Branched Fiber Bundles and Products of Knots

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0. Introduction

Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a complex polynomial mapping with an isolated singularity at the origin of $\mathbb{C}^{n+1}$. That is, $f(0) = 0$ and the complex gradient df has an isolated zero at the origin. The link of this singularity is defined by the formula $L(f) = V(f) \cap S^{2n+1}$. Here the symbol $V(f)$ denotes the variety of f, and S^{2n+1} is a sufficiently small sphere about the origin. Let $\mathbb{L}(f)$ denote the pair $\mathbb{L}(f) = (S^{2n+1}, L(f))$.

Given another polynomial $g : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}$, we may form $f+g : \mathbb{C}^{n+m+2} \longrightarrow \mathbb{C}$ and consider $\mathbb{L}(f+g)$.

In this paper we study a generalization of cyclic branched coverings which provides geometric constructions corresponding to $\mathbb{L}(f+g)$. That is, given two codimension two embeddings $\mathbb{K} = (S^k, K)$ and $\mathbb{L} = (S^\ell, L)$ so that either S^k or S^ℓ has a book structure (See section 1.) with binding K or L, then we construct a product $\mathbb{K} \otimes \mathbb{L} = (S^{k+\ell+1}, K \otimes L)$. For links of singularities, there is a natural book structure and one finds that $\mathbb{L}(f+g) \simeq \mathbb{L}(f) \otimes \mathbb{L}(g)$.

This product construction generalizes the results of [K1]. For $g = z^a$, it is identical with the a-fold cyclic suspension of [N]. Some of the results of this paper were announced in [K2].

The paper is organized as follows. Section 1 discusses open books and branched fibrations over D^2. These are our generalization of the standard ramified coverings of the disk given by $\mu_a : D^2 \longrightarrow D^2$, $\mu_a(z) = z^a$. They corres-pond to affine pencil of algebraic varieties with degenerate fiber over $0 \in D^2$.

In Section 2 we discuss branched fibrations along codimension-two submanifolds of a given manifold and discuss the properties of certain branched fibrations obtained by pull-back constructions. These generalize the cylic branched coverings.

Section 3 uses branched fibrations to define products of knots. We show that the product construction is commutative (Lemma 3.7) and give a cut-and-paste description of $\mathbb{K} \otimes \mathbb{L}$ (Proposition 3.8) as in [K2].

Section 4 relates the product construction to links of algebraic singularities.

Section 5 shows that if $\mathbb{K}$ and $\mathbb{L}$ each have book structure, then there is a book structure on $\mathbb{K} \otimes \mathbb{L}$. This is examined from two points of view.

Section 6 examines the embedding $K \otimes L \subset S^{k+l+1}$. We show that if $K = \partial F$, $F \subset S^k$ and $L = \partial G$, $G \subset S^l$, then $K \otimes L = \partial \mathbb{F}$ where $\mathbb{F}$ has the homotopy type of the join $F * G$ (Lemma 6.1). This leads to a calculation of the Seifert pairing for $K \otimes L$ (Proposition 6.2).

In section 7 we show that $\otimes$ is associative.

In section 8 we consider a generalized form of spinning. Given a spherical knot $K \subset S^k$ and $L \subset S^l$ the binding of a book structure B on S^l we define a new spherical knot $\mathcal{S}_B(K) \subset S^{k+l}$, the B-twist spin of K. This construction generalizes ordinary twist spinning. We the prove that S^{k+l} has a book structure with binding $\mathcal{S}_B(K)$ and leaf $K \otimes L - (\text{disk})$ (Theorem 8.2). This theorem is a generalization of Zeeman's theorem ([Z]) on ordinary twist spinning.

In sections 9 and 10 we use the product construction to give explicit isomorphisms for the [redacted] periodicity of the Levine knot cobordism groups. We compare our construction with those of G. Bredon and, S. Capell and J. Shaneson. Section 10 discusses $O(n)$-actions related to this periodicity.

Branched Fiber Bundles and Products of Knots

1. Foundations

In this section we discuss the general setting for branched fibrations and their relationship to open book structures and to singularities of mappings.

Definition 1.1. An open book structure on a closed compact manifold M is a smooth mapping $b: M \longrightarrow D^2$ such that

(i) the interior of D^2 consists of regular values of b.

(ii) letting $E = b^{-1}(\partial D^2)$, then the restriction $b| : E \longrightarrow S^1$ is a smooth fiber bundle with fiber F so that $\partial F \simeq b^{-1}(0)$.

A book structure $b: M \longrightarrow D^2$ will be denoted by the symbol (M, b). Two book structures (M, b) and (M', b') are said to be isomorphic ($(M,b) \simeq (M',b')$) if there is a diffeomorphism $h: M \longrightarrow M'$ such that $b' \circ h = b$.

Thus an open book structure on M yields a decomposition $M = E \cup N$ where E is a fiber bundle over S^1 with a trivialization of its boundary so that $\partial E \simeq K \times S^1 \simeq \partial N$, and $N \simeq K \times D^2$ where $K = b^{-1}(0)$. We refer to K as the binding of the book. The fibers of the bundle $E \longrightarrow S^1$ are the <u>leaves</u>.

Let $b : S^n \longrightarrow D^2$ be an open book structure on the n-sphere. Let $cb : D^{n+1} \longrightarrow D^2$ be defined by the formula $cb(rx) = r \cdot b(x)$ where $x \in S^n$ and $0 \leq r \leq 1$. *Note* that cb is a fiber bundle over $D^2 - \{0\}$ with fiber the leaf of the book, also $cb^{-1}(0) = CK$, the cone on the book's binding.
(For $r \neq 0$, $cb^{-1}(re^{i\theta}) = \{(r/s)x \mid s \geq r,\ x \in b^{-1}(se^{i\theta})\}$.)
This leads to the next definition.

<u>Definition 1.2.</u> A smooth map $\tau : D^{n+1} \longrightarrow D^2$ is a <u>branched-fibration</u> along $0 \in D^2$ if

(i) Every point of $D^2 - \{0\}$ is a regular value for τ.

(ii) $\tau^{-1}(0) \simeq CK$ where $K = \tau^{-1}(0) \cap S^n$.

(iii) $\tau^{-1}(0)$ is transverse to the spheres $S_\varepsilon = \{x \in D^{n+1} \mid \|x\| = \varepsilon\}$ for $0 < \varepsilon \leq 1$.

Thus an open book $b : S^n \longrightarrow D^2$ gives a branched fibration by smoothing the mapping $cb : D^{n+1} \longrightarrow D^2$. Conversely, the restriction $\tau| : S^n \longrightarrow D^2$ gives an open book structure on S^n when $\tau : D^{n+1} \longrightarrow D^2$ is a branched fibration along 0.

We wish to include certain low dimensional cases under these definitions. Thus if $[a] : S^1 \longrightarrow S^1$ is given by $[a](x) = x^a$, we call $[a]$ the open book structure on S^1 of degree a. It will also be referred to as the <u>empty knot</u> of degree a (since this book has no binding). The corresponding branched fibration is $\mu_a : D^2 \longrightarrow D^2$ where $\mu_a(z) = z^a$. That is, it is the standard a-fold ramified cover of D^2 with the origin as branch point.

Another way of obtaining examples is as follows. Let $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$ be a smooth map with an isolated singularity at $\underline{o} \in \mathbb{R}^{n+1}$ (and $f(\underline{o}) = 0$). Assume also that $f^{-1}(0)$ is transverse to sufficiently small spheres about $\underline{o} \in \mathbb{R}^{n+1}$. Choose $0 < \varepsilon << 1$ and let $0 < \delta << \varepsilon$. Set $E = \{x \in D^{n+1}_\varepsilon \mid \|f(x)\| \leq \delta\}$. Then we map E to the disk of radius δ by $\pi: E \longrightarrow D^2_\delta$, $\pi(x) = f(x)$. It follows from the construction that E is contractible. Hence $E \simeq D^{n+1}$ for $n \geq 4$. For $n \geq 5$ we may conclude that $\partial E \simeq S^n$.

<u>Definition 1.3</u>. Let $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$ be a smooth map (germ) with $f(\underline{o}) = 0$ and an isolated singularity at the origin. If $f^{-1}(0)$ is transverse to sufficiently small spheres about the origin, we say that f is a <u>tame isolated singularity</u>. For $0 < \varepsilon << 1$, define the <u>link of f</u> by $\mathrm{Link}(f) = S^n_\varepsilon \cap f^{-1}(0)$.

Thus we have shown that, for $n \geq 5$, there is a branched fibration over D^2, $\pi: E \longrightarrow D^2$ and associated open book $\pi| : \partial E \longrightarrow D^2$ so that $\mathrm{Link}(f)$ is the binding of the book and $(E, \partial E) \simeq (D^{n+1}, S^n)$. We do not assert that this book structure is on S^n_ε. However, if $f: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ is a (complex) polynomial mapping with an isolated singularity at the origin, then it follows from Milnor [M] that f is tame and that S_ε^{2n+1} inherits an open book structure. In fact, Milnor proves that the fibration of the complement of Link(f) is given by $\phi : S_\varepsilon^{2n+1} - \text{Link}(f) \longrightarrow S^1$ where $\phi(z) = f(z)/\|f(z)\|$.

It is of interest to know when two tame isolated singularities define the same book structure. To this end, we make the following definition.

Definition 1.4. Let $f, g : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$ be two tame isolated singularities. Assume that $n \geq 5$. We say that f and g are tame topologically equivalent if there exists a diffeomorphism $h : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}^{n+1} - \{0\}$ such that $g \circ h = f$ and so that for ε arbitrarily small, there exists $\varepsilon' << \varepsilon$ such that $f^{-1}(0) \cap D_\varepsilon^{n+1} - h^{-1}(g^{-1}(0) \cap D_{\varepsilon'}^{n+1})$ is a trivial h-cobordism. Thus equivalence preserves the cone structure in a weak sense.

Lemma 1.5. (i) If f and g are equivalent tame isolated singularities, then they define isomorphic open book structures on S^n.

(ii) Let $b : S^n \longrightarrow D^2$ be the open book structure corresponding to f. Let $\phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$ be the singularity germ of a smoothing of cb. Then f is equivalent to ϕ.

Proof. To prove (i), let $h : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ be as in Definition 1.4 so that $f = g \circ h$.

Choose $0 < \varepsilon << 1$ and $0 < \delta << \varepsilon$ so that $E = \{x \in D^{n+1}_\varepsilon \mid \|f(x)\| \leq \delta\} \xrightarrow{\pi} D^2_\delta$ is the branched fibration corresponding to f. Now choose $0 < \varepsilon' << \varepsilon$ so that $h^{-1}(D^{n+1}_{\varepsilon'}) \subset E$. Choose $0 < \delta' << \varepsilon'$ and define $E' = \{x \in D^{n+1}_{\varepsilon'} \mid \|g(x)\| \leq \delta'\}$. Thus

$$\begin{tikzcd} E \supset h^{-1}(E') \arrow[rr, "h"] \arrow[dr, "\pi|"'] & & E' \arrow[dl, "\pi'"] \\ & D^2_{\delta'} & \end{tikzcd}$$

[redacted] Let $W = f^{-1}(0) \cap D^{n+1}_\varepsilon - h^{-1}(g^{-1}(0) \cap D^{n+1}_{\varepsilon'})$. Since W is a trivial h-cobordism, we see that the restriction of π to $h^{-1}(\partial E')$ gives the same book structure as $\pi|\partial E$. Thus $(\partial E, \pi|\partial E) \simeq (h^{-1}\partial E', \pi|h^{-1}\partial E') \xrightarrow{\sim} (\partial E', \pi'|\partial E')$. This shows that f and g have isomorphic open books.

To prove (ii) we note the following: Given a branched fibration $\pi : D^{n+1} \longrightarrow D^2$ there is a diffeomorphism $h : D^{n+1} - \{0\} \longleftrightarrow S^n \times (0,1]$ so that $\pi(h(x,t)) = t \cdot \pi(x)$ for $(x,t) \in S^n \times (0,1]$. This is proved by using the local triviality of the bundle associated with π to construct a vector field on $D^{n+1} - \{0\}$ pointing away from $\pi^{-1}(0)$ and so that the limit of each integral curve, when extended backwards, is $0 \in D^{n+1}$. Thus any branched fibration over D^2 is a smoothing of cb, where $b = \pi| : S^n \longrightarrow D^2$ is the associated open book.

Now apply this remark to the branched fibration $\pi : E \longrightarrow D^2_\delta$ corresponding to f. We obtain $h : \partial E \times (0,1] \longrightarrow E - \{0\}$. Since $\partial E \simeq S^n$ we may regard this as $h : D^{n+1} - \{0\} \longrightarrow E - \{0\}$. Since $E \subset \mathbb{R}^{n+1}$ and

$E \simeq D^{n+1}$, there is no trouble obtaining an extension $h: \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}^{n+1} - \{0\}$. Define $g: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$ by $g(x) = f(h(x))$ for $x \neq 0$. Smooth g at the cone point. We now have that g is topologically equivalent to f and that the singularity germ ϕ of g is a smoothing of cb. This proves (ii).

<u>Remark.</u> Suppose $b: S^n \longrightarrow D^2$ and $r: D^{n+1} \longrightarrow D^2$ is a smoothing of cb. Regard r as a singularity germ $\phi: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$. Then the open book corresponding to ϕ is isomorphic to b. To see this let $E = \{x \in D^{n+1} \mid \|r(x)\| \leq \delta\}$ for $0 < \delta << 1$. Define $p: \partial_1 E \longrightarrow S^n$ by $p(x) = tx$ for the unique t such that $\|tx\| = 1$. Here $\partial_1 E = \{x \in E \mid \|r(x)\| = \delta\}$. Thus $p(\partial_1 E) = \{x \in S^n \mid \|f(x)\| \geqslant \delta\}$. So $S^n = p(\partial_1 E) \cup \partial_2 E$ where $\partial_2 E = \{x \in E \mid x \in S^n\}$. This shows that ∂E and S^n have the same book structure.

