Notes on Diagrammatic Matrix Algebra and Graphs
by L.K.

Lets first recall how matrix multiplication works. Matrices are arrays of elements of an arithmetic or an algebra. Here we will begin by assuming that the matrix elements occur in ordinary numbers (integers, rationals, reals or complex numbers) or their algebra. Two 2 x 2 arrays are multiplied by the following formula.

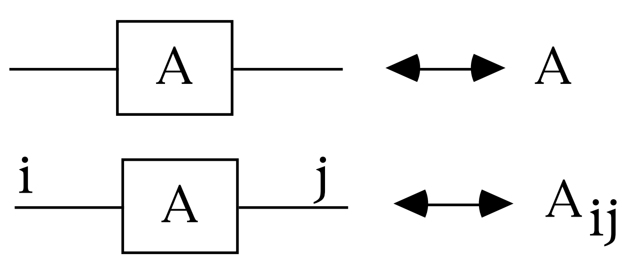
$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} a_{00}b_{00}+a_{01}b_{10} & a_{00}b_{01}+a_{01}b_{11} \\ a_{10}b_{00}+a_{11}b_{10} & a_{10}b_{01}+a_{11}b_{11} \end{pmatrix}$$

We denote a matrix $A = (A_{ij})$ by a global letter (A in this case), and by an indication of the form of the elements of the array, A_{ij}. The subscripts range over the set $\{0,1\}$ in the case of a 2 x 2 matrix, as shown above. The rule for multiplying two matrices is

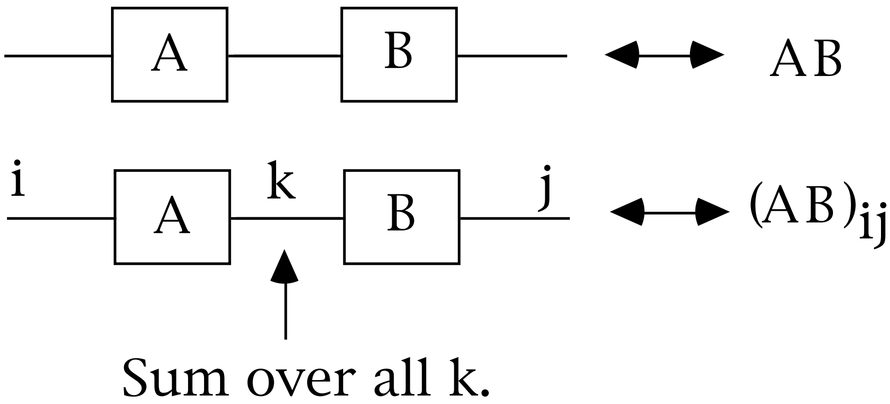
$$(AB)_{ij} = \Sigma_k A_{ik}B_{kj}.$$

where the summation is over the index set for the matrix size that we are using. Compare this formula with the arrangement of indices and sums in the explicit matrix product given above.

We now give a diagrammatic interpretation for matrix algebra. Each individual matrix is represented by a box with (input and output) lines that correspond to the matrix indices.



Matrix multiplication is represented by attaching the output line from one box to the input line of the other box.



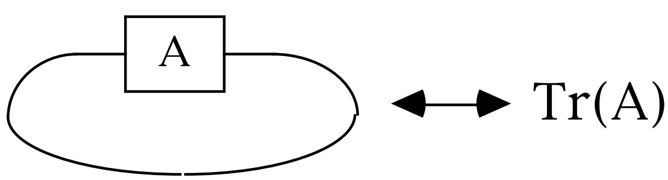
$$(AB)_{ij} = \Sigma_k A_{ik} B_{kj}$$

Lines tying one box to another correspond to internal indices in the matrix product, and so one sums over all possible choices of index for such internal lines.

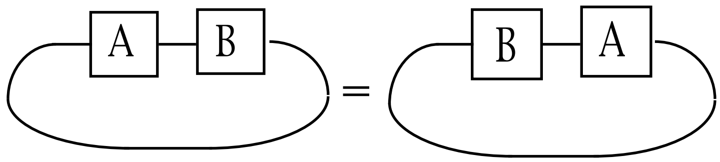
With these diagrammatic conventions in place, one can often make very efficient insight into properties of matrix composition. For example, the trace of a matrix A is given by the formula

$$Tr(A) = \Sigma_i A_{ii}.$$

Here is the diagram.



With this diagrammatic for the trace of A, we easily prove that **Tr(AB) = Tr(BA)** by putting two boxes in a circular connection pattern.



