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BRANCHED COVERINGS, OPEN BOOKS AND KNOT PERIODICITY

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§1. INTRODUCTION

THIS paper generalizes some properties of hypersurface singularities into the combined contexts of branched covering spaces and open book decompositions.

Perhaps the most striking corollary of this analysis is a completely topological construction of the Brieskorn manifolds $\Sigma(a_0, \dots, a_n) = V(f) \cap S^{2n+1}$ (for $f = z_0^{a_0} + \cdots + z_n^{a_n}$, z_i complex variables, $a_i \geq 1$ positive integers, $V(f) =$ zeros of f in $\mathbb{C}^{n+1}$). These manifolds have been of extraordinary interest in recent years, producing examples of exotic spheres, lens spaces, new group actions and so on.

To explain the approach note that $\Sigma(a_0, a_1, \dots, a_n)$ may be regarded as an a_n-fold cyclic branched covering space of S^{2n-1} with branch set $\Sigma(a_0, a_1, \dots, a_{n-1})$ (see [7] or [13]). Thus there is the following tower:

$$\begin{tikzcd}
 & & & \vdots \arrow[d] \\
 & & \Sigma(a_0, a_1, a_2, a_3) \arrow[r] \arrow[d] & S^7 \\
 & \Sigma(a_0, a_1, a_2) \arrow[r] \arrow[d] & S^5 & \\
\Sigma(a_0, a_1) \arrow[r] & S^3 & &
\end{tikzcd}$$

The vertical maps in this tower are branched coverings; the horizontal maps are the embeddings of the Brieskorn manifolds specified by their definition $V(f) \cap S^{2n+1} \subset S^{2n+1}$. These embeddings appear to depend upon the algebraic nature of the spaces involved.

However, one also knows [12] that there is a smooth fibration $\phi: S^{2n+1} - \Sigma(a_0, \dots, a_n) \to S^1$ and the embedding of Σ in S^{2n+1} may be regarded as its natural placement in $E \cup (\Sigma \times D^2) \simeq S^{2n+1}$ where $E = F \times [0, 1]/[h : F \times 1 \to F \times 0]$, F the closed fiber of ϕ, h the pasting map for

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the fibration over the circle. Thus the embedding $\Sigma \to S^{2n+1}$ arises from an open book structure [20] on the sphere with binding Σ and leaf F.

I generalize this situation as follows. Let S^{2n+1} have a *simple* open book structure (see Definition 2.3) with binding K and leaf F. Let $K(a)$ denote the a-fold cyclic cover of S^{2n+1} with branch set K; $F(a)$ stands for the a-fold cyclic cover of D^{2n+2} with branch set $\hat{F}$, $\hat{F}$ = the result of pushing F (keeping the boundary of F fixed) into D^{2n+2} via an inward normal vector field. Thus $\partial F(a) = K(a)$.

THEOREM. *Under the above conditions there is a good choice of diffeomorphism $H(a)$: $F(a) \to F(a)$ such that $H(a)|_{\partial F(a)} = 1_{\partial F(a)}$ and $S^{2n+3} \simeq E(a) \cup (K(a) \times D^2)$. Here $E(a)$ is the fiber bundle over the circle determined by $H(a)$. Thus S^{2n+3} inherits an open book structure with leaf $F(a)$ and binding $K(a)$.*

Here and throughout the paper $\simeq$ will denote diffeomorphism.

The construction may be iterated. Starting from an arbitrary simple book structure on S^{2n+1} with binding K, define inductively $K(a_1, a_2, \ldots, a_n) = K(a_1, \ldots, a_{n-1})(a_n)$. These are generalized Brieskorn manifolds.

In particular, if K is a torus link of type (a_0, a_1) then there is an easily described book structure on S^3 with binding K. This book coincides with the algebraic book with binding $\Sigma(a_0, a_1)$. Then $\Sigma(a_0, a_1, \ldots, a_n) = K(a_2, \ldots, a_n)$ giving a topological construction for Brieskorn manifolds (see § 7(a)).

The paper is organized as follows. §2 defines simple knots, open books and linking numbers. The Seifert pairing is discussed and an explicit form of Alexander duality [Lemma 2.2] is given. An open book with leaf F^{2n} has a "variation" map $\Delta: H_n(F, \partial F) \to H_n(F)$. Lemma 2.7 relates Δ and the Seifert pairing.

§3 shows (Proposition 3.3) that a simple open book $M^{2n+1} (n > 1)$ is a homotopy sphere if and only if the variation Δ is an isomorphism.

§4 and §5 construct the branched coverings $F(a)$ and $K(a)$. §5 gives a cut and paste description of $F(a)$ which shows that $F(a) \supset \hat{D}$, $\hat{D}$ a "fundamental domain" for the covering action $x: F(a) \to F(a)$ and (i) $F(a) = \bigcup_{i=0}^{a-1} x^i \hat{D}$, (ii) $\hat{D} \simeq D^{2n+2}$. Using this decomposition, Lemma 5.4 and Proposition 5.6 compute the intersection form on $H_{n+1}(F(a))$.