Thus we may conclude as follows.

<u>Corollary 1.6.</u> For $n > 5$, the set of isomorphism classes of open book structures on S^n is in 1-1 correspondence with the set of topological equivalence classes of tame isolated singularity germs $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$.

<u>Remark.</u> A motivating example for this section is the singularity $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ given by $f(z) = z_1^2 + z_2^2 + \cdots + z_n^2$. Then $f| : \mathbb{C}^n - \{0\} \longrightarrow \mathbb{C} - \{0\}$ is a fiber bundle with typical fiber $F_t = \{z \in \mathbb{C}^n \mid f(z) = t\}$, $0 < t \in \mathbb{R}$.

The singular fiber F_0 is contractible, while F_t contains the sphere $S_t^{n-1} = \{ z \in \mathbb{R}^n \subset \mathbb{C}^n \mid f(z) = t \}$ as a deformation retract. Thus $H_{n-1}(F_t) \cong \mathbb{Z}$ with generator S_t. As $t \to 0$, the cycle S_t collapses to the origin. We intend the notion of a branched fibration over D^2 as a topological analogue of this sort of degeneration in algebraic geometry.

2. Branched Fibrations

We now continue the discussion of branched fibrations, extending Definition 1.2 to branching over an arbitrary codimension two submanifold.

Definition 2.1. Let (S^n, b) be an open book structure on S^n with leaf F and binding K. Let M^m be a smooth manifold with codimension two submanifold $V^{m-2} \subset M^m$. We say that a smooth mapping $\pi : N \to M$ is a b-branched (or K-branched) fibration along V if

(i) $N - \pi^{-1}(V) \longrightarrow M - V$ is a smooth locally trivial fibration with fiber F.

(ii) for each $v \in V$, $\pi^{-1}(v) \cong CK$.

(iii) on normal disks to V, π is topologically equivalent to cb.

Remark. In (iii) we implicitly assume that the singularity germ corresponding to the restriction of π to a normal disk is tame (1.3). Topological equivalence is meant in the sense of definition 1.4. Note that it is legitimate to speak of cb (i.e. not worrying about smoothing it).

For example, let $r: D^{n+1} \longrightarrow D^2$ be a smoothing of $cb: D^{n+1} \longrightarrow D^2$. Then it follows from our definition that r is b-branched along $0 \in D^2$.

Another useful example is obtained by a pull-back construction. Given M and V as in definition 1.7, suppose that there is a smooth map $\alpha: M \longrightarrow D^2$, transverse to $0 \in D^2$ and so that $\alpha^{-1}(0) = V$. Then we may form the pull-back

$$\begin{tikzcd} N \arrow[r, "\hat{\alpha}"] \arrow[d, "r"] & D^{n+1} \arrow[d, "r"] \\ M \arrow[r, "\alpha"] & D^2 \end{tikzcd}.$$

The manifold N is b-branched along $V \subset M$. In this example, if $r = [a]: D^2 \longrightarrow D^2$, then N is a cyclic branched cover of M along V.

Let's pause for a moment and be more explicit about the differentiable structure on this pull-back construction. Since $N - r^{-1}(V) \xrightarrow{r} M - V$ is the pull-back of a smooth fiber bundle, it has a natural atlas. Thus, it is only necessary to specify charts for points $p \in r^{-1}(V)$. Note that $r^{-1}(V) \cong V \times CK$ where K is the binding of the book $(S^n, r|)$. Let $\tilde{V} \subset N$ denote those points in $r^{-1}(V)$ such that $\hat{\alpha}(\tilde{V}) = 0 \in D^{n+1}$. Let $b = r| : S^n \longrightarrow D^2$ be the open book structure corresponding to r. Recall that there is a diffeomorphism $h : D^{n+1}-\{0\} \longrightarrow S^n \times (0,1]$ so that the following diagram commutes (See the proof of 1.5 (ii)) :

$$\begin{tikzcd}
D^{n+1}-\{0\} \arrow[rr, "h"] \arrow[dr, "r"'] & & S^n \times (0,1] \arrow[dl, "cb"] \\
& D^2 &
\end{tikzcd}$$

where $cb(x,t) = tb(x)$. We now sepantely consider charts for points in $\pi^{-1}(V)-\tilde{V}$ and for points in $\tilde{V}$.

(a) Suppose $p \in \pi^{-1}(V)-\tilde{V}$. Then it follows from the construction that there are neighborhoods $\mathscr{U}$ of $\pi(p)$, $\tilde{\mathscr{U}}$ of p and $\mathcal{U}$ of $\pi(p)$ a neighborhood in V and charts g and $\tilde{g}$ so that the following diagram commutes:

$$\begin{tikzcd}
\tilde{\mathscr{U}} \arrow[r, "\tilde{g}"] \arrow[d, "\pi"] & \mathcal{U}\times(\mathcal{U}'\times(\varepsilon_1,\varepsilon_2)) \arrow[d, "1\times cb"] \\
\mathscr{U} \arrow[r, "g"] & \mathcal{U}\times\mathbb{R}^2
\end{tikzcd}$$

Here $\mathcal{U}' \subset S^n$ is an open set ; $(\varepsilon_1,\varepsilon_2)$ is an open interval in $(0,1]$.
We include the charts $(\tilde{\mathscr{U}}, \tilde{g})$ in the atlas on N.

(b) Suppose $p \in \tilde{V}$. Then there are neighborhoods $\mathscr{U}$ of $\pi(p)$ in M, $\tilde{\mathscr{U}}$ of p in N, $\mathcal{U}$ of p in V so that we obtain the next diagram.

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r, "\tilde{g}"] \arrow[d, "\pi"] & U \times \mathbb{R}^{n+1} \arrow[d, "1 \times \phi"] \\ \mathcal{U} \arrow[r, "g"] & U \times \mathbb{R}^2 \end{tikzcd}$$

Here $\phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$ is obtained by restricting π to the inverse image of a small neighborhood of 0. Thus ϕ is a function germ corresponding to π. Including the charts $(\tilde{\mathcal{U}}, \tilde{g})$ for $p \in \tilde{V}$ completes the atlas for N.

<u>Remark</u>. Since charts of type (b) depend upon π it is not at all clear whether the differentiable structure of the pull-back depends only on the topological type of π. In fact, this is probably false.

Let $N(\alpha, \pi)$ denote the pull-back construction discussed above. Note that given an open book $b : S^n \longrightarrow D^2$, we may analogously define $N(\alpha, cb)$ with corresponding differentiable structure. Now the projection $N(\alpha, cb) \longrightarrow M$ is no longer smooth along $V \subset M$. Again, it is not clear that the differentiable structure of $N(\alpha, cb)$ is dependent only on the isomorphism class of (S^n, b).

We can say the following.

<u>Proposition 2.2</u>. 1) Suppose $r, r' : D^{n+1} \longrightarrow D^2$ are branched fibrations along 0. Then, $N(\alpha, r)$ and $N(\alpha, r')$ are homeomorphic whenever r and r' are topologically equivalent. If there is a diffeomorphism $h : D^{n+1} \longrightarrow D^{n+1}$ so that $r = r' \circ h$, then $N(\alpha, r) \cong N(\alpha, r')$.

2) Suppose (S^n, b) and (S^n, b') are isomorphic book structures. Then $N(\alpha, cb)$ is homeomorphic to $N(\alpha, cb')$. If the book isomorphism $h : (S^n, b) \longrightarrow (S^n, b')$ extends to a diffeomorphism of D^{n+1}, then $N(\alpha, cb) \cong N(\alpha, cb')$.

The proof is obvious.

<u>Remark</u>. Call two book structures (S^n, b) and (S^n, b') <u>isotopic</u> if there is a map $H : S^n \times I \longrightarrow S^n$ so that H is differentiable, each $h_t = H(\ , t)$ is a diffeomorphism, $h_0 = 1_{S^n}$, and h_1 is a book isomorphism from (S^n, b) to (S^n, b'). Thus isotopic book structures give diffeomorphic branched fibrations.

<u>Proposition 2.3</u> If a branded fibration $r : D^{n+1} \longrightarrow D^2$ is obtained from cb via $r = f \circ cb$ where $f : D^2 \longrightarrow D^2$, then $N(\alpha, r) \cong N(\alpha, cb)$.

<u>Proof</u>. This follows directly from our definition of the differentiable structure.

It should also be remarked that in the case of the books $[a]: S^1 \longrightarrow S^1$ (that is, for branched cyclic coverings) we do, by the remarks above, obtain uniqueness of differentiable structure. This fact will help to pin-point the differentiable structure of other branched fibrations.

The next few lemmas delineate further properties of the pull-back construction.

<u>Lemma 2.4</u>. Let $V^{m-2} \subset M^m$ be a proper embedding of smooth manifolds. Assume that M is 2-connected. Then

(a) There exists a smooth map $\alpha: M \longrightarrow D^2$ with 0 a regular value and $V = \alpha^{-1}(0)$.

(b) The map α is unique up to homotopy preserving property (a).

<u>Proof.</u> Since M is 2-connected, it follows from a Mayer-Vietoris sequence that $H^1(M-V) \simeq \mathbb{Z}$. Interpet $H^1(M-V)$ as $[M-V, S^1]$ where $[\ ,\ ]$ denotes homotopy classes of maps. Choose a homotopy class $a: M-V \longrightarrow S^1$ representing $1 \in \mathbb{Z}$. There is then no problem in choosing a smooth map $\alpha: M \longrightarrow D^2$ so that $\alpha|M-V \simeq a$ and α satisfies property (a). The homotopy-uniqueness is clear.

__Lemma 2.5__. Let $V_1^{m-4} \subset V_2^{m-2} \subset M^m$ be proper embeddings of smooth manifolds. Assume that M and V_2 are each 2-connected. Let $i : V_2 \longrightarrow M$ denote the given inclusion. Then there exists an embedding of pairs $j : (V_2, V_1) \longrightarrow (M, V_2)$ so that

(a) $j(V_2)$ is transverse to V_2 with $j(V_2) \cap V_2 = V_1$.

(b) j is isotopic to i through maps satisfying (a).

(c) j is unique up to isotopy through maps satisfying (a) and (b).

__Proof__. For existence, note that V_2 has a neighborhood diffeomorphic to $V_2 \times D^2$ in M. Thus we may write $V_2 \subset V_2 \times D^2 \subset M$. Let $\alpha : V_2 \longrightarrow D^2$ be a map as in Lemma 2.4 so that $\alpha^{-1}(0) = V_1$. Define $\hat{\alpha} : V_2 \longrightarrow (V_2 \times D^2) \subset M$ by $\hat{\alpha}(x) = (x, \alpha(x))$. Then $\hat{\alpha} : (V_2, V_1) \longrightarrow (M, V_2)$ and satisfies property (a). To see (b), define $i_t : V_2 \longrightarrow (V_2 \times D^2) \subset M$ by $i_t(x) = (x, t\alpha(x))$. Then $i_0 = i$ and $i_1 = \hat{\alpha}$. This provides the isotopy. Part (c) is a tautology.

~~Corollary 2.6. Given a book structure (S^n, b) and inclusions $V_1 \subset V_2 \subset M$ as in Lemma 2.5 then there is an inclusion of k-branched fibrations~~

$$\begin{tikzcd} N_1 \arrow[r, hook] \arrow[d] & N_2 \arrow[d] \\ V_2 \arrow[r, "\hat{\alpha}"] & M \end{tikzcd}$$

<u>Remark.</u> Given $V \subset M$ as in Lemma 2.4 and a book $b: S^n \to D^2$, it follows that the pull-back $N(\alpha, cb)$ is actually independent of α. As we have already observed, the differentiable structure on $N(\alpha, cb)$ is the same as $N(\alpha, \nu)$ when the smoothing $\nu : D^{n+1} \to D^2$ is obtained from a homeomorphism $h: D^2 \to D^2$ so that $\nu = h \circ cb$. Therefore, we shall now denote the pull-back $N(\alpha, \nu)$ by $M(V, b)$. Thus

$$\begin{tikzcd}
M(V,b) \arrow[r] \arrow[d] \arrow[dd, bend right, "\pi"'] & D^{n+1} \arrow[d, "cb"] \arrow[dd, bend left, "\nu"] \\
M \arrow[d] & D^2 \arrow[d, "b"] \\
M \arrow[r, "\alpha"] & D^2
\end{tikzcd}$$

The branched fibration $\pi: M(V,b) \to M$ is uniquely determined by $V \subset M$ and the book (S^n, b).

<u>Corollary 2.6.</u> Given a book structure $b: S^n \to D^2$ and inclusions $V_1 \subset V_2 \subset M$ as in Lemma 2.5, then there is an inclusion of branched-fibrations

$$\begin{tikzcd}
V_2(V_1, b) \arrow[r, "\hat{j}"] \arrow[d, "\pi'"] & M(V_2, b) \arrow[d, "\pi"] \\
V_2 \arrow[r, "j"] & M
\end{tikzcd}$$

where j is the map constructed in Lemma 2.5.

<u>Proof.</u> Restrict the branched fibration π to $j(V_2)$. Since $j(V_2) \cap V_2 = V_1$, we

see that the restricted fibration branches along V_1. It is then easy to check that the restriction is, in fact $V_2(V_1, b)$.

Thus we obtain canonical inclusions of branched fibrations. A useful example for 2.6 is as follows. Let $K \subset S^k \subset S^{k+2}$ where K is a codimension 2 closed submanifold of S^k and $S^k \subset S^{k+2}$ is the standard (unknotted) inclusion. Then we obtain, from 2.5, a mapping $j : S^k \to S^{k+2}$ so that $j(S^k) \cap S^k = K$. Taking the book $[a]$ we obtain

$$\begin{array}{ccc} S^k(K,[a]) & \xhookrightarrow{\hat{j}} & S^{k+2}(S^k,[a]) \\ \downarrow & & \downarrow \\ S^k & \xrightarrow{j} & S^{k+2} \end{array}$$

But $S^k(K,[a])$ is the a-fold branched cyclic cover of S^k along K, and $S^{k+2}(S^k,[a]) \simeq S^{k+2}$ since branching along an unknotted sphere has no effect. Therefore $\hat{j}$ embeds the branched covering into S^{k+2}. Letting $K \otimes [a]$ stand for $\hat{j}S^k(K,[a])$ we obtain a pair $(S^{k+2}, K\otimes[a])$ for each pair (S^k, K). This will be referred to as the <u>a-fold cyclic suspension</u> of (S^k, K). (See [N].)