$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

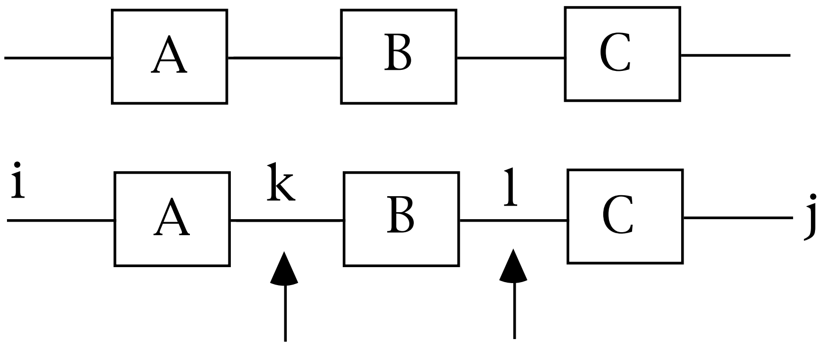
You should compare this with the algebraic proof:

$$\mathrm{Tr}(AB) = \Sigma_i (AB)_{ii} = \Sigma_i \Sigma_j A_{ij}B_{ji}$$

$$= \Sigma_j \Sigma_i B_{ji}A_{ij} = \Sigma_j (BA)_{jj} = \mathrm{Tr}(BA).$$

Associativity

Another application of these diagrams is to the proof that matrix multiplication is associative. For we see that the diagram for the product of three matrices A, B and C is given by



Sum over all k. Sum over all l.

$$(ABC)_{ij} = \sum_{k,l} A_{ik} B_{kl} C_{lj}$$

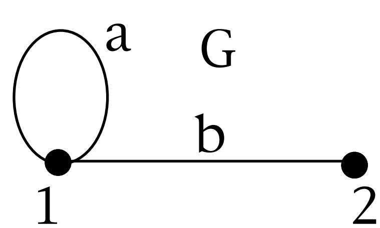
Matrix Multiplication is associative because the products of matrix entries are associative.

Adjacency Matrix of a Graph

Given a graph G, we define the adjacency matrix A(G) to be an n xn matrix where n = #V(G) = the number of nodes of the graph G. Letting A = A(G), then A is defined by the equation

A_{ij} = the number of edges in G with endpoints i and j.

For example, let G be the graph shown below with V(G) = {1,2} and E(G) = {a,b} where a is a loop at 1 and b has endpoints 1 and 2.



Then we have

$$A = A(G) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Walk Theorem.
Let $A = A(G)$ be the adjacency matrix of a graph G.
Let $B = A^m$ for some $m = 1,2,3,\ldots$. Then B_{ij} is equal to the number of walks of length m in G from node i to node j.

Proof. Since

$$(A^m)_{ij} = \Sigma_{k1,k2,\ldots,k(m-1)} (A_{ik1} A_{k1k2} \ldots A_{k(m-1)j})$$

we see that each term in the sum for $(A^m)_{ij}$ counts the number of walks that could happen in the pattern

i ----> k1 ----> k2 ----> .... ----> k(m-1) -----> j.

These add up and count the total number of walks. The proves the Theorem. //

Exercise. (a) Let A be the adjacency matrix for the example G just before the statement of the Walk Theorem. (G has two nodes 1 and 2, a loop at 1 and an edge from 1 to 2.) Compute the first few powers A^k, and verify that the entries do count walks of length k on the graph G.
(b) Using the matrix A of part (a) find recursive formulas for the entries of A^m as a function of m. Hint: Examine the first few powers of A to find patterns. Then prove your patterns by induction on m.
(c) Use the method of generating functions, via the finding the inverse matrix $(I - At)^{-1}$, to determine the values of the walks on G.
(d) Find the characteristic polynomial $C_G(t) = Det(A - tI)$ and use it to find the eigenvalues of G, and go through the eigenvalue calculations for th walks on G as in Class Notes #1.

(e) Now do the same work as in steps (c) and (d) above for the adjacency matrix for the graph $G(n,m,p)$ with vertices 1 and 2 and n loops at 1, m edges from 1 to 2 and p loops at 2. Write down the adjacency matrix for $G(n,m,p)$ and find the generating series, the characteristic polynomial, the eigenvalues and find recursion relations for the walks from 1 to 1 of all lengths.
(f) Read the web site
<http://www.math.harvard.edu/archive/21b_fall_03/goodwill/>
(It is linked on the Graph Theory course webpage.)
Analyse the graph given there, using what you know about the adjacency matrix. Can you figure out the rest of what is going on mathematically on that page?