While §4 and §5 are independent of the book structure, §6 assumes that F is the leaf of a book for S^{2n+1}. If h is the monodromy for this book structure then h has an extension $h: D^{2n+2} \to D^{2n+2}$ (Lemma 6.1). Then the monodromy $H(a): F(a) \to F(a)$ is defined (Definition 6.4). Essentially, $H(a) = x \circ \hat{h}$ where $\hat{h}: F(a) \to F(a)$ via $h: \hat{D} \to \hat{D}$ and the decomposition of §5. Theorem 6.6 then shows that the open book determined by $H(a)$ is a homotopy sphere by computing the variation and applying the criterion of §3.

While this completes the proof of the theorem stated in this introduction, actually more is true. Lemma 6.7 computes the Seifert pairing associated with $F(a) \to S^{2n+3}$. This computation may be used to find invariants for generalized Brieskorn manifolds.

§7 contains applications to the classification of book structures on spheres, knot cobordism periodicity for fibered knots, identification of algebraic and topological book structures, and codimension one foliations of spheres.

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§2. LINKING NUMBERS, INTERSECTION NUMBERS, SEIFERT PAIRING

Definition 2.1. A *simple knot* is a pair (S^{2n+1}, K^{2n-1}) where K^{2n-1} is an oriented $(n-2)$-connected submanifold of S^{2n+1}, and there exists an embedding of an $(n-1)$-connected oriented manifold $F^{2n} \subset S^{2n+1}$ such that $\partial F^{2n} = K^{2n-1}$. If K^{2n-1} is a homotopy sphere, the knot is said to be *spherical.* (S^{2n+1} is also oriented and all manifolds will be smooth.)

In case $n = 1$ a simple knot denotes a collection of disjoint circles embedded in S^3. When spherical, it is a knot in the usual sense of the word.

Definition 2.2. An *open book structure* on a closed manifold M^{2n+1} with *binding* K^{2n-1} and *leaf* F^{2n} is a decomposition $M^{2n+1} = E \cup (K \times D^2)$ where $\partial E = K \times S^1$; E is a $(2n+1)$-manifold with boundary which fibers over the circle via $\phi : E \to S^1$, fiber F and $\phi \mid \partial E : K \times S^1 \to S^1$ is projection on the second factor.

Open book structures on manifolds have been considered by various authors [see 5, 19, 20].

Definition 2.3. An open book M^{2n+1} is said to be *simple* if K^{2n-1} is $(n-2)$-connected, F^{2n} is $(n-1)$-connected. A simple book structure on S^{2n+1} will be called a *simple fibered knot.*

Remark. It follows from the definition that the leaf of a simple book has homotopy type of a wedge of n-spheres. For $n \neq 2$ F will in fact be a handlebody. For $n = 2$ we shall *assume* that every two-dimensional homology class can be represented by a (combinatorially) embedded sphere. (This will be true for $F(a)$, F the leaf of an S^3-book, $F(a)$ as described in the introduction.)

Suppose $a, b \subset S^{2n+1}$ are disjoint embedded n-spheres. Assume orientations are chosen for a, b and S^{2n+1}. We define the *linking number* of a and b to be $l(a, b) = \langle ca, cb \rangle$ where ca and cb are radial cones in D^{2n+2} with apex the origin $0 \in D^{2n+2}$. The symbol $\langle\,,\rangle$ denotes intersection number and is well-defined since ca and cb are representatives for the generators of $H_{n+1}(D^{2n+2}, a \cup b)$ and one may choose piecewise linear representatives in their homology classes which intersect transversely. The vanishing of $H_*(D^{2n+2}, a \cup b)$ for $* > n + 1$ then assures that any pair of transverse representatives have the same intersection number. (See the *PL* general position Theorem 5.3 of [15].)

It follows that $l(a, b) = \langle a, B \rangle$ where B is an $(n+1)$-chain on S^{2n+1}, $\partial B = b$ and $\langle\,,\rangle$ denotes intersection number in S^{2n+1}. Also, $l(a, b) = (-1)^{n+1} l(b, a)$.

Given a simple knot $K^{2n-1} \subset S^{2n+1}$ with spanning surface F^{2n} there is a bilinear pairing

(Seifert pairing) $\theta : H_n(F) \times H_n(F) \to \mathbb{Z}$, $\theta(a, b) = l(i_* a, b)$ where $i_* =$ "push off F into $S^{2n+1} - F$ in the positive normal direction".

One also has an intersection pairing $\langle\, , \rangle : H_n(F) \times H_n(F) \to \mathbb{Z}$. This may be defined via the nonsingular Poincaré–Lefschetz duality pairing $f: H_n(F) \times H_n(F, \partial F) \to \mathbb{Z}$. Let $j: H_n(F) \to H_n(F, \partial F)$ be the map induced by inclusion. Then $\langle a, b\rangle = f(a, j(b))$.

There is a well-known relationship between θ and $\langle\, , \rangle$ [see 10]. We include a proof since the technique is useful.

LEMMA 2.1. $(-1)^{n+1}\langle a, b\rangle = \theta(a, b) + (-1)^n\theta(b, a)$.