The next section considers branched fibrations, arising from knots $K \subset S^k$, which generalize the cyclic suspensions.

3. Products of Knots and Books

Definition 3.1. A knot is a pair $\mathbb{K} = (S^k, K)$ where $K \subset S^k$ is a codimension two submanifold.

Given $K \subset S^k$, there is a homotopy-unique map (in the sense of Lemma 2.4) $\alpha : S^k \longrightarrow D^2$ with 0 as a regular value, and $\alpha^{-1}(0) = K$. If α could be made into a fibration on the complement of a tubular neighborhood of K, then we would have an open book structure for S^k. Conversely, the inclusion of the binding of a book structure on a sphere yields a knot.

Given a knot $\mathbb{K} = (S^k, K)$ and an open book (S^n, b) we shall define a product knot $\mathbb{K} \otimes \mathbb{b} = (S^{n+k+1}, K \otimes b)$. This will generalize the cyclic suspension mentioned in the previous section. We first define the manifold $K \otimes b$ and then discuss its embedding in S^{n+k+1}.

Suppose $V \subset M$ with M 2-connected so that we can form $M(V, b) \xrightarrow{\pi} M$. Note that the boundary decomposes as follows: $\partial(M(V,b)) \cong \partial M(\partial V, b) \cup (M \times L)$ where L is the binding of the book (S^n, b). Of course, $\partial(\partial M(\partial V, b)) \cong \partial M \times L$ and the union in the first formula refers to union along the boundary. The differentiable structure on the boundary of $M(V,b)$ is specified by its construction. In this case, we could also take $\partial M(\partial V, b)$ and $M \times L$, and, with the usual straightening of angles define the differentiable structure on the union. In many of our constructions it will be most convenient to have the differentiable structure specified automatically from the definitions.

To define $K \otimes b$ we proceed as follows.

<u>Definition 3.2</u>. Given a knot $\mathbb{K} = (S^k, K)$ and a book (S^n, b), choose a properly embedded codimension two submanifold $F \subset D^{k+1}$ so that $\partial F = K \subset S^k$. Then $K \otimes b = \partial(D^{k+1}(F, b)) = S^k(K, b) \cup (D^{k+1} \times L)$. ($L$ is the binding of (S^n, b)).

If the book is the empty knot $(S^1, [a])$, then $K \otimes [a] = S^k(K, [a])$, the a-fold branched cyclic cover of S^k along K.

To obtain the embedding $K \otimes b \subset S^{n+k+1}$ we need a lemma.

<u>Lemma 3.3</u>. Let $D^{k+1} \subset D^{k+3}$ be the standard (unknotted) embedding. Then for any book (S^n, b) we have

$$D^{k+3}(D^{k+1}, b) \simeq D^{k+n+2}.$$

<u>Proof</u>. Regard $D^{k+3} = D^{k+1} \times D^2 \supset (D^{k+1} \times 0) = D^{k+1}$. Then the representative map $\alpha : D^{k+3} \longrightarrow D^2$ is given by projection onto the D^2 factor. Hence pulling back the branched fibration over D^2 gives us $D^{k+3}(D^{k+1}, b) \simeq D^{k+1} \times D^{n+1} \cong D^{k+n+2}$.

<u>Definition 3.4</u>. Under the same hypotheses as 3.3 let $D^{k+1} \subset D^{k+3}$ be the standard inclusion. Apply Lemma 2.5 to the triple $F \subset D^{k+1} \subset D^{k+3}$, obtaining $j : D^{k+1}(F, b) \hookrightarrow D^{k+3}(D^{k+1}, b)$. Taking boundaries and using Lemma 3.3, we obtain $j : K \otimes b \hookrightarrow S^{k+n+1}$. This defines $\mathbb{K} \otimes \mathbb{b}$.

Note that this definition reduces to the cyclic suspension when $(S^n, b) = (S^1, [a])$.

Here is a more symmetrical description of $\mathbb{K} \otimes \mathbb{b}$.

Lemma 3.5. Let $\mathbb{K} = (S^k, K)$ be a knot and (S^n, b) a book. Let $\gamma : D^{k+1} \longrightarrow D^2$ be a smooth map with $\gamma^{-1}(0) = F$, 0 a regular value and $\partial F = K$. Let $\nu : D^{n+1} \longrightarrow D^2$ be a standard smoothing of cb (as in the remark after 2.5. Use these maps to form the pull-back

$$\begin{tikzcd} D^{k+1}(F, b) \arrow[r] \arrow[d] & D^{n+1} \arrow[d, "\nu"] \\ D^{k+1} \arrow[r, "\gamma"] & D^2 \end{tikzcd}.$$

Thus $D^{k+1}(F, b) \subset D^{k+1} \times D^{n+1}$.

Then $(S^{k+n+1}, K \otimes b) \simeq (\partial(D^{k+1} \times D^{n+1}), \partial(D^{k+1}(F, b)))$.

Proof. We have already identified $K \otimes b$ as the boundary of the pull-back. Thus it suffices to check the embedding. Recall the method of Lemma 2.5. We have the triple $F \subset D^{k+1} \subset D^{k+3} = D^{k+1} \times D^2$. The map γ gives rise to $\hat{\gamma} : D^{k+1} \longrightarrow D^{k+3}$, $\hat{\gamma}(x) = (x, \gamma(x))$. The map $\alpha : D^{k+3} \longrightarrow D^2$ is $\alpha : D^{k+1} \times D^2 \longrightarrow D^2$, $\alpha(x, z) = z$. Thus $\alpha \circ \hat{\gamma} = \gamma$. Using these maps to construct the pull-backs, the conclusion of 3.5 follows easily.

<u>Definition 3.6</u>. If (S^n, b) and (S^m, b') are books we may define a <u>knot</u> $\mathbb{b}\otimes\mathbb{b}' = (S^{n+m+1}, b\otimes b')$ by regarding the first book as defining a knot $(S^n, b^{-1}(0))$ and taking the knot product.

We shall see later that there is a natural book structure on S^{n+m+1} with binding $b\otimes b'$. Thus the product of books becomes a book.

<u>Lemma 3.7</u>. Given books (S^n, b) and (S^m, b') then $\mathbb{b}\otimes\mathbb{b}' \simeq \mathbb{b}'\otimes\mathbb{b}$.

<u>Proof</u>. Let $\tau : D^{n+1} \longrightarrow D^2$ and $\tau' : D^{m+1} \longrightarrow D^2$ be branched fibrations corresponding to b and b'. Let $f_{\varepsilon'} : D^2 \longrightarrow D^2$ be a diffeomorphism of D^2 which fixes a collar about the boundary and so that $f_{\varepsilon'}(\varepsilon') = 0$ (for $0 < \varepsilon' << 1$). We may assume that $f_{\varepsilon'}$ is isotopic to the identity through maps $f_\varepsilon : D^2 \longrightarrow D^2$ $(0 \leq \varepsilon \leq \varepsilon')$ so that $f_0 = 1_{D^2}$ and $f_\varepsilon(\varepsilon) = 0$. Let $\tau_\varepsilon = f_\varepsilon \circ \tau$ and $\tau'_\varepsilon = f_\varepsilon \circ \tau'$. Then for $0 < \varepsilon \leq \varepsilon'$, $\tau_\varepsilon : D^{n+1} \longrightarrow D^2$ has the origin as a regular value and $\tau_\varepsilon^{-1}(0) \subset D^{n+1}$ is a submanifold with boundary isotopic to the binding of b. Consequently we may use τ_ε and τ' to form $\mathbb{b}\otimes\mathbb{b}'$:

$$\begin{array}{ccc} X_\varepsilon & \longrightarrow & D^{m+1} \\ \downarrow & & \downarrow \tau \\ D^{n+1} & \xrightarrow{\ \tau_\varepsilon\ } & D^2 \end{array}$$

Thus $\mathbb{b}\otimes\mathbb{b}' = (\partial(D^{n+1}\times D^{m+1}), \partial X_\varepsilon)$ by Lemma 3.5. Letting ε approach 0 isotops ∂X_ε in $\partial(D^{n+1}\times D^{m+1})$. Of course X_0 acquires a singularity, but this occurs in a collar away from its boundary. Thus we obtain $\mathbb{b}\otimes\mathbb{b}' = (\partial(D^{n+1}\times D^{m+1}), \partial X_0)$. Since X_0 is defined symmetrically in terms of b and b', this proves that $\mathbb{b}\otimes\mathbb{b}' \simeq \mathbb{b}'\otimes\mathbb{b}$.

<u>Remark</u>. If $b=[k]$ and $b'=[\ell]$, the empty knots of degree k and ℓ respectively. Then $\mathbb{b}\otimes\mathbb{b}' = (S^3, [k]\otimes[\ell])$, a torus link of type (k,ℓ). Lemma 3.7 is a generalization of the fact that a link of type (k,ℓ) is equivalent to a link of type (ℓ,k).

<u>Notation</u>. Given a knot $\mathbb{K}$ and an open book (S^k, b) with binding L, we shall sometimes denote $\mathbb{K}\otimes\mathbb{b}$ by $\mathbb{K}\otimes\mathbb{L}$ (and $\mathbb{K}\otimes b$ by $\mathbb{K}\otimes L$ where it is understood that $\mathbb{L}$ has an open book structure. Also, we understand that $\mathbb{b}\otimes\mathbb{K}$ and $\mathbb{L}\otimes\mathbb{K}$ both stand for $\mathbb{K}\otimes\mathbb{b}$. Thus products of books is commutative, while [redacted] the product of a knot and a book is commutative by <u>definition</u>.

The rest of this section is devoted to giving a cut and paste description of $\mathbb{K}\otimes\mathbb{L}$.

Let $\mathbb{K} = (S^k, K)$ be a knot. Since codimension two submanifolds of a sphere have trivial normal bundle, we may write $S^k = E_K \cup (K\times D^2)$ where $\partial E_K = K\times S^1$. There exists a map $\alpha : E_K \longrightarrow S^1$ so that α represents the generator for $H^1(E_K)$ and $\alpha|\partial E_K : K\times S^1 \longrightarrow S^1$ is projection on the second factor. If (S^ℓ, b) is a book structure with binding $L \subset S^\ell$ we may write $S^\ell = E_L \cup (L \times D^2)$ and choose $\beta : E_L \longrightarrow S^1$ satisfying identical conditions and so that β is a smooth fiber bundle corresponding to the book structure.

Thus we may form the pull-back $E_K \times_{S^1} E_L$:

$$\begin{array}{ccc} E_K \times_{S^1} E_L & \longrightarrow & E_L \\ \downarrow & & \downarrow \beta \\ E_K & \xrightarrow{\alpha} & S^1 \end{array} .$$

This is a well-defined manifold, and $\partial(E_K \times_{S^1} E_L) \simeq (K \times E_L) \cup (L \times E_K)$.

<u>Proposition 3.8</u>. Let $\mathbb{K}$ and $\mathbb{L}$ be as above. Then, as a manifold,

$$K \otimes L \simeq (K \times D^{\ell+1}) \cup (E_K \times_{S^1} E_L) \cup (D^{k+1} \times L)$$

where boundary identifications are made as follows:

$$\partial(K \times D^{\ell+1}) \simeq (K \times E_L) \cup (K \times D^2 \times L)$$
$$\partial(D^{k+1} \times L) \simeq (E_K \times L) \cup (K \times D^2 \times L).$$

Thus the second factors are identified with each other, while the first factors are glued to $\partial(E_K \times_{S^1} E_L)$.

The embedding $j : K \otimes L \hookrightarrow S^{k+\ell+1}$ may be described as follows. Define the following maps:

$j_1 : K \times D^{\ell+1} \longrightarrow S^k \times D^{\ell+1}$, $j_1(w, y) = ((w, r(y)), y)$
$(w, r(y)) \in K \times D^2 \subset S^k$.

$j_2 : E_K \times_{S^1} E_L \subset E_K \times E_L \subset S^k \times S^\ell$.

$j_3 : D^{k+1} \times L \longrightarrow D^{k+1} \times S^\ell$, $j_3(x, w) = (x, (w, \alpha(x)))$
$(w, \alpha(x)) \in L \times D^2 \subset S^\ell$.

Here $\nu : D^{\ell+1} \longrightarrow D^2$ is the branched fibration corresponding to the book (S^n, b), and $\alpha : D^{k+1} \longrightarrow D^2$ is the map transversal to 0 so that $\alpha^{-1}(0) = F$, $\partial F = K$.

<u>Remark</u>. We make no attempt to directly specify the differentiable structure via pasting. This proposition simply observes that this decomposition is a result of our definitions.

<u>Proof</u>. Recall the construction for $K \otimes L$. Let $M = D^{k+1} \supset F$ with $\partial F = K$. Then $K \otimes L = \partial(M(F,b)) = \partial M(K,b) \cup (D^{k+1} \times L)$. But $\partial M(K,b) = \pi^{-1}(E_K) \cup \pi^{-1}(K \times D^2) = \pi^{-1}(E_K) \cup (K \times D^{\ell+1})$ where $\pi : \partial M(K,b) \longrightarrow S^k$ is the pull-back branched fibration. Now $\pi| : \pi^{-1}(E_K) \longrightarrow E_K$ and $\alpha(E_K) \subset S^1$. Hence we have the diagram

$$\begin{tikzcd}
\pi^{-1}(E_K) \arrow[r] \arrow[d, "\pi|"] & E_L \arrow[d, "\beta"] & \subset D^{\ell+1} \arrow[d, "\nu"] \\
E_K \arrow[r, "\alpha"] & S^1 & \subset D^2 .
\end{tikzcd}$$

Hence $\pi^{-1}(E_K) = E_K \times_{S^1} E_L$. Therefore $K \otimes L = (K \times D^{\ell+1}) \cup (E_K \times_{S^1} E_L) \cup (D^{k+1} \times L)$. This proves the first part.

