind, consider a node i with conductances $c_{i1}, c_{i2}, \ldots, c_{in}$ on the edges incident to this node (Fig. A.1). Here c_{ij} labels an edge joining nodes i and j. Then let E_i denote the voltage at node i with respect to some fixed reference node (the *ground*) in the network (the voltages are only determined up to a fixed constant). Then the current in the edge labeled c_{ik} is $(E_i - E_k)c_{ik}$. ($EC = I$, where C is the conductance.) Hence we have by Kirchoff's law,

$$\sum_{k=1}^{n} (E_i - E_k)c_{ik} = 0.$$

This equation will be true without exception at all nodes but *two*. These special nodes v, v' are the ones where we have set up a current source on a special edge between them as shown in Fig. A.2. We can assume that the

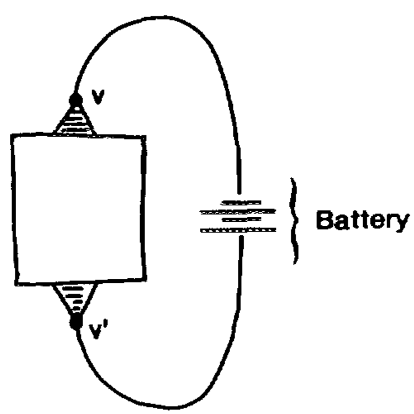

FIGURE A.2

battery edge delivers a fixed current I_0 and voltage E_0. Then the set of equations for voltages and currents takes the form

$$\begin{pmatrix} -I_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = M \begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_m \end{pmatrix},$$

where the nodes in the graph are labeled $0, 1, \ldots, m, 0'$, with 0 labeling v and $0'$ labeling v'. We take $0'$ to be the ground, whence $E_{0'} = 0$. We take M to be the matrix for the system of equations for nodes $0, 1, 2, \ldots, m$. Then, if M is invertible, we have

$$\begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_m \end{pmatrix} = M^{-1} \begin{pmatrix} -I_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

M is the famous Kirchoff matrix. It is a remarkable fact that the determinant of M enumerates the spanning trees in G (no battery edge in the sense that, up to sign, $\det(M)$ is the sum over these trees of the products of the conductances along the edges of the tree. This is the matrix tree theorem (see [29, 25]).

EXAMPLE. Consider the equations for the network of Fig. A.3:

$$\begin{array}{lll}
\text{(node 0)}, & -I_0 = & (E_1 - E_0)a + (E_2 - E_0)b \\
\text{(node 1)}, & 0 = & (E_0 - E_1)a + (E_2 - E_1)c + (E_{0'} - E_1)d \\
\text{(node 2)}, & 0 = & (E_0 - E_2)b + (E_1 - E_2)c + (E_{0'} - E_2)e.
\end{array}$$

Thus

$$\begin{pmatrix} -I_0 \\ 0 \\ 0 \end{pmatrix} = M \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix},$$

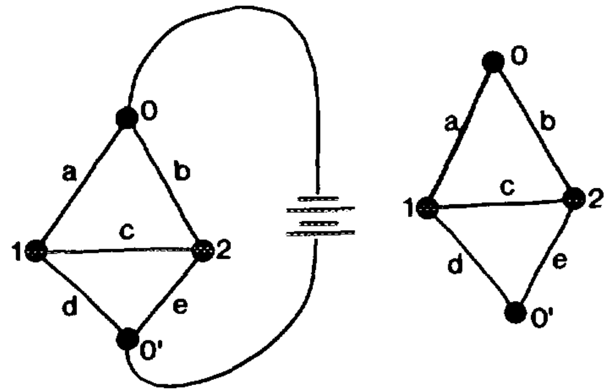

FIGURE A.3

where

$$M = \begin{pmatrix} -a-b & a & b \\ a & -a-c-d & c \\ b & c & -b-c-e \end{pmatrix}.$$

The matrix M is obtained from a matrix K defined for the graph G without the special edge

$$0 \cdot \text{———}\!\mid\!\mid\!\mid\!\text{———} \cdot 0'.$$

The matrix K is a node—node matrix with nondiagonal entry k_{ij} equal to the sum of the labels (conductances) on the edges connecting nodes i and j, if such edges exist, and 0 if there is no edge. The diagonal entries k_{ii} are the negative sum of the labels of edges incident to node i. Hence

$$K = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 0' \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 0' \end{array} \\ \begin{pmatrix} -a-b & a & b & 0 \\ a & -a-c-d & c & d \\ b & c & -b-c-e & e \\ 0 & d & e & -d-e \end{pmatrix} \end{array}.$$

Thus we see that M is obtained from K by removing the $0'$ row and the $0'$ column.

Now go back to our system of equations and solve for E_0:

$$E_0 = \frac{\operatorname{Det}\begin{bmatrix} -I_0 & a & b \\ 0 & -a-c-d & c \\ 0 & c & -b-c-e \end{bmatrix}}{\operatorname{Det} M}$$

$$= -I_0 \frac{\operatorname{Det}\begin{bmatrix} -a-c-d & c \\ c & -b-c-e \end{bmatrix}}{\operatorname{Det} M}.$$

Since $I_0 = CE$, where C is the conductance from v to v' (0 to $0'$), we have

$$C = \frac{-\operatorname{Det} M}{\operatorname{Det}\begin{bmatrix} -a-c-d & c \\ c & -b-c-e \end{bmatrix}}.$$

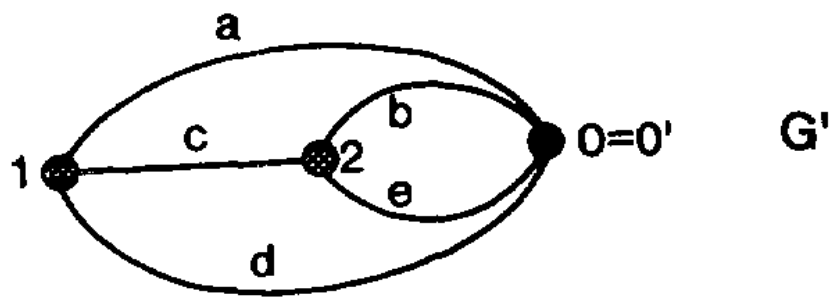

FIGURE A.4

Let G' be the graph obtained by identifying v with v' (Fig. A.4). The Kirchoff matrix K' is given by

$$K' = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} -a-b-d-e & a+d & b+e \\ a+d & -a-c-d & c \\ b+e & c & -b-c-e \end{pmatrix} \end{array}$$

and we take

$$M' = \begin{pmatrix} -a-c-d & c \\ c & -b-c-e \end{pmatrix}.$$

Thus $\mathrm{Det}(M')$ enumerates the trees in G'.

Therefore, we see that the conductance is given by the ratio of tree summations

$$C(G, v, v') = \left|\mathrm{Det}(M)/\mathrm{Det}(M')\right| = w(G)/w(G').$$

This is the full electrical background to our combinatorics and topology.

Remark. That the determinants of minors of the Kirchoff matrix enumerate spanning trees in the graph has been the subject of much study. In the electrical context, it is worth mentioning the Wang algebra [8] and the work of Bott and Duffin [9] that clarified some of these issues in terms of Grassman algebra.

These notes are an excerpt from

Knots, Tangles, and Electrical Networks

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