Proof. Note that via a normal vector field to F we have diffeomorphisms $i_t : F \to S^{2n+1}$ for $-1 \leq t \leq +1$. Here positive t denotes translation in the positive normal direction. We may assume that $i_t | \partial F = 1_{\partial F}$ so that for $F_t = i_t(F)$, $F_t \cap F_{t'} = K$ for $t \neq t'$. Let $i_* = i_{1/2}$, $i^* = i_{-1/2}$ so that $i^* i_* = i_* i^* = 1_F (F \equiv F_0)$. Hence $l(i_* a, b) = l(i^* i_* a, i^* b) = l(a, i^* b)$. Now if $B = \bigcup_{t \in [-1/2, 1/2]} i_t(b)$, then B is an $(n+1)$-chain on S^{2n+1} and, up to sign, $\langle a, b\rangle = \pm\langle a, B\rangle$ where the second intersection number is in S^{2n+1}. Choosing the orientation for B so that $\langle a, b\rangle = \langle a, B\rangle$ means that we regard $B \simeq b \times [-1/2, 1/2]$. Then $\partial B = (-1)^n (b \times (1/2) - b \times (-1/2)) = (-1)^n(i_* b - i^* b)$. By our definition of linking numbers,

$$\begin{aligned}\langle a, B\rangle &= l(a, \partial B) = (-1)^n l(a, i_* b - i^* b)\\ &= (-1)^n((-1)^{n+1} l(i_* b, a) - l(a, i^* b)) = -\theta(b, a) + (-1)^{n+1}\theta(a, b).\end{aligned}$$

Thus $(-1)^{n+1}\langle a, b\rangle = \theta(a, b) + (-1)^n\theta(b, a)$ proving the lemma.

LEMMA 2.2 (Alexander duality). *The linking pairing* $l: H_n(F) \times H_n(S^{2n+1} - \bar{F}) \to \mathbb{Z}$ *is non-singular. (Here* $\bar{F} = F -$ *(collar neighborhood of* ∂F*).) Moreover, if* $\{\hat{a}_i\}$ *is a basis for* $H_n(F, \partial F)$*, dual to a basis* $\{a_i\}$ *for* $H_n(F)$ *in the sense that* $f(a_i, \hat{a}_j) = \delta_{ij}$*, then* $l(a_i, A_j) = \delta_{ij}$ *where* $A_j = (-1)^n(i_* \hat{a}_j - i^* \hat{a}_j)$*. Thus* $\{A_i\}$ *is an Alexander dual basis for* $H_n(S^{2n+1} - \bar{F})$.

Proof. Alexander duality implies that the pairing is non-singular. Using the notation of the previous lemma, let $B_j = \bigcup_{t \in [-1/2, 1/2]} i_t(\hat{a}_j)$. Orient this $(n+1)$-chain so that $\langle a_i, B_j\rangle = f(a_i, \hat{a}_j)$ for all i. Since $i_t | \partial F = 1_{\partial F}$, ∂B_j is an n-cycle on $S^{2n+1} - \bar{F}$. Furthermore, $\partial B_j = A_j$. Thus $\delta_{ij} = f(a_i, \hat{a}_j) = \langle a_i, B_j\rangle = l(a_i, \partial B_j) = l(a_i, A_j)$. This proves the lemma.

LEMMA 2.3. *The following sequence is exact.* $0 \to H_n(K) \to H_n(F) \xrightarrow{j} H_n(F, \partial F) \to H_{n-1}(K) \to 0$. *With respect to the bases* $\{a_i\}$ *and* $\{\hat{a}_j\}$, j *has matrix* N *where* $N_{ij} = \langle a_i, a_j\rangle$.

Proof. Exactness follows from the homology sequence of the pair $(F, \partial F)$, $K = \partial F$, and the given connectivity conditions.

This completes a summary of the algebra associated with a simple knot. Now suppose that $K^{2n-1} \subset S^{2n+1}$ is a simple *fibered* knot with fiber F. Thus S^{2n+1} has an open book structure and $S^{2n+1} = E \cup (K \times D^2)$, $\phi : E \to S^1$. The fibration ϕ is determined by a diffeomorphism $h : F \to F$ such that $h | \partial F = 1\partial F$. The map h will be referred to as the *monodromy* of the open book. In what follows, we shall use the notation developed above for bases an pairings.

Note that h may be regarded as $h_1 : F_0 \to F_0$ where $h_t : F_0 \to F_t$, $F_t = \phi^{-1}(e^{2\pi i t})$, $t \in \mathbb{R}$, $h_{t_1+t_2} = h_{t_1} \circ h_{t_2}$. Thus we may take $i_* = h_{1/2}$, $i^* = h_{-1/2}$ so that $i^* i_* = i_* i^* = h_0 = 1_F$. Each map h_t is the identity when restricted to ∂F_0. Thus $hi^* = h_1 \circ h_{-1/2} = h_{1/2} = i_*$. The following lemmas are direct consequences of the definitions.

LEMMA 2.4. $\theta(a, b) = (-1)^{n+1}\theta(b, ha)$.

LEMMA 2.5. $(-1)^{n+1}\langle a, b\rangle = \theta(a, (I-h)b)$.

LEMMA 2.6. *Let* $V = (V_{ij})$ *(Seifert matrix) where* $V_{ij} = \theta(a_i, a_j)$. *Then* $i^* a_j = \sum_{i=1}^{r} V_{ij} A_i$ *in* $H_n(S^{2n+1} - \bar{F})$.