The second part follows in the same manner by breaking up the diagram

$$\begin{tikzcd}
M(F,b) \arrow[r] \arrow[d] & D^{\ell+1} \arrow[d, "\nu"] \\
D^{k+1} \arrow[r, "\alpha"] & D^2
\end{tikzcd}$$

and examining the embedding

$$K \otimes L = \partial(M(F,b)) \subset \partial(D^{\ell+1} \times D^{k+1}) = S^{k+\ell+1} .$$

4. Links of Complex Hypersurface Singularities

Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a polynomial mapping with an isolated singularity at the origin. Recall that one defines the link of the singularity, $\mathbb{L}(f) \subset S^{2n+1}$, by the formula $L(f) = f^{-1}(0) \cap S^{2n+1}$. Here S^{2n+1} denotes a sphere about the origin of small radius. We also write $\mathbb{L}(f) = (S^{2n+1}, L(f))$.

Milnor proved that the book structure associated with such a singularity occurs naturally on S^{2n+1}. The fibering of the complement is given by $\phi : S^{2n+1} - L(f) \longrightarrow S^1$, $\phi(x) = f(x)/|f(x)|$. Thus we shall let $\mathbb{L}(f)$ denote this knot with its open book structure.

Given another singularity $g : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}$ we may form $f+g : \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} \longrightarrow \mathbb{C}$ by $(f+g)(x,y) = f(x) + g(y)$. In this section we show that $\mathbb{L}(f+g) \simeq \mathbb{L}(f) \otimes \mathbb{L}(g)$.

Lemma 4.1. Let f and g be complex polynomial mappings, $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$, $g : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}$, each having an isolated singularity at the origin. Choose $0 < \delta << \varepsilon << 1$. Let neighborhoods N_f and N_g be chosen as follows:

$$N_f = \{x \in \mathbb{C}^{n+1} \mid \|x\| \leq \varepsilon,\ |f(x)| \leq \delta\}$$
$$N_g = \{y \in \mathbb{C}^{m+1} \mid \|y\| \leq \varepsilon,\ |g(y)| \leq \delta\}$$

Then $\mathbb{L}(f+g) \simeq (\partial(N_f \times N_g),\ (f+g)^{-1}(0) \cap \partial(N_f \times N_g))$.

Proof. By a slight modification of [[M], Lemma 5.9] (compare with our proof of Lemma 1.5 (ii).) there is a vector field on $D_{\varepsilon'} - \{0\}$ ($D_{\varepsilon'} = \{(x,y) \in \mathbb{C}^{n+m+2} \mid \|(x,y)\| = \varepsilon' = \sqrt{2}\,\varepsilon\}$) which points away from the origin and is tangential to $(f+g)^{-1}(0)$. By moving points of $\partial(N_f \times N_g)$ along the integral curves of this vector field we obtain a diffeomorphism (after smoothing corners of $N_f \times N_g$) $h : \partial(N_f \times N_g) \longrightarrow S_{\varepsilon'}$ and $h((f+g)^{-1}(0) \cap \partial(N_f \times N_g)) = L(f+g) \subset S_{\varepsilon'}$.

This proves the lemma.

<u>Proposition 4.2.</u> Let f and g be complex polynomial mappings as in 4.1. Then $\mathbb{L}(f+g) \simeq \mathbb{L}(f) \otimes \mathbb{L}(g)$.

<u>Proof.</u> Let $K = L(f)$. We first represent $\partial(N_f \times N_g) - (K \times D^2 \times D^{2m+2})$ as a branched fibration over S^{2m+1}.

Let $M = \{x \mid \|x\| = \varepsilon, |f(x)| \leq \delta\} \times N_g \subset N_f \times N_g$. Then $M \simeq (K \times D^2 \times D^{2m+2})$. The branched fibration is given by $\pi : \partial(N_f \times N_g) - M \longrightarrow$ $\longrightarrow \partial(D^2_\delta \times N_g) \simeq S^{2m+3}$ where $\pi(x,y) = (f(x), y)$.

Restricting π to $(f+g)^{-1}(0) \cap (\partial(N_f \times N_g) - M)$, we see that it branches along $(x + g(y) = 0) \cap \partial(D^2_\delta \times N_g)$.

Thus, letting $\tau : N_f \longrightarrow D^2_\delta$ by $\tau(x) = f(x)$ we have that τ is the branched fibration corresponding to the book structure on $\mathbb{L}(f)$. We have shown That $\partial(N_f \times N_g) - \mathring{M} \simeq \mathcal{S}^{2m+3}(\mathcal{S}^{2m+1}, \tau)$ and $(f+g)^{-1}(0) \cap (\partial(N_f \times N_g) - \mathring{M}) \simeq j\mathcal{S}^{2m+1}(L(g), \tau)$ where $j : S^{2m+1}(L(g), \tau) \hookrightarrow \mathcal{S}^{2m+3}(\mathcal{S}^{2m+1}, \tau)$ is the inclusion corresponding to the triple $L(g) \subset S^{2m+1} \subset \mathcal{S}^{2m+3}$ (Lemma 2.5).

Since $\mathbb{L}(f) \otimes \mathbb{L}(g) \simeq (\mathcal{S}^{2m+3}(\mathcal{S}^{2m+1}, \tau) \cup (K \times D^2 \times D^{2m+2})$, $j(S^{2m+1}(L(g), \tau)) \cup j(K \times D^{2m+2}))$ we have shown that $\mathbb{L}(f+g) \simeq \mathbb{L}(f) \otimes \mathbb{L}(g)$.

<u>Corollary 4.3.</u> Let f and g be as in 4.2. Then write $S^{2n+1} = E_K \cup (K \times D^2)$ and $S^{2m+1} = E_L \cup (L \times D^2)$ where $K = L(f)$ and $L = L(g)$. Define $\phi : E_K \longrightarrow S^1$, $\psi : E_L \longrightarrow S^1$ via $\phi(x) = f(x)/|f(x)|$, $\psi(x) = g(y)/|g(y)|$. Then $L(f+g) \simeq$ [redacted] $L(f) \otimes L(g) \simeq (K \times D^{2m+2}) \cup (E_K \times_{S^1} E_L) \cup (D^{2n+2} \times L)$ with identifications as in [redacted] Proposition 3.8. *Similarly*, the embedding $L(f+g) \subset \mathcal{S}^{2m+2n+3}$ may be described as in 3.8.

<u>Remarks</u>. (1) We could have proved proposition 4.2 by exhibiting $N_f \times N_g$ as a branched fibration over $D^2 \times N_g$ ($\pi : N_f \times N_g \longrightarrow D^2 \times N_g$, $\pi(x,y) = (P(f(x)), y)$ where $P : D^2 \longrightarrow D^2$ displaces the origin as in 3.7).

(2) Proposition 4.2 may be extended to arbitrary tame isolated singularities as follows. Let $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$ and $g : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^2$ be tame isolated singularities. Define N_{f+g} via $N_{f+g} = \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \mid \|(x,y)\| \leq \epsilon',\ |f(x)+g(y)| \leq \delta'\}$. Choosing ϵ' and δ' sufficiently small, choose $0 < \delta \ll \epsilon \ll \delta' \ll \epsilon'$ and use ϵ and δ to form N_f and N_g. Then $N_f \times N_g \subset N_{f+g}$. By <u>definition</u> $\mathbb{L}(f+g) = (\partial N_{f+g}, (f+g)^{-1}(0) \cap \partial N_{f+g})$. A vector field argument shows that $(\partial(N_f \times N_g), (f+g)^{-1}(0) \cap \partial(N_f \times N_g)) \simeq \mathbb{L}(f+g)$. Then the argument of 4.2 shows that $\mathbb{L}(f+g) \simeq \mathbb{L}(f) \otimes \mathbb{L}(g)$.

(3) Using remark (2) we may now conclude that for $n, m \neq 4, 5$ the product of books is a book. The reasoning goes as follows. Let $\mathbb{B}$ and $\mathbb{B}'$ be books. Let f and g be corresponding singularity germs. Then $f+g$ is a tame germ and determines a book structure on a sphere with corresponding knot $\mathbb{L}(f+g)$. But $\mathbb{L}(f+g) \simeq \mathbb{L}(f) \otimes \mathbb{L}(g) \simeq \mathbb{B} \otimes \mathbb{B}'$. Hence $\mathbb{B} \otimes \mathbb{B}'$ carries a book structure.

In the next section we shall see how the book structure on a product arises in the category of books.

5. Exteriors of Products and Products of Books

In this section we will examine the complement of a knot product. It will then be apparent that the product of two books is a book. Other structural details will emerge.

Lemma 5.1. Let $\mathbb{K} = (S^k, K)$ be a knot and $\alpha : S^k \longrightarrow D^2$ a smooth map with $0 \in D^2$ a regular value and $\alpha^{-1}(0) = K$. We may assume that $\alpha^{-1}[0,1] = F$ is a smooth spanning manifold for K. Furthermore there exists a smooth extension $a : D^{k+1} \longrightarrow D^2$ so that
(i) $a|S^k = \alpha$
(ii) a is topologically equivalent to $c\alpha$.
(iii) $a^{-1}(0) \cong CK$
(iv) $a^{-1}(t) \cong F_t$ where for $0 < t < 1$ F_t is a properly embedded submanifold with $\partial F_t \cong K$. $(F_1 = F)$.

Proof. Smooth the cone on α.

We may also assume that there is a function $\mathbb{R} \times D^{k+1} \longrightarrow D^{k+1}$ $(r, x) \longrightarrow r * x$ so that $a(r*x) = r\,a(x)$. We shall simply write rx for $r*x$ when this leads to no confusion.

As in 3.7, if $P_\varepsilon : D^2 \longrightarrow D^2$ is a diffeomorphism which displaces the center then $P_\varepsilon \circ a$ will have the origin as a regular value. Instead of using this shift function we shall write $a(x) - \varepsilon$.

Let $b : S^\ell \longrightarrow D^2$ be a book with corresponding knot $L \subset S^\ell$. Let $b : D^{\ell+1} \longrightarrow D^2$ also stand for the construction of Lemma 5.1 applied to b. Thus $b : D^{\ell+1} \longrightarrow D^2$ is the branched fibration for this book. Then, if $\mathcal{M} = D^{k+1}$ we have the pull-back

$$\begin{tikzcd} \mathcal{M}(F_\epsilon, b) \arrow[r] \arrow[d] & D^{\ell+1} \arrow[d, "b"] \\ D^{k+1} \arrow[r, "a-\epsilon"] & D^2 \end{tikzcd}$$

and $K \otimes L = \partial \mathcal{M}(F_\epsilon, b)$.

Now $\mathcal{M}(F_\epsilon, b) = \{(x,y) \in D^{k+1} \times D^{\ell+1} \mid a(x) - b(y) = \epsilon\}$. Since it will be more convenient to replace the formula $a(x) - b(y) = \epsilon$ with $a(x) + b(y) = \epsilon$, we shall do so. The results are the same up to orientations.

Proposition 5.2. Using the notation above, let

$$\mathbb{E} = \{(x,y) \in D^{k+1} \times D^{\ell+1} \mid |a(x) + b(y)| = \epsilon\}.$$

Then (i) $\mathbb{E} \xrightarrow{\pi} D^{k+1} \times S^1$, $\pi(x,y) = (x, (a(x)+b(y))/\epsilon)$ exhibits $\mathbb{E}$ as a branched fibration along $\{(x,\lambda) \in D^{k+1} \times S^1 \mid a(x) = \lambda\epsilon\} = j(E_K)$ where $j : E_K \longrightarrow D^{k+1} \times S^1$ by $j(x) = (\epsilon x, d(x))$.

(ii) $\mathbb{E} \simeq E_{K\otimes L} = S^{k+\ell+1} - (K \otimes L \times \mathring{B}^2)$.

(iii) Hence if both $\mathbb{K}$ and $\mathbb{L}$ are books then $\mathbb{K} \otimes \mathbb{L}$ has a book structure with fibration of the complement given by $E_{K\otimes L} \simeq \mathbb{E} \xrightarrow{pr \circ \pi} S^1$ where $pr : D^{k+1} \times S^1 \longrightarrow S^1$ is projection on the second factor.

Proof. (i) is clear.

To prove (ii) define $\rho : \mathbb{E} \longrightarrow \partial(D^{k+1} \times D^{\ell+1}) = S^{k+\ell+1}$ by $\rho(x,y) = (rx, ry)$ for the unique $r \geq 1$ such that $(rx, ry) \in S^{k+\ell+1}$. Then we find that the image $\rho(\mathbb{E}) = \{(x,y) \in S^{k+\ell+1} \mid |a(x) + b(y)| \geq \epsilon\}$.

Since $\{(x,y) \in S^{k+l+1} \mid |a(x)+b(y)| \leq \epsilon\} \cong (\mathbb{K} \otimes \mathbb{L}) \times D^2$, we have $\rho : \mathbb{E} \xrightarrow{\cong} E_{\mathbb{K}\otimes\mathbb{L}}$.

(iii) If both $\mathbb{K}$ and $\mathbb{L}$ are books, then a and b are each branched fibrations. If $p = pro\circ\pi : \mathbb{E} \longrightarrow S^1$, then

$$p^{-1}(\lambda) = \{(x,y) \in D^{k+1} \times D^{l+1} \mid a(x) + b(y) = \lambda\epsilon\}$$

$$= \{(x,y) \in D^{k+1} \times D^{l+1} \mid b(y) = (\lambda\epsilon - a(x))\}$$

The map $\alpha : D^{k+1} \longrightarrow D^2$ given by $\alpha(x) = \lambda\epsilon - a(x)$ has $0 \in D^2$ as a regular value. Hence $p^{-1}(\lambda) \cong D^{k+1}(\alpha^{-1}(0), b)$. That is, $p^{-1}(\lambda) = M(F_{\epsilon\lambda}, b)$. Thus $p : \mathbb{E} \longrightarrow S^1$ is a fiber bundle with each fiber diffeomorphic to $M(F_\epsilon, b)$.