(See Class Notes #1)

Graph Isomorphism and Matrices

Let G be a finite graph with n nodes, and $A = A(G)$ its adjacency matrix. This means that we have labeled the nodes of G from the set $\{1,2,\ldots,n\}$ and we then define the matrix A via
A_{ij} = *the number of edges in G from node i to node j.*

Note that we can obtain many different adjacency matrices depending upon the labeling of the nodes. Lets figure out how A will change if we reorder the nodes of the graph. Suppose we take a permutation $\sigma:\{1,2,\ldots,n\} \dashrightarrow \{1,2,\ldots,n\}$.
And we use the nodes in the order

$$\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(n).$$

Then we would have a new adjacency matrix B with

$$B_{ij} = A_{\sigma(i)\sigma(j)}.$$

In fact the isomorphism problem for graphs can be stated in terms of matrices in just this way.

Given adjacency matrices A and B of the same size ($n \times n$), does there exist a permutation σ such that $B_{ij} = A_{\sigma(i)\sigma(j)}$?

As you can see, this problem is a search problem among $n!$
different
possibilities and so becomes computationally hard very quickly as n increases. For this reason, we look for methods involving matrix algebra to get at least partial information about the problem.

Given a permutation σ as above, we can define an n x n permutation matrix P via the formula

$$P_{ij} = \delta_{\sigma(i),j}$$

where $\delta_{a,b} = 1$ when $a = b$ and 0 otherwise.

For example, the matrix P below corresponds to the permutation $\sigma = (1234)$ in cycle notation. That is
$\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 4,\ \sigma(4) = 1.$

P =

0	1	0	0
0	0	1	0
0	0	0	1
1	0	0	0

For a permutation matrix, it is easy to see that the inverse matrix P^{-1} is given by the formula

$$P^{-1}{}_{ij} = \delta_{i,\sigma(j)}.$$

From this you can easily prove the following

Theorem. *Let A be given as a specific* $n \times n$ *matrix* A_{ij}.
Let B be defined as above via

$$B_{ij} = A_{\sigma(i)\sigma(j)}$$

for a permutation $\sigma:\{1,2,\ldots,n\} \text{-----}> \{1,2,\ldots,n\}$.
Let P be the permutation matrix corresponding to σ.
Then $B = P\,A\,P^{-1}$.

Proof. *Exercise* (we will do it in class).//

We have seen in the class notes that it is useful for understanding the powers of A (and hence the walks on G) to use the matrix $C_A(t) = Det(A - t\,I),$ the characteristic polynomial of $A.$ We have also pointed out that for any invertible matrix P (n x n) we have that if $B = P\,A\,P^{-1},$ then

$$C_B(t) = C_A(t).$$

This means that

Theorem. *The characteristic polynomial of any adjacency matrix $A(G)$ is an invariant of the graph isomorphism class of $G.$ That is, if G is isomorphic to G' and A and A' are any choices of adjacency matrix for G and G' respectively, then*

$$C_{A(G)}(t) = C_{A(G')}(t).$$

In turn, this Theorem means that you can show two graphs are not isomorphic if you show that they have distinct characteristic polynomials. The *spectrum* of a graph is the set of roots of its characteristic polynomial. Again, by the same argument, *isomorphic graphs have the same spectrum.*

Exercise. Create examples to illustrate these results.

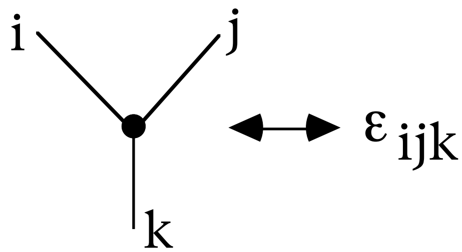
The Epsilon Matrix

One of my favorite matrices is the "epsilon tensor" ε_{ijk}.
This matrix has three indices, each of which can take the values 1, 2 or 3. The values of the epsilon are as follows

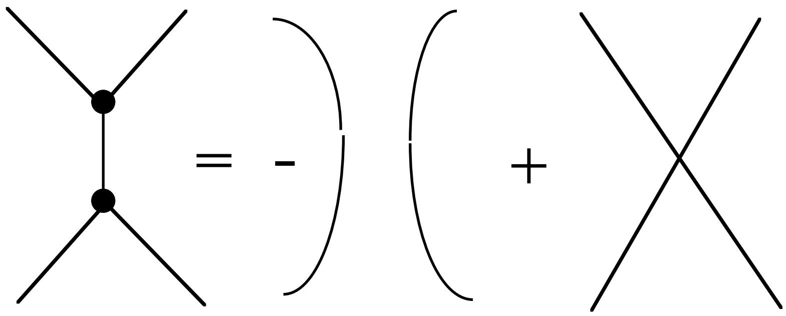
$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1.$$