Now consider the action of h on $H_n(F, \partial F)$. *Since* $h|\partial F = 1_{\partial F}$, $I - h$ *induces a map* $\Delta : H_n(F, \partial F) \to H_n(F)$.

Remark. To avoid complexity of notation, the same symbol will often be used for a map, the induced map on homology and the matrix of the latter with respect to a basis.

LEMMA 2.7. *For a simple fibered knot the Seifert matrix V is unimodular and the matrix of* Δ *(also denoted by* Δ*) with respect to the bases* $\{\hat{a}_i\}$ *and* $\{a_i\}$ *is given by*

$$\Delta = (-1)^{n+1}V^{-1}.$$

Proof. Recall that
$$\begin{aligned} A_i &= (-1)^n(i_*\hat{a}_i - i^*\hat{a}_i) \\ &= (-1)^n i^*(h\hat{a}_i - \hat{a}_i). \end{aligned}$$
Thus
$$A_i = (-1)^{n+1} i^* \Delta(\hat{a}_i).$$

Lemma 2.6 shows that V is the matrix of i^* with respect to the bases $\{a_i\}$ and $\{A_i\}$. Hence the last formula reads $I = (-1)^{n+1}V\Delta$. Thus V is unimodular and the lemma follows.

Note that this last lemma shows that the monodromy of an open book structure on S^{2n+1} determines the Seifert pairing. Classical formulas for the Picard–Lefschetz transformation in algebraic geometry [see 9,14] compute the variation Δ, and hence a Seifert pairing, for special singularities.

§3. WHEN IS A SIMPLE BOOK A FIBERED KNOT?

We wish to find conditions on a simple open book M^{2n+1} which insure that M^{2n+1} is a homotopy sphere. In particular, one wants conditions on the monodromy.

Note that for $n > 1$ a simple book is simply connected. Thus, for $n > 1$, one need only find conditions for M^{2n+1} to be a homology sphere.

PROPOSITION 3.3. *Let* M^{2n+1} *be a simple open book with leaf F, monodromy h and* $n > 1$. *Let* $\Delta : H_n(F, \partial F) \to H_n(F)$ *be the variation described in* §2. *Then* M^{2n+1} *is a homotopy sphere if and only if* Δ *is an isomorphism.*

The proof of this proposition will occur at the end of the section after some preliminary discussion.

Recall the Wang sequence [12] for a fibration $\phi : E \to S^1$ (fiber F^{2n}($n-1$)-connected, monodromy h). $0 \to H_{n+1}(E) \to H_n(F) \xrightarrow{I-h} H_n(F) \to H_n(E) \to 0$. This sequence is exact.

The map on the right is induced by inclusion $F \subset E$. The map $H_{n+1}(E) \to H_n(F)$ is the composite $H_{n+1}(E) \to H_{n+1}(E, F) \simeq H_{n+1}(F \times [0, 1], F \times \{0, 1\}) \simeq H_n(F)$.

LEMMA 3.1. *Let M^{2n+1} be a simple open book with $n > 1$, leaf F and bundle E as above. Then the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 \to & H_n(\partial F) & \to & H_n(F) & \xrightarrow{j} & H_n(F, \partial F) & \xrightarrow{\bar\partial} & H_{n-1}(\partial F) & \to 0 \\ & \downarrow{\scriptstyle \times S^1} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \times S^1} & \\ 0 \to & H_{n+1}(E) & \to & H_n(F) & \xrightarrow{I-h} & H_n(F) & \longrightarrow & H_n(E) & \to 0. \end{array}$$

Here $\times S^1 : H_*(\partial F) \to H_{*+1}(E)$ denotes the composite $H_*(\partial F) \xrightarrow{\times S^1} H_{*+1}(\partial F \times S^1) \xrightarrow{=} H_{*+1}(\partial E) \to H_{*+1}(E)$. Thus, if Δ is an isomorphism then the two sequences are isomorphic.

Proof. The middle square commutes since it follows from the definition of Δ that $\Delta \circ j = I - h$. Commutativity of the left-hand square follows directly from the definition of $H_{n+1}(E) \to H_n(F)$ in the Wang sequence.

To see commutativity of the right hand square, suppose $x \in H_{n-1}(\partial F)$ and $X \in H_n(F, \partial F)$ so that $\bar\partial X = x$. Let $\tilde X \in C_n(F, \partial F)$ be a relative chain representing X. By translating $\tilde X$ around the bundle E one obtains an $(n + 1)$-chain $B \in C_{n+1}(E)(B = \bigcup_{0 \le t \le 1} h_t(\tilde X))$ such that $\partial B = \Delta\tilde X - (\partial\tilde X \times S^1)$ (viewing $C_*(F, \partial F) \subset C_*(E)$). Thus $\Delta\tilde X$ is homologous to $\partial\hat X \times S^1$ in $H_n(E)$. This completes the proof of the lemma.