This gives the open book structure on $\mathbb{K} \otimes \mathbb{L}$ once we identify $\mathbb{E} \cong E_{\mathbb{K}\otimes\mathbb{L}}$.

This is a very useful result [redacted] and deserves some comment. First note that we now have a very nice interpetation of $E_{K\otimes L}$ as the L-branched fibration of $D^{k+1} \times S^1$ along $j(E_K)$ where $j : E_K \longrightarrow D^{k+1} \times S^1$ is, essentially, the embedding obtained in Lemma 2.5. That is, $E_K \subset S^k$, $S^{k+2} = (D^{k+1} \times S^1) \cup (S^k \times D^2)$, and by 2.5 the triple $K \subset S^k \subset S^{k+2}$ gives an embedding $j : S^k \hookrightarrow S^{k+2}$ with $j(S^k) \cap S^k = K$. The embedding $j : E_K \to D^{k+1} \times S^1$ is just the restriction.

[redacted]

For ease of discussion let us restrict ourselves to ordinary branched coverings. The usual embedding $K\otimes[a] \subset S^{k+2}$ is obtained as follows. Use the embedding $j: S^k \longrightarrow S^{k+2}$. Branch S^{k+2} along S^k giving $\pi: S^{k+2} \longrightarrow S^{k+2}$ (since S^{k+2} branched along S^k is just S^{k+2} again). Then $K\otimes[a] = \pi^{-1}(j(S^k)) \subset S^{k+2}$.

This leads to the following notion. We know that $j(S^k) \subset S^{k+2}$ is the trivial knot. Why not branch S^{k+2} along $j(S^k)$ and take the inverse image of the "axis" S^{k+2}? That is let $\pi': S^{k+2} \longrightarrow S^{k+2}$ denote the a-fold branched cover along $j(S^k)$. Then we have the pair $K\otimes[a] \cong \pi'^{-1}(S^k) \subset S^{k+2}$. Do these two constructions yield the same knot?

Lemma 5.2 assures us that the answer is <u>yes</u>. This also follows from the next lemma.

<u>Lemma 5.3 (Interchange)</u>. Let $K \subset S^k$ be a knot and $j: S^k \longrightarrow S^{k+2}$ the embedding corresponding to the triple $K \subset S^k \subset S^{k+2}$. Then there exists an orientation preserving homeomorphism $h: S^{k+2} \longrightarrow S^{k+2}$ so that $h(S^k) = j(S^k)$ and $h(j(S^k)) = S^k$.

<u>Proof (sketch)</u>. We show that it is possible to simultaneously isotop $j(S^k)$ to S^k and S^k to $j(S^k)$ keeping their intersection fixed. We use a modified picture of these embeddings as follows: View $S^{k+2} = \mathrm{Spin}(D^{k+1})$. That is,

$$S^{k+2} \cong (D^{k+1} \times S^1)/\sim \quad \text{where } (x,\lambda) \sim (x',\lambda')$$

for all [redacted] $x \in S^k = \partial D^{k+1}$ and $\lambda, \lambda' \in S^1$.

Let $D^{k+1} \times \lambda \subset S^{k+2} = \mathrm{Spin}(D^{k+1})$ be denoted by $D\lambda$. Thus $S^{k+2} = \bigcup D\lambda$ and all the $D\lambda$'s have the same boundary S^k. Let $\dot{i} : S^k \longrightarrow D^{k+1}$ so that $\dot{i}|K$ = inclusion and $\dot{i}(S^k - K) \subset \mathrm{Interior}(D^{k+1})$. This may be accomplished by pushing inward along a vector field. Let $\phi : S^k - K \longrightarrow S^1$ so that for each $\lambda \in S^1$, $\overline{\phi^{-1}(\lambda)}$ has collared boundary K and singularities (if any) away from this collar. Let $F_\lambda = \overline{\phi^{-1}(\lambda)}$.

Then $\mathring{j} : S^k \longrightarrow S^{k+2}$ may be represented by $\mathring{j}(x) = (\dot{i}x, \phi(x)) \in \mathrm{Spin}(D^{k+1})$. Note that $j(F_\lambda) \subset D\lambda$ is the same as $\dot{i}(F_\lambda) \subset D^{k+1}$.

To perform the isotopy, slide all the "fibers" $j(F_\lambda)$ toward the axis $S^k \subset S^{k+2}$. Leave $K \subset S^k$ fixed. At the same time, begin sliding fibers $F_\lambda \subset S^k$ (the axis) into D_λ^{k+1}, leaving their boundaries fixed. If one isotopy is denoted I_t and the other I'_t then we require that $I_t(F_\lambda) \cap I'_t(F_\lambda) = K$ for all t. This can be accomplished since F_λ is codimension two in D^{k+1}. This completes the proof sketch of the interchange lemma.

Remark. It is interesting to examine the interchange lemma in the three sphere. It states that if K_a is a torus knot of type $(a,1)$ represented as a braid about an axis circle S^1, then there is an isotopy interchanging these. (See also [6].)

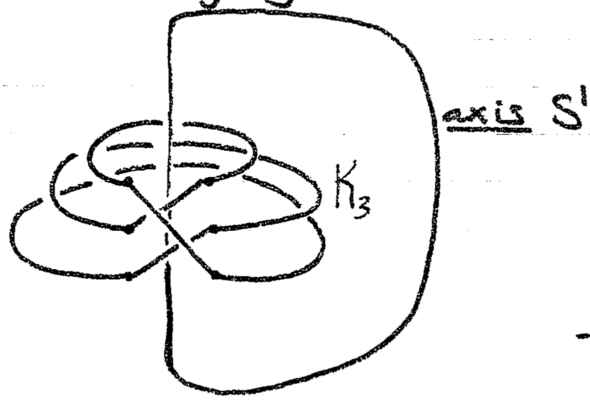


Figure 1

The b-fold branched cyclic covering of S^3 along K_a lifts the standard book structure on S^3 to a book structure on S^3 with leaf F_λ a b-fold branched covering of D^2_λ branched along the a points $D^2_\lambda \cap K_a$. If $\pi: S^3 \longrightarrow S^3$ denotes this covering, then either 5.2 or the interchange lemma assure us that $\pi^{-1}(S^1) \simeq [a] \otimes [b] \subset S^3$ is the usual embedding of the (a,b) torus knot.

If we wish to look at this example from the viewpoint of 5.2, let $a: D^2 \longrightarrow D^2$, $b: D^2 \longrightarrow D^2$, $a(x) = x^a$, $b(x) = x^b$ be the branched fibrations corresponding to the books $[a]$ and $[b]$. Let $\rho(x,y) = (r^{1/a}x, r^{1/b}y)$. Then $\mathbb{E} = \{(x,y) \in D^2 \times D^2 \mid |x^a + y^b| = \varepsilon\}$ and $\pi: \mathbb{E} \longrightarrow D^2 \times S^1$ is a b-fold branched cover along $\{(\sqrt{\varepsilon}\,\mu, \mu^a)\}_{\mu \in S^1} \subset D^2 \times S^1$. This is a torus braid of type $(a,1)$. Thus, we see (again) that $[a] \otimes [b] \subset S^3$ is obtained by taking the inverse image of the axis under the branched covering $\pi: S^3 \to S^3$ with branch set the $(a,1)$ braid.

<u>Remark</u>. With a little extra care we can replace the homeomorphism h of 5.3 by a diffeomorphism. Lemma 5.3 can then be used to give an alternate proof for Proposition 5.2. We may always regard the book structure induced on $\mathbb{K} \otimes \mathbb{L}$ as arising by lifting the trivial book structure on S^{k+2} via branching along $j(S^k)$.

Here is a proof sketch of the result, using this method of inducing book structures. We use the Spin decomposition for ease of visualization. By keeping closer track of tubular neighborhoods, everything can be done in the differentiable category.

<u>Proposition 5.3</u>. Let $K \subset S^k$ and $L \subset S^l$ be knots with book structure. Let $a: D^{k+1} \longrightarrow D^2$ and $b: D^{l+1} \longrightarrow D^2$ be the corresponding branched fibrations. Then $K \otimes \mathbb{L}$ has book structure with leaf the b-fold branched fibration of D^{k+1} along F^*. F^* denotes the result of pushing a fiber F for $K \subset S^k$ in D^{k+1},

leaving the boundary fixed.

Proof. The argument is a direct generalization of our remarks about cyclic branched covers.

First, a few remarks about spinning: We may regard D^{N+2} as the result of spinning D^{N+1}. That is, let $\partial D^{N+1} = D^N_- \cup D^N_+$. Then $D^{N+2} \approx (D^{N+1} \times S^1)/\sim$ where $(D^N_- \times \lambda) \sim (D^N_- \times \lambda')$ for all $\lambda, \lambda' \in S^1$. We write this decomposition as $D^{N+2} = Sp(D^{N+1})$. Thus $\partial Sp(D^{N+1}) = Spin(D^N)$. Just as $S^{N-1} \subset Spin(D^N)$ is called the axis, we refer to $D^N_- \subset Sp(D^{N+1})$ as the axis of D^{N+2}.

Then $K \otimes L$ may be described as follows. Let $M(K,b) \xrightarrow{\pi} S^k$ be the b-branched fibration along $K \subset S^k$. As usual, $K \otimes L = M(K,b) \cup (D^{k+1} \times L)$. View $S^{k+2} = Spin(D^{k+1})$ and let $\hat{S}^k \subset S^{k+2}$ denote $j(S^k) \subset S^{k+2}$ as described in the proof of the interchange lemma. Now $S^{k+\ell+1} = M(\hat{S}^k, b) \cup (D^{k+3} \times L)$. The inclusion $i : M(K,b) \to M(\hat{S}^k, b)$ is induced by $S^k \subset S^{k+2}$. Let $i_1 : D^{k+1} \hookrightarrow D^{k+3}$ be the embedding of the axis for $D^{k+3} = Sp(D^{k+2})$. Then $i : D^{k+1} \times L \hookrightarrow D^{k+3} \times L$ is defined by $i(x, \ell) = (i_1 x, \ell)$. This defines an embedding $i : K \otimes L \hookrightarrow S^{k+\ell+1}$. One can verify, using the interchange lemma, that this is $K \otimes L$.

Now the map $\pi : M(\hat{S}^k, b) \longrightarrow S^{k+2}$ lifts the usual book structure $S^{k+2} = Spin(D^{k+1}) = \cup D_\lambda$ to a book structure on $\mathcal{M} = M(\hat{S}^k, b)$. The decomposition reads $\mathcal{M} = \cup M_\lambda$ where $\pi : M_\lambda \longrightarrow D_\lambda$ is the the b-branched fibration along F^*_λ. Since $\partial D_\lambda = \partial D_{\lambda'}$ for $\lambda, \lambda' \in S^1$, we have $\pi^{-1}(\partial D_\lambda) = \pi^{-1}(\partial D_{\lambda'})$. But $\pi^{-1}(\partial D_\lambda) = M(K,b)$ and $\partial M_\lambda = \partial \pi^{-1}(D_\lambda) = M(K,b) \cup (D_\lambda \times L)$. Hence $\partial \mathcal{M} = L \times Sp(D^{k+1})$ where $Spin(D^{k+1}) = (\cup D_\lambda)/\sim$.

Regard $D^{k+3} = Sp(D^{k+2}) = \bigcup D_\lambda^{k+2}$ where $\partial_+ D_\lambda^{k+2} = D_\lambda^{k+1} = D_\lambda$ and $\partial_- D_\lambda^{k+2} = D^{k+1}$.

Then $\mathfrak{M} \cup (L \times D^{k+3}) = \bigcup (M_\lambda \cup D_\lambda^{k+2}) = \bigcup M_\lambda'$

gives the appropriate book structure on S^{k+l+1}. Here $M_\lambda \cup D_\lambda^{k+2}$ denotes union along D_λ.

This completes the proof of the Proposition.

6. Fiber Structure and Seifert Pairing

This section continues the discussion of section 5. Given $K \otimes L \subset S^{k+l+1}$, we use 5.2 to examine submanifolds of S^{k+l+1} with boundary $K \otimes L$.

We continue using the notation of section 5. $K \subset S^k$ is an arbitrary knot with spanning manifold F and map $a: D^{k+1} \longrightarrow D^2$ as in 5.1 ($(0,$ [redacted] $\subset D^2$ are regular values for a; $a^{-1}(0) \simeq CK$, $a^{-1}(1) = F$.). The knot $L \subset S^l$ is the link of a branched fibration $b: D^{l+1} \longrightarrow D^2$.