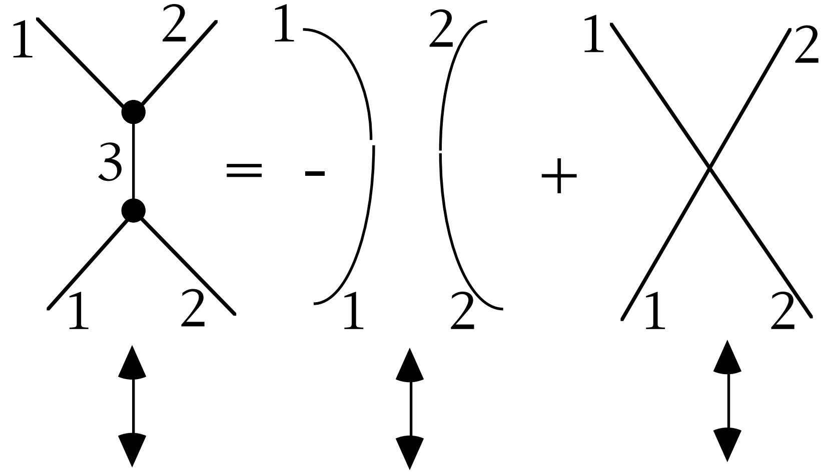
Otherwise (if there is any repetition of indices) the epsilon is zero.
Note that epsilon is invariant under cyclic permutation of the indices.
We diagram epsilon by using a trivalent vertex.



There is a magic identity about the epsilon, which translates into diagrammatic language as



A single line represents the identity matrix. That is, when the two endpoints of the line have the same index value, then the value of the matrix element is one, otherwise it is zero. You can see the truth of this diagrammatic identity by assigning some values to the lines. For example:

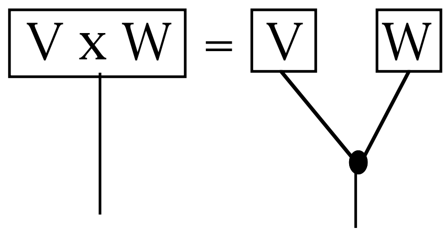


$(1)(-1) = -1 \quad -(1)(1) = -1 \quad (0)(0)=0$

Now the *cross product* of two three dimensional vectors is defined by the epsilon:

$$(V \times W)_k = \sum_k \varepsilon_{ijk} V_i W_j.$$

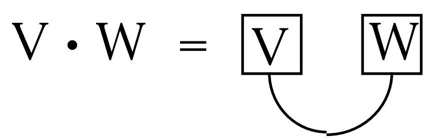
Here one sums over the repeated index **k**. Note that a vector, having only one index is represented by a box with one line. In diagrams the vector cross product is given as follows.



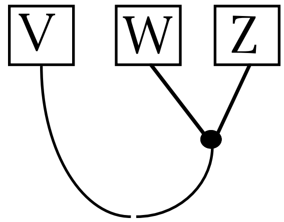
Similarly, the *dot product* of two vectors is given by the formula

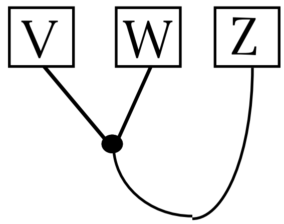
$$V \bullet W = \sum_k V_k W_k.$$

In diagrams, we have:



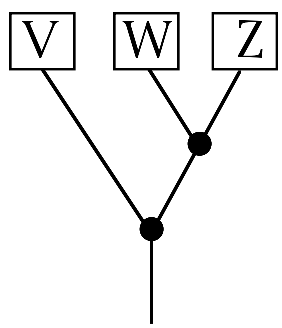
Now we are prepared to see some identities about the vector cross product and the dot product.

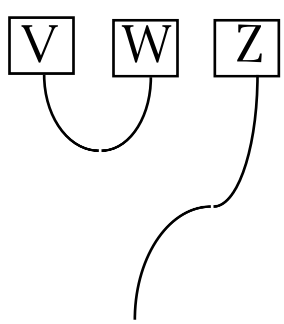
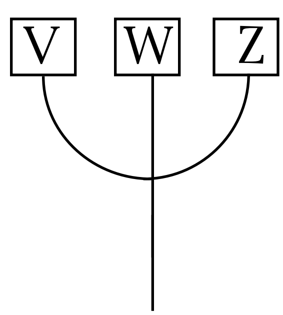
$V \cdot (W \times Z) =$ 

$(V \times W) \cdot Z =$ 

$$V \cdot (W \times Z) = (V \times W) \cdot Z$$

The diagrams deform to one another in the plane. The epsilon is invariant under cyclic permutation of its indices.
Here is one that uses the basic epsilon identity.

$V \times (W \times Z) =$ 

$= -$  $+$ 

$$= -(V \cdot W)Z + (V \cdot Z)W$$

Vector algebra becomes transparent through the use of diagrammatic matrices.