LEMMA 3.2. *Let M^{2n+1} be a simple open book, $n > 1$, E, F and h as above. If Δ is an isomorphism then $i' : H_n(\partial F) \to H_n(E)$, induced by inclusion, is the zero map.*

Proof. Consider $i : H_n(\partial E; Q) \to H_n(E; Q)$. A standard argument [see 11] shows that $\dim(\operatorname{Ker} i) = 1/2 \dim(H_n(\partial E; Q))$. Here dim refers to vector space dimension over Q = the rationals. But $\partial E = \partial F \times S^1$ and $H_n(\partial F \times S^1) = H_n(\partial F) \oplus H_{n-1}(\partial F)$ so that $H_n(\partial E) \to H_n(E)$ is given by $H_n(\partial F) \oplus H_{n-1}(\partial F) \xrightarrow{i' \oplus (\times S^1)} H_n(E)$. By Lemma 3.1, $\times S^1 : H_{n-1}(\partial F) \to H_n(E)$ is an isomorphism. Since $\dim(H_{n-1}(\partial F; Q)) = \dim(H_n(\partial F; Q))$ and $H_n(\partial F)$ is free, we conclude that i' is the zero map.

Proof (of 3.3). We need only check that if Δ is an isomorphism then M is a homology sphere. Write $M = E \cup (\partial F \times D^2)$ with $E \cap (\partial F \times D^2) = \partial E = \partial F \times S^1$ and apply the Mayer–Vietoris sequence. Note that $H_*(E) = 0$ for $* \neq 0, n, n + 1, 2n + 1$; $H_*(\partial F) = 0$ for $* \neq 0, n - 1, n, 2n - 1$; $H_*(\partial F \times S^1) = H_*(\partial F) \oplus H_{*-1}(\partial F) = 0$ for $* \neq 0, n - 1, n, n + 1, 2n - 1, 2n$. Thus the relevant sections of the sequence are of the form

$$H_{*+1}(M) \to H_*(\partial F \times S^1) \to H_*(E) \oplus H_*(\partial F \times D^2) \to H_*(M).$$

The center map is given by

$$\begin{gathered} H_*(\partial F) \oplus H_{*-1}(\partial F) \to H_*(E) \oplus H_*(\partial F) \\ (x, y) \to (i(x) + y \times S^1, x) \end{gathered}$$

where $i : H_*(\partial F) \to H_*(E)$ is induced by inclusion and $\times S^1$ is the map discussed above. It then follows from Lemmas 3.1 and 3.2 that this is an isomorphism often enough to insure that $H_*(M) = 0$ for $* \neq 0,\ 2n+1$. Thus M is a homology sphere and hence a homotopy sphere.

§4. CONSTRUCTION OF BRANCHED COVERINGS

In this section and the next $K^{2n-1} \subset S^{2n+1}$ is a simple knot with spanning surface F^{2n}. No open book structure is assumed.

Using the normal field of S^{2n+1} in D^{2n+2} one can push F into the interior of D^{2n+2} obtaining a diffeomorphic copy of F lying near ∂D^{2n+2}. Joining the boundary of this copy to K by normal trajectories one obtains a manifold $\hat{F} \subset D^{2n+2}$ such that $\hat{F} \cap \partial D^{2n+2} = \partial\hat{F} = K$. Note that $\hat{F}$ is a manifold with corners. By the usual technique of "straightening the angle" one may assume that $\hat{F}$ is a smooth submanifold of D^{2n+2} and $\hat{F} \simeq F$.

Since $\hat{F}$ has trivial normal bundle in D^{2n+2} we can extend the inclusion to an embedding $\psi: \hat{F} \times D^2 \to D^{2n+2}$. Let $X = D^{2n+2} - \psi(\hat{F} \times \mathring{D}^2)$($\mathring{D}$ = interior of D). Then X is a manifold with boundary (and corners).

LEMMA 4.1. $H_1(X) \simeq \mathbb{Z}$.

Proof. Note that $D^{2n+2} = (\hat{F} \times D^2) \cup_\phi X$ where $\phi = \psi | \hat{F} \times S^1$. Let $j: D^2 \to \hat{F} \times D^2$, $j(z) = (x, z)$ for some fixed $x \in \hat{F}$. In the diagram below the bottom vertical maps are induced by j.

$$\begin{tikzcd}
0 = H_2(D^{2n+2}) \arrow[r] & H_2(D^{2n+2}, X) \arrow[r, "\bar{\partial}", "\simeq"'] & H_1(X) \arrow[r] & H_1(D^{2n+2}) = 0. \\
& H_2(\hat{F} \times D^2, \hat{F} \times S^1) \arrow[u, "\simeq"'] \arrow[r] & H_1(\hat{F} \times S^1) \arrow[u, "\phi"'] & \\
& H_2(D^2, S^1) \arrow[u, "\simeq"'] \arrow[r, "\partial", "\simeq"'] & H_1(S^1) \arrow[u] &
\end{tikzcd}$$

Hence $H_1(X) \simeq \mathbb{Z}$.