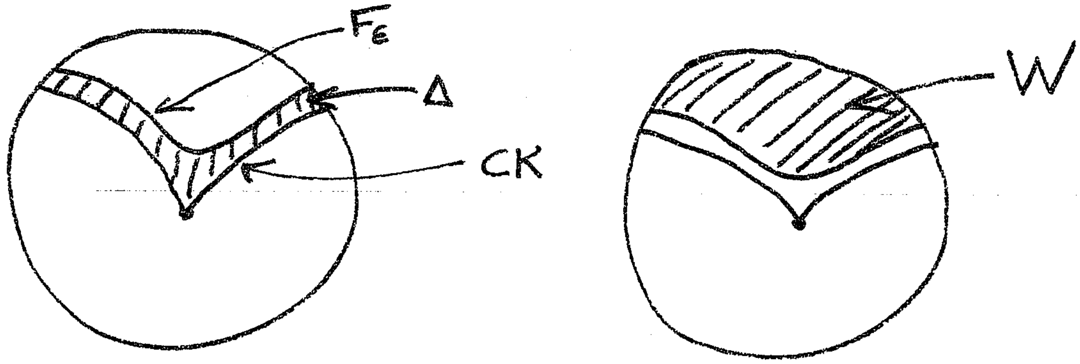
As we saw in section 5, $E_{K \otimes L} \xleftarrow{\rho_\simeq} \mathbb{E}$ where $\mathbb{E}$ is the b-branched fibration of $D^{k+1} \times S^1$ along $j(E_K)$. Let $\mathbb{F}^* \subset \mathbb{E}$ be defined by $\mathbb{F}^* = \pi^{-1}(D^{k+1} \times 1)$. Here $\pi: \mathbb{E} \longrightarrow D^{k+1} \times S^1$ is the branched fibration of 5.2. Then $\pi: \mathbb{F}^* \longrightarrow D^{k+1}$ is a branched fibration along $F_\varepsilon = a^{-1}(\varepsilon)$. Since ε is a regular value, $\mathbb{F}^*$ is a manifold. $\rho: \mathbb{F}^* \longrightarrow \mathbb{F} = \{(x,y) \in \partial(D^{k+1} \times D^{l+1}) \mid a(x)+b(y) \geq \varepsilon\}$. This shows us that $\mathbb{F}$ is a submanifold of S^{k+l+1} with $\partial \mathbb{F} \simeq K \otimes L \subset S^{k+l+1}$.

<u>Lemma 6.1</u>. There is a subset $\mathfrak{F} \subset \mathbb{F}$ so that

1) $\mathfrak{F} \subset \mathbb{F}$ is a homotopy equivalence.
2) $\mathfrak{F}$ is homotopy equivalent to $F * G$ where G is the leaf of the book structure on S^l.
3) The inclusion $\mathfrak{F} \subset S^{k+l+1}$ is homotopy equivalent to $F * G \subset S^k * S^l$. The symbol, $*$, denotes the join of these spaces.

<u>Proof</u>. Note that $\mathbb{F}^* = \{(x,y) \in D^{k+1} \times D^{\ell+1} \mid a(x) + b(y) = \varepsilon\}$ and $\pi : \mathbb{F}^* \longrightarrow D^{k+1}$ is given by the formula $\pi(x,y) = x$.

Let $\Delta = \{x \in D^{k+1} \mid 0 \le a(x) \le \varepsilon\}$ and $W = \{x \in D^{k+1} \mid \varepsilon \le a(x) \le 1\}$. Thus W is a submanifold of D^{k+1} and $\partial W \simeq F_\varepsilon \cup F$. We may think of W as the trace of pushing F into D^{k+1}. (See Figure 2)



<u>Figure 2</u>

Let $\hat{D}$ be the result of splitting D^{k+1} along $W - F_\varepsilon$. Thus we have an identification map $q : \hat{D} \longrightarrow D^{k+1}$ and $q^{-1}(W) = \hat{W}_+ \cup \hat{W}_-$ with $\hat{W}_+ \cap \hat{W}_- = F_\varepsilon$. In fact (see [ ]) $\hat{D} \simeq D^{k+1}$ with $q^{-1}(W) \simeq$ a submanifold of ∂D^{k+1} obtained by translating F normal to itself in S^k.

Recall that π is a bundle away from the breach set. If we pull back this bundle by the map q it becomes trivial. There is a [redacted] collapse $\psi : G \longrightarrow CL$ induced by the branched fibration $b : D^{\ell+1} \longrightarrow D^2$. Combining these we obtain a surjection $\hat{q} : \hat{D} \times G \longrightarrow \mathbb{F}^*$ so that $\hat{q}(x, g) = \psi g \in CL = \pi^{-1}(q(x))$ for $x \in \hat{F}_\varepsilon$ and $\hat{q}(x_+, g) = \hat{q}(x_-, h(g))$ where $q(x_+) = q(x_-) \in W$ and $h : G \longrightarrow G$ is the gluing map for the fibration over S^1 associated with the book structure on $\mathcal{L}$. There are no other identifications outside

$\hat{W} \times \mathbb{G}$. Thus we have the following diagram.

$$\begin{tikzcd} \hat{D} \times \mathbb{G} \arrow[r, "\hat{q}"] \arrow[d] & \mathbb{F}^* \arrow[d, "\pi"] \\ \hat{D} \arrow[r, "q"] & D^{k+1} \end{tikzcd}.$$

Since $\bar{q}^{-1}(\Delta)$ is homotopy equivalent to $CF \subset D^{k+1}$, it is then not hard to see that $\pi^{-1}(\Delta) \subset \mathbb{F}^*$ is a homotopy equivalence.

Let $\mathbb{F} = \rho(\pi^{-1}(\Delta)) \subset \mathbb{F}$. Then $\mathbb{F} = \{(x,y) \in \partial(D^{k+1} \times D^{l+1}) \mid a(x)+b(y) \geq \varepsilon,\ 0 \leq a(x), b(x)\}$. It is clear from our description that $\mathbb{F}$ has the homotopy type of $F * \mathbb{G}$. Similar analysis shows that $\mathbb{F} \subset S^{k+l+1}$ is equivalent to $F * \mathbb{G} \subset S^k * S^l$. This completes the proof.

<u>Remark</u>. If $\mathbb{K}$ also has an open book structure with gluing map $h' : F \longrightarrow F$, then a similar argument shows that the corresponding gluing map $h'' : \mathbb{F} \longrightarrow \mathbb{F}$, when restricted to $F * \mathbb{G}$ is given by $h' * h : F * \mathbb{G} \longrightarrow F * \mathbb{G}$.

Lemma 6.1 will allow us to extract algebraic information about $K \otimes L \subset S^{k+l+1}$. Recall that if $k = 2n+1$ and $l = 2m+1$, then we have <u>Seifert pairings</u>

$$\Theta_K : \mathcal{H}_n(F) \times \mathcal{H}_n(F) \longrightarrow \mathbb{Z}$$
and
$$\Theta_L : \mathcal{H}_m(\mathbb{G}) \times \mathcal{H}_m(\mathbb{G}) \longrightarrow \mathbb{Z}.$$

Here $\mathcal{H}_*$ denotes the torsion free part of the reduced homology.

The Seifert pairing is defined by the formula $\theta(x,y) = \ell(i_*x, y)$ where i_*x is the result of translating x into the complement of the spanning manifold along the positive normal direction. The symbol $\ell(\ ,\ )$ denotes linking number in the ambient sphere. Actually, this pairing depends upon the choice of spanning manifold for the knot. When we write Θ_K, Θ_L and $\Theta_{K\otimes L}$, we understand that these pairings are taken for the specific manifolds F, $\mathbb{G}$, and $\mathbb{F}$ respectively.

<u>Proposition 6.2</u>. Let $\mathbb{K}$ and $\mathbb{L}$ be given as above with $K = \partial F$, $L = \partial \mathbb{G}$ and $K\otimes L = \partial \mathbb{F}$. Suppose that $k = 2n+1$, $\ell = 2m+1$, and suppose that $\mathbb{G}$ is $(m-1)$ connected with $\mathcal{H}_m(\mathbb{G})$ generated by (combinatorially) embedded spheres.

Then $\Theta_{K\otimes L} \simeq (-1)^{(n+1)(m+1)}\, \Theta_K \otimes \Theta_L$.

That is, $\mathcal{H}_{n+m+1}(\mathbb{F}) \simeq \mathcal{H}_n(F) \otimes \mathcal{H}_m(\mathbb{G})$ and $\Theta_{K\otimes L}(a\otimes a', b\otimes b') = (-1)^{(n+1)(m+1)}\Theta_K(a,b)\,\Theta_L(a',b')$ for $a,b \in \mathcal{H}_n(F)$ and $a',b' \in \mathcal{H}_m(\mathbb{G})$.

(This result can be generalized. Its present form is sufficient for our purposes.)

<u>Proof</u>. The isomorphism $\mathcal{H}_n(F)\otimes\mathcal{H}_m(\mathbb{G}) \longrightarrow \mathcal{H}_{n+m+1}(F*\mathbb{G})$ is given by $[a]\otimes[b] \longmapsto [a*b]$. It is an isomorphism by our hypotheses. Since $\mathcal{H}_{n+m+1}(F*\mathbb{G}) \simeq \mathcal{H}_{n+m+1}(\mathbb{F})$ this checks the decomposition of the homology of $\mathbb{F}$. The rest follows from the definition of the Seifert Pairing, the fact that $\mathbb{F} \subset S^{k+\ell+1}$ is equivalent to $F*\mathbb{G} \subset S^k * S^\ell$, and the following lemma.

<u>Lemma 6.3</u>. Let α, β be disjoint n-cycles in S^k and α', β' be disjoint m-spheres embedded in S^l. Then, in $S^k * S^l = S^{k+l+1}$,

$$\ell(\alpha * \alpha', \beta * \beta') = (-1)^{(n+1)(m+1)} \ell(\alpha, \beta)\, \ell(\alpha', \beta') .$$

<u>Proof</u>. By definition, $\ell(x, y) = \langle x, Y \rangle$ where $\langle , \rangle$ denotes intersection number in the appropriate sphere, and $\partial Y = y$. Note that $\partial(A * B) = (\partial A) * B \pm A * (\partial B)$. Thus, if $\partial B = \beta$, then $\partial(B * \beta') = \beta * \beta'$. Hence $\ell(\alpha * \alpha', \beta * \beta') = \langle \alpha * \alpha', B * \beta' \rangle$.

Regarding $A * B = (CA \times B) \cup (A \times CB)$ we see that this last intersection occurs only at the right-hand cone point where α' and β' are collapsed to points. Assuming that α and B intersect transversely, we see that for each $x \in \alpha \cap B$ there is a contribution $\varepsilon(x) \cdot \langle c\alpha', c\beta' \rangle$. Since (using the Lefschetz orientation convention) coordinates for $c\alpha'$ and for B must be transposed, we see $\varepsilon(x) = (-1)^{(n+1)(m+1)} \eta(x)$ where $\eta(x)$ is the local contribution to $\langle \alpha, B \rangle$. That is, $\langle \alpha, B \rangle = \sum_{x \in \alpha \cap B} \eta(x)$.

$$\text{Hence } \ell(\alpha * \alpha', \beta * \beta') = (-1)^{(n+1)(m+1)} \langle \alpha, B \rangle \cdot \langle c\alpha', c\beta' \rangle$$

$$= (-1)^{(n+1)(m+1)} \ell(\alpha, \beta)\, \ell(\alpha', \beta') .$$

This completes the proof of the proposition.

<u>Example</u>. If we apply this Proposition to the cyclic suspensions then we obtain the following result.

<u>Proposition 6.3</u>. Let $K \subset S^{2n+1}$ be an arbitrary knot with spanning manifold F and corresponding Seifert pairing $\Theta : \mathcal{H}_n(F) \times \mathcal{H}_n(F) \longrightarrow \mathbb{Z}$. Then the a-fold cyclic suspension $K \otimes [a] \subset S^{2n+3}$ spans a manifold $F_a \subset S^{2n+3}$ with F_a the homotopy type of $F * (\mathbb{Z}/a\mathbb{Z})$ and Seifert pairing $\Theta_a \simeq (-1)^{(n+1)} \Theta \otimes \Lambda_a$ where Λ_a denotes the Seifert pairing for the empty knot of degree a.

Since the empty knot has spanning manifold consisting of a-points in S^1 it is easy to see that Λ_a has matrix form

$$\Lambda_a = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & -1 \\ & & & & 1 \end{bmatrix} \quad (\text{size } (a-1) \times (a-1)).$$

This general result about the Seifert pairing may be used to deduce the Signature periodicity theorem of [N] by using the method outlined in ([P], section 3).

By iteration we find that the Seifert pairing of $[a_0] \otimes [a_1] \otimes \cdots \otimes [a_n]$ is $\pm \Lambda_{a_0} \otimes \Lambda_{a_1} \otimes \cdots \otimes \Lambda_{a_n}$. As we shall see in the next section this is the (well-known) computation of the Seifert pairing for the Brieskorn knots.

7. Products are Associative

This section is devoted to the proof that the knot product operation is associative. We proceed by the following lemmas.

Lemma 7.1. Let $K \subset S^k$, $L \subset S^l$ and $T \subset S^t$ be given with book structures on at least two of these knots. Then as knots, $(\mathbb{K} \otimes \mathbb{L}) \otimes \mathbb{T} \simeq (\mathbb{K} \otimes \mathbb{T}) \otimes \mathbb{L}$.

Proof. Since $\otimes$ is commutative, it will suffice to assume that $\mathbb{L}$ has a book structure. Let $\tau: D^{l+1} \longrightarrow D^2$ be the branched fibration for $\mathbb{L}$. We will use the following notation: Given a knot $K \subset S^k$, $\pi: M(K, \tau) \longrightarrow S^k$ is the τ-branched fibration of S^k along K.

Recall how we constructed the embedding $K \otimes T \subset S^{k+t+1}$. The triple $K \subset S^k \subset S^{k+2}$ induces an embedding $K \otimes T \subset S^k \otimes T \cong S^{k+t+1}$. Using Proposition 3.8, we have $K \otimes T = (K \times D^{t+1}) \cup (E_K \times_{S^1} E_T) \cup (D^{k+1} \times T)$ and $S^{k+t+1} \cong S^k \otimes T = (S^k \times D^{t+1}) \cup ((D^{k+1} \times S^1) \times_{S^1} E_T) \cup (D^{k+1} \times D^2 \times T)$. Then the pieces embed as follows:

(i) $K \times D^{t+1} \longrightarrow S^k \times D^{t+1}$ is the inclusion $\times 1_{D^{t+1}}$.

(ii) $E_K \times_{S^1} E_T \longrightarrow (D^{k+1} \times S^1) \times_{S^1} E_T$ is induced by $j: E_K \longrightarrow D^{k+1} \times S^1$ where $j(x) = (ix, \phi(x))$ (i pushes E_K into the interior of D^{k+1} by a small amount).

(iii) $D^{k+1} \times T \longrightarrow D^{k+1} \times D^2 \times T$ is the inclusion into $D^{k+1} \times 0 \times T$.