Let $\pi: \tilde{X} \to X$ be the regular covering space corresponding to the kernel of $\pi_1(X) \to H_1(X) \to \mathbb{Z}/a\mathbb{Z}$ for a given choice of positive integer a. The embedding ϕ induces an embedding $\tilde{\phi}: F \times S^1 \to \tilde{X}$ so that the following diagram commutes.

$$\begin{tikzcd}
\hat{F} \times S^1 \arrow[r, "\tilde{\phi}"] \arrow[d, "1 \times \lambda_a"'] & \tilde{X} \arrow[d, "\pi"] \\
\hat{F} \times S^1 \arrow[r, "\phi"] & X
\end{tikzcd} \qquad \lambda_a(x) = x^a,\ x \in S^1.$$

Definition 4.2. $F(a) = (\hat{F} \times D^2) \cup_{\tilde{\phi}} \tilde{X}$

$K(a) = \partial F(a)$.

Thus $F(a)$ is the a-fold cyclic cover of D^{2n+2} with branch set $\hat{F}$, and its boundary $K(a)$ is the a-fold cyclic cover of S^{2n+1} branching along K. They are both smooth manifolds with differentiable structure independent of the choices of tubular neighborhoods, and so on, involved in their construction. Verification of this last point will be omitted.

We wish to investigate the topology of $F(a)$ and $K(a)$ The first task will be to give a cut and paste description of $F(a)$.

§5. CUTTING, PASTING AND THE INTERSECTION FORM FOR $F(a)$

The first result of this section gives a decomposition of $F(a)$. This decomposition is then used to find the intersection form on $H_{n+1}(F(a))$.

Definition 5.1. Let W, W_-, W_+ denote the following submanifolds of S^{2n+1}: $W = \bigcup_{-1 \le t \le +1} i_t(F)$, $W_- = \bigcup_{-1 \le t \le 0} i_t(F)$, $W_+ = \bigcup_{0 \le t \le +1} i_t(F)$. Here the family of embeddings $i_t : F \to S^{2n+1}$, $-1 \le t \le +1$, is defined in §2. Note that $i_t | \partial F$ is the identity. There is an involution $T: W \to W$ such that $T(W\pm) = W_{\mp}$, $T(W\pm) \cap W_{\pm} = F$. Given $i_t(x) \in W$, $T(i_t(x)) = i_{-t}(x)$; thus $T|_F = 1_F$.

PROPOSITION 5.2. *Let $x : F(a) \to F(a)$ denote the generating covering translation. Thus x has order a and is the identity when restricted to the branch set $F \subset F(a)$. Then there is a "fundamental domain" $\hat{D} \subset F(a)$ such that*

(i) *$\hat{D}$ has the structure of smooth submanifold so that $\hat{D} \simeq D^{2n+2}$.*
(ii) *$F(a) = \bigcup_{i=0}^{a-1} x^i \hat{D}$.*
(iii) *Identifying $\partial\hat{D}$ with ∂D^{2n+2} and using the notation of 5.1, one has $x^i \hat{D} \cap x^{i+1} \hat{D} = x^i W_- = x^{i+1} W_+$, $x^i \hat{D} \cap x^j \hat{D} = F$ for $|i-j| \not\equiv 0, 1 \pmod a$. The identification $x^i W_- = x^{i+1} W_+$ is given by*

$$x^i(p) = x^{i+1}T(p), \ T \textit{ as in } 5.1.$$

Proof. Recall that $F(a) = (\hat{F} \times D^2) \cup_{\hat{\phi}} \tilde{X}$ and note an alternate description of $\tilde{X}$: Since $H_1(X) \simeq \mathbb{Z}$, there is a map $\alpha : X \to S^1$ such that $\alpha | \hat{F} \times S^1$ is projection on the second factor and $\tilde{X}$ is the pullback

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\alpha}} & S^1 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \lambda_a} \\ X & \xrightarrow{\alpha} & S^1 \end{array}.$$

(The map α is a generator for $H^1(X) = [X, S^1]$. That $\alpha | \hat{F} \times S^1$ is projection follows from the proof of 4.1.)

By choosing a differentiable representative for α and making it transverse to a point on S^1 we obtain a splitting of X along a codimension 1 submanifold. Then $\tilde{X}$ is a union of a split copies of X.

Since $\hat{F}$ was obtained by sliding F into the interior of D^{2n+2} via an inward normal vector field, we actually constructed a submanifold $\overline{W}^{2n+1} \subset D^{2n+2}$ such that $\partial \overline{W} = \hat{F} \cup F$. (Essentially, $\overline{W}$ is the union of all the trajectories running from F to $\hat{F}$.) We may assume that the embedding $\psi: \hat{F} \times D^2 \to D^{2n+2}$ is transverse to $\overline{W}$ along $\hat{F} \times S^1$ and that $\psi(\hat{F} \times D^2)$

intersects $\overline{W}$ in a collar neighborhood of $\hat{F}$ in W. Thus $\overline{W}' = \overline{W} \cap X$ is a submanifold of X such that $\partial \overline{W}' \simeq (F \times *) \cup F'$ where $F' \subset S^{2n+1}$ and $\partial F' = K \times *$, $* \in S^1$.

Now *define* $\alpha : X \to S^1$ as follows. First set $\alpha | F \times S^1$ to be projection on the second factor. Next, define $\alpha(\overline{W}') = *$. It is then an easy exercise in obstruction theory to see that this extends to $\alpha : X \to S^1$.

Thus $\overline{W}'$ may be used to effect the splitting of X.