The reader should note that this embedding has been expressed abit differently from the version in 3.8. The same arguments apply to give this embedding.

Consider $\pi: M(K \otimes T, \tau) \longrightarrow S^{k+t+1}$. View it over each of the pieces.

Then (i) $\pi^{-1}(S^k \times D^{t+1}) \cong M(K,\nu) \times D^{t+1}$

(ii) $\pi^{-1}((D^{k+1}\times S^1)\times_{S^1} E_T) \cong E_{K\otimes L} \times_{S^1} E_T$
This follows from [redacted] Proposition 5.2.
It is the key to this proof.

(iii) $\pi^{-1}(D^{k+1}\times D^2 \times T) \cong D^{k+1}\times D^{\ell+1}\times T$.

Thus $M(K\otimes T,\nu)$ = the union of these pieces.

Now $(K\otimes T)\otimes L = M(K\otimes T,\nu)\cup(D^{k+t+1}\times L)$

$$\cong ((M(K,\nu)\cup(D^{k+1}\times L))\times D^{t+1})\cup(E_{K\otimes L}\times_{S^1}E_T)\cup(D^{k+\ell+2}\times T)$$

$$\cong ((K\otimes L)\times D^{t+1})\cup(E_{K\otimes L}\times_{S^1}E_T)\cup(D^{k+\ell+2}\times T)$$

$$\cong (K\otimes L)\otimes T.$$

Thus we have shown that $(K\otimes L)\otimes T \cong (K\otimes T)\otimes L$.

To see that this is an equivalence of knots, apply the same arguments simultaneously to the embeddings in the form of the diagram below.

$$\begin{tikzcd}
(K\otimes L)\otimes T \arrow[r, hook] \arrow[d, "\cong"] & (S^{k+2}\otimes L)\otimes T \cong S^{k+\ell+t+2} \arrow[d, "\cong"] \\
(K\otimes T)\otimes L \arrow[r, hook] & (S^{k+2}\otimes T)\otimes L \cong S^{k+\ell+t+2}
\end{tikzcd}$$

This completes the proof that
$(\mathbb{K}\otimes\mathbb{L})\otimes\mathbb{T} \cong (\mathbb{K}\otimes\mathbb{T})\otimes\mathbb{L}$.

<u>Theorem 7.2</u>. The product operation is associative. That is, given three knots $\mathbb{K}$, $\mathbb{T}$, and $\mathbb{L}$ so that at least two have book structures, then
$$(\mathbb{K}\otimes\mathbb{T})\otimes\mathbb{L} \simeq \mathbb{K}\otimes(\mathbb{T}\otimes\mathbb{L}).$$

<u>Proof</u>. This follows directly from commutativity of $\otimes$ and the Lemma.

$$\begin{aligned}(\mathbb{K}\otimes\mathbb{L})\otimes\mathbb{T} &\simeq (\mathbb{K}\otimes\mathbb{T})\otimes\mathbb{L}\\ &\simeq (\mathbb{T}\otimes\mathbb{K})\otimes\mathbb{L}\\ &\simeq (\mathbb{T}\otimes\mathbb{L})\otimes\mathbb{K}\\ &\simeq \mathbb{K}\otimes(\mathbb{T}\otimes\mathbb{L})\\ &\simeq \mathbb{K}\otimes(\mathbb{L}\otimes\mathbb{T}).\end{aligned}$$

<u>Remark</u>. Note that this associativity property shows that given a knot $\mathbb{K}$ and books $\mathbb{B}_1\ \mathbb{B}_2, \ldots, \mathbb{B}_n$ then to form $(((\mathbb{K}\otimes\mathbb{B}_1)\otimes\mathbb{B}_2)\otimes\cdots)\otimes\mathbb{B}_n$ it suffices to form the book $\mathbb{B} = \mathbb{B}_1\otimes\cdots\otimes\mathbb{B}_n$ and the take $\mathbb{K}\otimes\mathbb{B}$.

For example, it follows from section 4 that $\mathbb{L}ink(z_0^{a_0}+z_1^{a_1}+\cdots+z_n^{a_n}) \simeq [a_0]\otimes\cdots\otimes[a_n]$. Hence $\mathbb{K}\otimes[a_0]\otimes\cdots\otimes[a_n]$ may be interpeted as the result of a sequence of cyclic suspensions or as the tensor product of $\mathbb{K}$ with the Brieskorn book $L(\Sigma z_i^{a_i})\subset S^{2n+1}$.

8. Spinning a Generalization of Zeeman's Theorem

In this section, the product construction will be used to produce a generalization of a theorem of Zeeman [Z] about twist spinning knots.

Definition 8.1. Let M^n be a closed manifold. Then $\mathcal{P}(M)$ (punctured M) is defined to be the manifold with boundary S^{n-1}, obtained by removing an n-disk from M.

Theorem 8.2 (Zeeman). Let $K \subset S^k$ be a spherical knot. Then S^{k+1} has an open book structure with binding $\mathcal{A}a(K)$ the a-twist spin of K $(a = 1, 2, 3, 4, \ldots)$, and leaf $F_\omega \simeq \mathcal{P}(K \otimes [a])$.

This is Zeeman's result, phrased in our language. We generalize this as follows. Let $a: S^m \longrightarrow D^2$ be an open book structure on S^m. Then we shall define the a-twist spin of K, $\mathcal{A}a(K) \subset S^{n+m}$, another spherical knot. We then show that S^{n+m} has book structure with binding $\mathcal{A}a(K)$ and leaf $F_\omega \simeq \mathcal{P}(K \otimes a)$.

It is necessary to recall the definition of twist spinning. First recall the notion of spinning a knot: Given a spherical knot $K \subset S^n$ we obtain a knotted disk pair $(D^n, \mathcal{D})$ by removing a small ball from S^n which intersects K transversely in an unknotted disk. Thus we may assume that $\partial\mathcal{D} = S^{n-3} \subset S^{n-1} = \partial D^n$, the standard embedding. The knot K is then spun through many dimensions (no twisting here) to $\mathcal{A}(K) \subset S^{n+m}$ via the following definition.

<u>Definition 8.3</u>. Given $K \subset S^n$ and $(D^n, \mathcal{D})$ as above, define the <u>(m) spin</u> of K, $\mathcal{S}(K) \subset S^{n+m}$ by the formula

$$\mathcal{S}(K) = (\mathcal{D} \times S^m) \cup (S^{n-3} \times D^{m+1}) \subset (D^n \times S^m) \cup (S^{n-1} \times D^{m+1}) = S^{n+m}.$$

Thus 1-spinning a knot in S^3 corresponds to holding the ends of the arc $\mathcal{D}$ fixed while spinning the arc in a direction orthogonal to the given 3-space directions. Standard twist-spinning is a generalization of 1-spinning: The disc is twisted some number of times while it spins. [redacted]

To be more specific, note that S^1 acts as a group of rotations of D^n, fixing $D^{n-2} \subset D^n$ pointwise. That is, if $D^n = D^2 \times D^{n-2}$ and $\lambda \in S^1$, then $\lambda(z, x) = (\lambda z, x)$ for $(z, x) \in D^2 \times D^{n-2}$. Thus we may twist $\mathcal{D} \subset D^n$ by any $\lambda \in S^1$, obtaining $\lambda\mathcal{D} \subset D^n$.

<u>Definition 8.4</u> Let $a = 0, 1, 2, 3, \ldots$ and $K \subset S^n$ be as above. Then the <u>a-twist spin</u> of K is the knot $\mathcal{S}_a(K) \subset S^{n+1}$ defined by

$$\mathcal{S}_a(K) = \{(\lambda^a d, \lambda) \mid \lambda \in S^1, d \in \mathcal{D}\} \cup (S^{n-3} \times D^{m+1});$$

$$\mathcal{S}_a(K) \subset (D^n \times S^1) \cup (S^{n-1} \times D^{m+1}) = S^{n+1}.$$

Of course, $\mathcal{S}_0(K) \subset S^{n+1}$ is the ordinary spin of K through one dimension. It follows from Zeeman's theorem that $\mathcal{S}_1(K) \subset S^{n+1}$ is unknotted. It bounds a disk which may be constructed directly.

[redacted]

It will be helpful to recall why the 1-twist spin bounds a disk.

Lemma 8.5. Let $K \subset S^n$ be a spherical knot. Then S^{n+1} has a book structure with leaves Γ_λ ($\lambda \in S^1$) so that $\Gamma_\lambda \simeq D^n$ and $\partial \Gamma_\lambda = \mathcal{A}_1(K) \subset S^{n+1}$.

Proof. There is a map $\alpha : D^n - \mathcal{D} \longrightarrow S^1$ representing a generator for $H^1(D^n - \mathcal{D}; \mathbb{Z})$. We may choose it so that $F_\theta = \overline{\alpha^{-1}(\theta)}$ has singularities (if any) away from its collared boundary, and $\partial F_\theta = \mathcal{D} \cup D^{n-2}_\theta$. (Here D^{n-2}_θ corresponds to $D^{n-2} \times \theta$ in $S^{n-1} = (D^{n-2} \times S^1)/\sim$.)

The action of S^1 on D^n moves these fibers so that $\partial(\lambda F_\theta) = \lambda \mathcal{D} \cup F_{\lambda\theta}$.

$$\text{Let } \Gamma_\lambda = \left(\bigcup_{\theta \in S^1} (\theta F_{\theta^{-1}\lambda}) \times \theta \right) \cup (D^{n-2}_\lambda \times D^2).$$

Then $\Gamma_\lambda \subset (D^n \times S^1) \cup (S^{n-1} \times D^2)$ is a submanifold and $\partial \Gamma_\lambda = \mathcal{A}_1(K) \subset S^{n+1}$. As in (   ) Γ_λ may be identified with $S^n - N(\mathcal{D})$, where $N(\mathcal{D})$ is a regular neighborhood of $\mathcal{D}$ in S^n. This completes the proof sketch of the lemma.

Remark. Zeeman uses the same construction as in 8.5 to prove his theorem.

Since the book structure of 8.5 is a fibering by disks, we may choose an isomorphism with the standard book structure on S^{n+1}. It then follows that there is a book structure on D^{n+2} with binding $\Gamma \subset D^{n+2}$, a properly embedded disk, so that $\partial \Gamma = \mathcal{A}_1(K)$. Let the leaves of this book be denoted W_λ. Thus $\partial W_\lambda = \Gamma \cup \Gamma_\lambda$ and the book restricts to the book of 8.5 along ∂D^{n+2}.

<u>Definition 8.6</u>. Let $a: S^m \longrightarrow D^2$ be an open book with binding $L \subset S^m$, and $S^m = E_L \cup (L \times D^2)$. Let $K \subset S^n$ be a spherical knot, and $\mathcal{D} \subset D^n$ the corresponding disc pair.

Let

$$\Delta = \{(a(x)d, x) \mid d \in \mathcal{D}, x \in E_L\} \cup (\partial\mathcal{D} \times D^{m+1}).$$

Then $\partial\Delta = L \times \mathcal{A}_1(K) \subset (D^n \times (L \times S^1)) \cup (S^{n-1} \times (L \times D^2))$.

Since $\mathcal{A}_1(K) = \partial\Gamma$ with $\Gamma \subset D^n \times D^2 = D^{n+2}$ (remark above) we may regard $L \times \Gamma \subset S^{n+m}$ and define the <u>a-twist spin</u> of $K \subset S^n$

to be the knot $\mathcal{A}_a(K) = \Delta \cup (L \times \Gamma) \subset S^{n+m}$.

<u>Remark</u>. $\mathcal{A}_a(K) \subset S^{n+m}$ is spherical. If $a: S^1 \to S^1$ $(m=1)$ represents the empty knot of degree $a = 1, 2, \ldots$, then we recover the ordinary a-twist spin.

<u>Theorem 8.7</u>. Let $a: S^m \to D^2$ be a book with binding L, and let $K \subset S^n$ be a spherical knot. Then S^{n+m} has a book structure with binding $\mathcal{A}_a(K)$ and leaf $\varphi(K \otimes a)$.

<u>Proof</u>. We use the same notation as in the proof of 8.5.

$$\text{Let} \quad X_\lambda = \left(\bigcup_{x \in E_L} (a(x) F_{a(x)^{-1}\lambda}) \times x \right) \cup (D^{n-2}_\lambda \times D^{m+1}).$$

Then $X_\lambda \subset S^{n+m}$ is a submanifold with the following properties.

(i) If $a: S^1 \to S^1$ $(m=1)$ is the identity map, then $X_\lambda = \Gamma_\lambda$.

(ii) For any m, $\partial X_\lambda = \Delta \cup (L \times \Gamma_\lambda)$ (Δ as in 8.6).

(iii) For $\lambda \neq \lambda'$, $X_\lambda \cap X_{\lambda'} = \Delta$.

Now note that $X_\lambda \subset S^{n+m} - (D^n \times (L \times D^2))$ while $L \times W_\lambda \subset D^n \times (L \times D^2)$. Since $\partial W_\lambda = \Gamma_\lambda \cup \Gamma$ we may form $Y_\lambda = X_\lambda \cup (L \times W_\lambda)$.

(iv) $\partial Y_\lambda = \Delta \cup (L \times \Gamma) = \mathcal{A}_a(K)$

(v) The manifolds Y_λ form the leaves of a book structure on S^{n+m} with binding $\mathcal{A}_a(K)$.

Thus it remains to show that $Y_\lambda \cong \varphi(K \otimes a)$. To see this, note

(vi) $L \times W_\lambda \cong D^{n+1} \times L$

(vii) $X_\lambda \cong ((K - \mathring{D}) \times D^{m+1}) \cup (E_K \times_{S^1} E_L)$

Thus $Y_\lambda \cong K \otimes L - \mathring{D} \times D^{m+1} \cong K \otimes L - \text{disc} \cong \varphi(K \otimes L)$.

This completes the proof.