Since $F(a) = \tilde{X} \cup (\hat{F} \times D^2)$, it may be described as follows: Let $\hat{D} = D^{2n+2}$ split along $(\overline{W} - \hat{F})$ so that $\partial\hat{D} = \hat{S}^{2n+1} \cup \overline{W}_- \cup \overline{W}_+$ where $\overline{W} \cap \overline{W}_+ \simeq \hat{F}$, $\overline{W}_- \simeq \overline{W}_+ \simeq W$, $\partial \overline{W}_- = \hat{F} \cup F_-$, $\partial \overline{W}_+ = \hat{F} \cup F_+$, $\partial F \pm = K$, $F_\pm \simeq F$. Similarly $\hat{S} = S^{2n+1}$ split along F, and $\partial\hat{S} = F_- \cup F_+$, $F_- \cap F_+ = K$.

Then $F(a) = \bigcup_{i=0}^{a-1} x^i\hat{D}$ and the $x^i\hat{D}$ intersect according to the schema in the statement of the proposition with $W_\pm$ replaced by $\overline{W}_\pm$.

It remains to show that $\hat{D} \simeq D^{2n+2}$, $\hat{S} \cup \overline{W}_- \cup \overline{W}_+ \simeq S^{2n+1}$ and that $\overline{W}_- \cup \overline{W}_+ = W$ under this identification. It is clear abstractly that $\overline{W}_\pm \simeq W_\pm$. Furthermore, the quotient space of D^{2n+2} under identification of $x \in W_-$ with $Tx \in W_+$ is clearly diffeomorphic to D^{2n+2} with W_+ and W_- going to a submanifold of D^{2n+2} diffeomorphic to $\overline{W}$. Thus $\hat{D} = D^{2n+2}$ split along $\overline{W} \simeq D^{2n+2}$.

Using this identification, the proposition follows.

Remarks. (1) Smoothing details for the above proof have been suppressed. The submanifold $\hat{D}$, as defined, is a manifold with corner along F. It is clear that upon following the above proof and straightening the angle that $\hat{D} \simeq D^{2n+2}$.

(2) A simple example for this proposition is given by a solid torus double branch covering D^3 (same geometry even though 3 is odd). Let $T = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1| \leq 1, |z_2| = 1\}$ and $x : T \to T$ by $x(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$ where $\bar{z}$ denotes the complex conjugate of z. Then $\hat{D} = \{(z_1, z_2) \in T | \operatorname{Im}(z_2) \geq 0\}$ and $T = \hat{D} \cup x\hat{D}$, $T/x = \hat{D}/x = D^3$ and $\hat{D} \simeq D^3$. Figure 1 illustrates this example. Note that here K = four points in S^2 and F = two disjoint arcs joining pairs of points.

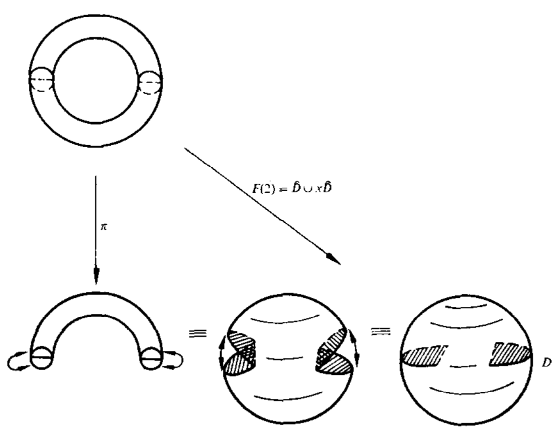

FIG. 1

(3) From now on $\hat{D}$ will be denoted simply by D.

To describe the topology of $F(a)$, let $CF \subset D$ be the join of $F \subset \partial D$ with the center of D. Similarly, $x^iCF \subset x^iD$. Thus $F(a) \supset F'(a)$ where $F'(a) = CF \cup xCF \cup \cdots \cup x^{a-1}CF$ and $x^iCF \cap x^jCF = F$ for $i \not\equiv j \pmod a$.

LEMMA 5.3. *$F'(a) \subset F(a)$ is a homotopy equivalence.*

Proof. Let $D' \subset D$ be a smaller concentric disk. Let $D'' = D' \cup F \times [0, 1]$ where $F \times [0, 1]$ is radially embedded in the annulus between D and D' ($F \times t \subset$ a sphere concentric to ∂D) and $F \times 0 = F \subset \partial D$. Let $F''(a) = D'' \cup xD'' \cup \cdots \cup x^{a-1}D''$. Thus $F''(a) \supset F'(a)$. Now note that $F''(a) \subset F(a)$ is a retract of $F(a)$ and $F'(a) \subset F''(a)$ is a homotopy equivalence. Hence $F'(a) \subset F(a)$ is a homotopy equivalence.

We know that $H_n(F)$ has a basis represented by embedded spheres. Let these be $\{a_1, a_2, \ldots, a_r\}$. Let $ca_i \subset CF$ denote the radial cone over a_i in D similarly $x^jca_i \subset x^iCF$. These cones fit together to form many $(n + 1)$-spheres in $F(a)$. Let $\Sigma a_j = ca_j \cup xca_j$ and, as an element of the chain group $C_{n+1}(F(a))$, $\Sigma a_j = ca_j - xca_j$. Note that one may regard $\Sigma : C_n(F) \to C_{n+1}(F(a))$.