<u>Remark</u>. We may now turn around and obtain the open book structure on S^{n+m} in another way. Since $\mathcal{A}_1(K) \subset S^{n+1}$ so that $\mathcal{A}_1(K) \cap (D^n \times \theta) = \partial \mathcal{D} \times \theta$, we may form a ~~generalized~~ a-branched fibration of S^{n+1} along $\mathcal{A}_1(K)$ and this will induce a book structure on the resulting product. Since $\mathcal{A}_1(K)$ is trivial the product $\mathcal{A}_1(K) \otimes a \cong S^{n+m}$.

As in 5.3, the inclusion of $S^{n-1} \subset S^{n+1}$ (the axis for the standard book structure) induces an embedding $S^{n-3} \otimes a \simeq S^{n+m-2} \xrightarrow{t} S^{n+m}$. This embedding is the binding of the induced book on S^{n+m}. It follows from our theorem that $(S^{n+m}, t(S^{n+m-2})) \simeq (S^{n+m}, \mathcal{A}_a(K))$.

The a-twist spin of K is obtained by a-branching S^{n+1} along $\mathcal{A}_1(K)$ and taking the "inverse image" of the axis $S^{n-1} \subset S^{n+1}$.

9. Geometric Knot Periodicity

Let C_n denote the Levine cobordism (concordance) group of spherical knots in S^{2n+1}. It is well known that for $n \geqslant 3$, $C_n \simeq C_{n+4}$ ([L]). Recently, [redacted] constructions for this periodicity isomorphism have been given by G. Bredon ([B]) and by S. Capell and J. Shaneson ([C]). We shall add a third such construction to the list.

Definition 9.1. Let $\Lambda \subset S^3$ denote the link of two unknotted circles as depicted in Figure 3. This is a fibered link. In fact, $\Lambda = [2] \otimes [2] = \mathbb{L}\mathrm{ink}(z_1^2 + z_2^2)$. It accordingly has Seifert matrix of form $\Lambda_2 \otimes \Lambda_2 = (1) \otimes (1) = (1)$.

Theorem 9.2. Let $\omega : C_n \longrightarrow C_{n+4}$ by the formula $\omega(\mathbb{K}) = \mathbb{K} \otimes \Lambda$ where Λ is the fibered link of 9.1. Then ω is an isomorphism for $n \geqslant 3$.

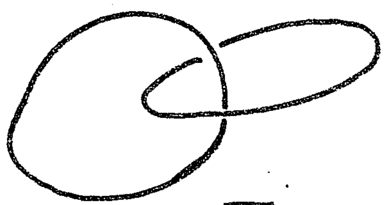
Proof. By ( ) it is sufficient to see that $\omega(\mathbb{K})$ and $\mathbb{K}$ have the same Seifert Pairing, up to sign. Since Θ_Λ has matrix (1), we see from 6.2 that

$$\Theta_{\mathbb{K} \otimes \Lambda} = (-1)^{n+1}\, \Theta_{\mathbb{K}} .$$

This proves the theorem.

It is interesting to note that for n even, $\omega(\mathbb{K})$ has the same Seifert Pairing and that for n-odd $\Theta_{\mathbb{K}\otimes\Lambda} = -\Theta_{\mathbb{K}}$. This means that for n-odd $\omega(\mathbb{K})$ should be replaced by its mirror image to obtain the most satisfactory isomorphism.

It is also worth noting that $\omega(\mathbb{K})$ is obtained from $\mathbb{K}$ by performing two order-2 cyclic suspensions. Iteration of ω amounts to tensoring with the Brieskorn books $\mathbb{L}(z_1^2 + \cdots + z_\ell^2)$.



$\Lambda \subset S^3$

Figure 3

The knots obtained by iterating ω become progessively more and more symmetric.

<u>Lemma 9.3</u>. There is a (topological) action of $O(n)$ on S^{k+2n} leaving $K \otimes \Lambda_n \subset S^{k+2n}$ invariant (where $K \subset S^k$, $\Lambda_n = \mathrm{Link}(z_1^2 + \cdots + z_n^2)$).

<u>Proof</u>. First note that $O(n)$ acts on $\mathbb{C}^n$ via $\mathbb{C}^n \cong \{x + iy \mid x, y \in \mathbb{R}^n\}$ and $g \cdot (x + iy) = gx + igy$ for $g \in O(n)$. This action restricts to S^{2n-1} an leaves Λ_n invariant. It is also compatible with the book structure corresponding to this singularity, so that $S^{2n-1} = E\Lambda_n \cup (\Lambda_n \times D^2)$ and $O(n)$ leaves $E\Lambda_n$ invariant and acts on $\Lambda_n \times D^2$ via $g(\lambda, z) = (g\lambda, z)$ (Λ_n is identified as an orbit of the $O(n)$-action).

Now

$$K \otimes \Lambda_n = (K \times D^{2n}) \cup (E_K \times_{S^1} E\Lambda_n) \cup (D^{k+1} \times \Lambda_n)$$

and each piece inherits an action of $O(n)$: (i) $g(k, z) = (k, gz)$ $\quad (k, z) \in K \times D^{2n}$

(ii) $O(n)$ acts on the second factor in each of the other pieces.

This produces an action of $O(n)$ on $K\otimes\Lambda_n$. Since this pasting viewpoint doesn't show that the action is smooth, we only say now that it is topological.

Now note that $\mathbb{K}\otimes\Lambda_{n+1} = \mathbb{K}\otimes(\Lambda_n\otimes[z]) = (\mathbb{K}\otimes\Lambda_n)\otimes[z]$. Hence $\mathbb{Z}_2$ acts on $K\otimes\Lambda_{n+1}$ as the matrix $\mathrm{diag}(1,\ldots,1,-1)\in O(n)$ and $K\otimes\Lambda_{n+1}/\mathbb{Z}_2 \simeq S^{k+2n}$ while the set of fixed points is $K\otimes\Lambda_n \simeq (K\otimes\Lambda_{n+1})^{\mathbb{Z}_2}$. Thus

$$\mathbb{K}\otimes\Lambda_n = (S^{k+2n}, K\otimes\Lambda_n) = \left(\frac{K\otimes\Lambda_{n+1}}{\mathbb{Z}_2}, (K\otimes\Lambda_{n+1})^{\mathbb{Z}_2}\right).$$

Since the last pair has the required $O(n)$ action, this proves the lemma.

<u>Remark</u>. It is not hard to show that $(K\otimes\Lambda_n)/O(n)$ is homeomorphic to D^{k+1} with the set of fixed points corresponding to $K\subset\partial D^{k+1}$. Thus, topologically, we have constructed the regular $O(n)$-manifolds of ( ). [redacted]

To see that the action of $O(n)$ can be made smooth requires a closer look at the differentiable structure on $K\otimes\Lambda_n$. We shall discuss this point in the last section.

We close this section with a problem. Capell and Shaneson have described the geometric knot periodicity as follows. Given $K\subset S^k$, form $K\times\mathbb{CP}^2\subset S^k\times\mathbb{CP}^2$ ($\mathbb{CP}^2$ = complex projective space). They then show that, up to concordance, this codemension-two embedding in $S^k\times\mathbb{CP}^2$ may be obtained from the standard embedding $S^{k-2}\times\mathbb{CP}^2\subset S^k\times\mathbb{CP}^2$ by taking connected sum with

a "local knot" $K' \subset S^{k+4}$. They then define $\omega' : C_n \longrightarrow C_{n+4}$ by $\omega'((S^k, K)) = (S^{k+4}, K')$ and show that it is an isomorphism.
<u>The problem</u>: Construct the Capell-Shaneson concordance explicitly in such a way that $(S^{k+4}, K') \simeq (S^{k+4}, K \otimes \Lambda)$. That is, show that our construction corresponds to theirs <u>explicitly</u>.

10. Smooth $O(n)$-Actions

In this section we examine the $O(n)$-action on $K \otimes \Lambda n$ more carefully, showing how it may be done in the differentiable category. For this, it is helpful to look at the models for some standard $O(n)$ actions.

Let $O(n)$ act on $\mathbb{C}^n$ as in 9.3. Then following Bredon ([B]) we have that $\mathbb{C}^n/O(n) \simeq \{(y, v) \in \mathbb{R} \times \mathbb{C} \mid y \geq |v|\} = B$ with orbit map $\pi: \mathbb{C}^n \longrightarrow B$ given by the formula $\pi(z) = (\|z\|^2, (z_1^2 + \cdots + z_n^2))$. Thus B is an [redacted] unbounded solid cone and hence homeomorphic to $\mathbb{R}^+ \times$ [redacted] $\mathbb{C}$ where $0 \times \mathbb{R}^2$ corresponds to the boundary of the cone. One then obtains a smooth orbit map $p: \mathbb{C}^n \longrightarrow \mathbb{R}^+ \times \mathbb{C}$.

Now $\pi(D^{2n})$ is homeomorphic to the cone CD^2 with $\pi(S^{2n-1}) = D^2$. In $\mathbb{R}^+ \times \mathbb{C}$, CD^2 corresponds to a subset D^3_+ as in Figure 4. We write $\partial D^3_+ = D^2 \cup E^2$. Here $E^2 \subset 0 \times \mathbb{C}$ and corresponds to the spherical orbits and the fixed point $0 \in \mathbb{C}^n$.

Figure 4

Consider the orbit map $\pi: S^{2n-1} \longrightarrow D^2$. This is not a book structure on S^{2n-1}. However if $g: D^2 \longrightarrow D^2$ collapses a collar [redacted] neighborhood of the boundary (radially), then $g \circ \pi: S^{2n-1} \longrightarrow D^2$ is a book structure. In fact it is the Milnor book structure for the singularity $z_1^2 + z_2^2 + \cdots + z_n^2$.

Let $\mathrm{proj}: CD^2 \longrightarrow D^2$ be the projection with formula $\mathrm{proj}(x,t) = tx$. Let $cq: CD^2 \longrightarrow CD^2$ be given by $cq(x,t) = (qx, t)$. Then $(\mathrm{proj})\circ cq: CD^2 \to D^2$ and let $p': D^3_+ \longrightarrow D^2$ be a smooth version of this map. Then $p'' = p' \circ p : D^{2n} \longrightarrow D^2$ is the branched fibration corresponding to the Σz_i^2 book structure on S^{2n-1}. We have the following diagram

$$\begin{tikzcd} D^{2n} \arrow[dr, "p"] \arrow[dd, "p''"'] & \\ & D^3_+ \cong D^{2n}/O(n) \arrow[dl, "p'"] \\ D^2 & \end{tikzcd}$$

In this way, the orbit map and branched fibration are related.

Bredon's method for constructing $O(n)$-manifolds uses a pull-back construction. Roughly, he proceeds as follows. Given a knot $K \subset S^k$, let $M = D^{k+1} \supset V$ with $\partial V = K$. Then map $\beta: D^{k+1} \longrightarrow D^3_+$ so that (i) $\beta| : S^k \longrightarrow E^2$ is transverse to $0 \in E^2$, and $\beta|^{-1}(0) = K$, (ii) $\beta(\text{Interior } D^{k+1}) \subset D^3_+ - E^2$. Form the pull-back

$$\begin{tikzcd} M(K) \arrow[r] \arrow[d, "\bar{p}"] & D^{2n} \arrow[d, "p"] \\ D^{k+1} \arrow[r, "\beta"] & D^3_+ \end{tikzcd}$$

This is a smooth $O(n)$-manifold with three orbit types so that $\bar{p}: M(K) \longrightarrow D^{k+1}$ is the orbit map. The set of fixed points goes to $K \subset S^k = \partial D^{k+1}$ under $\bar{p}$.

<u>Lemma 10.1</u>. *Let $M(K)$ be the $O(n)$-manifold constructed as above. Then $M(K)$ is diffeomorphic to $K \otimes \Lambda n$.*

<u>Proof</u>. Let $\alpha = p' \circ \beta : D^{k+1} \longrightarrow D^2$. Then we may assume that β has been so chosen that α has $0 \in D^2$ as a regular value and $\alpha^{-1}(0) = V \subset D^{k+1}$. Thus α may be used to construct $K \otimes \Lambda n$.

We form the branched-fibration of D^{k+1} along V.

$$\begin{tikzcd} N = N(V, p'') \arrow[r] \arrow[d, "\pi_N"] & D^{2n} \arrow[d, "p''"] \\ D^{k+1} \arrow[r, "\alpha"] & D^2 \end{tikzcd}$$

By definition, $K \otimes \Lambda_n = \partial N$. Thus we must show that $M(K) \simeq \partial N$.

Now $M(K) = \{ (x,y) \in D^{k+1} \times D^{2n} \mid \beta(x) = p(y) \}$
and $\beta(x) = p(y) \Rightarrow p'\beta(x) = p'p(y) \Rightarrow \alpha(x) = p''(y)$.
Hence $M(K) \subset N$.

$$\begin{tikzcd} M(K) \arrow[r] \arrow[d, "\pi_M"] & D^{2n} \arrow[d, "p"] \arrow[dd, bend left, "p''"] \\ D^{k+1} \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & D^3_+ \arrow[d, "p'"] \\ & D^2 \end{tikzcd}$$

Note that $M = M(K) \simeq \pi_M^{-1}(S^k) \cup \pi_M^{-1}(\mathrm{Int}(D^{k+1}))$.

Since $\beta| : S^{k+1} \longrightarrow E^2$ we see that $\pi_M^{-1}(S^k) \simeq \pi_N^{-1}(S^k)$.

On the other hand $\pi_M^{-1}(\mathrm{Int}(D^{k+1})) \simeq D^{k+1} \times \Lambda_n$. That is, there is only one orbit type over $\mathrm{Int}(D^{k+1})$ and the corresponding bundle is trivial. In the above decomposition of M, this is not a union along boundaries; however the annular collapse included in p' carries the $\cup$ in M to $M \simeq \pi_N^{-1}(S^k) \cup (D^{k+1} \times \Lambda_n)$ where this is a union along the boundary. Thus $M \simeq \pi_N^{-1}(S^k) \cup (D^{k+1} \times \Lambda_n) \simeq \partial N$.
Hence $M \simeq K \otimes \Lambda_n$ as desired. //