LEMMA 5.4. *The homology group $H_{n+1}(F(a))$ is free of rank $r \cdot (a - 1)$. A basis is given by $\mathscr{B} = \{x^i\Sigma a_j \mid 0 \le i \le a - 2, j = 1, \ldots, r\}$.*

Proof. $H_{n+1}(F(a)) \simeq H_{n+1}(F'(a))$ and the lemma is clear for the latter.

Next, we wish to determine the intersection pairing $\langle\,,\rangle : H_{n+1}(F(a)) \times H_{n+1}(F(a)) \to \mathbb{Z}$.

Definition 5.5. $d(\Sigma\alpha) = ci^*\alpha - xci_*\alpha$. Here $i^* = i_{-1/2}$, $i_* = i_{+1/2}$ with i_* as usual. Note that $Ti^* = i_*$ and thus $d(\Sigma\alpha)$ is a cycle on $F(a)$. It is clearly homologous to $\Sigma\alpha$ via the family $d_t(\Sigma\alpha) = ci_{-t}\alpha - xci_{+t}\alpha$, $0 \le t \le 1/2$. (See Fig. 2.) Similarly, $d(x^i\Sigma\alpha) = x^id(\Sigma\alpha)$.

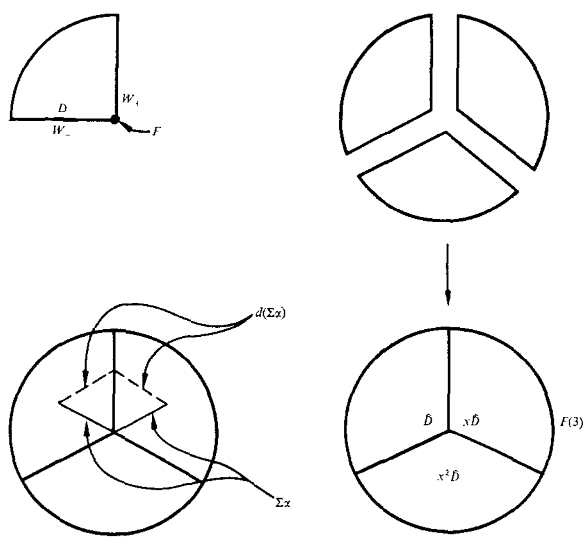

FIG. 2

Since cones are taken radially, $x^i\Sigma\alpha$ and $d(x^i\Sigma\beta)$ can intersect only at the apex of cones, and hence in at most two points.

PROPOSITION 5.6. *On the basis $\mathscr{B}$ the intersection form is given by*

$$\langle x^j\Sigma a_i, x^{j'}\Sigma a_{i'}\rangle = \begin{cases} \theta(a_i, a_{i'}) + (-1)^{n+1}\theta(a_{i'}, a_i) & j = j' \\ (-1)^n\theta(a_{i'}, a_i) & j = j' + 1 \\ \quad -\theta(a_i, a_{i'}) & j + 1 = j' \\ \quad 0 & \textit{otherwise.} \end{cases}$$

Here $\theta\colon H_n(F) \times H_n(F) \to \mathbb{Z}$ is the Seifert pairing.

Proof. This calculation involves the deformation of Definition 5.5 and the definition of linking numbers in S^{2n+1} in terms of intersection numbers in D^{2n+2}. Since it is straightforward, we do the case $j = j'$ and leave the rest to the reader.

$$\begin{aligned} \langle x^j\Sigma a_i, x^j\Sigma a_{i'}\rangle &= \langle \Sigma a_i, \Sigma a_{i'}\rangle \\ &= \langle \Sigma a_i, d(\Sigma a_{i'})\rangle \\ &= \langle ca_i - xca_i, ci^*a_{i'} - xci_*a_{i'}\rangle \\ &= \langle ca_i, ci^*a_{i'}\rangle + \langle ca_i, ci_*a_{i'}\rangle \\ &= l(a_i, i^*a_{i'}) + l(a_i, i_*a_{i'}) \\ &= \theta(a_i, a_{i'}) + (-1)^{n+1}\theta(a_{i'}, a_i). \end{aligned}$$

The other cases follow similarly.

COROLLARY 5.7. *With respect to the basis $\mathscr{B}$, $\langle\, ,\rangle$ has matrix $N(a) = V(a) + (-1)^{n+1}V(a)^t$ where $V(a) = V \otimes L_a$, $V =$ Seifert matrix $V_{ij} = \theta(a_i, a_j)$,*

$$L_a = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \bigcirc \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & \bigcirc & & & 1 \end{bmatrix} ((a-1) \times (a-1)).$$

Here $\otimes$ denotes tensor product of matrices.

Proof. This is simply a restatement of (5.6) in matrix terms.

It is also useful to have an explicit basis for $H_{n+1}(F(a), \partial F(a))$. Let $\{\hat{a}_1, \ldots, \hat{a}_r\}$ be a dual basis for $H_n(F, \partial F)$ so that $f(a_i, \hat{a}_j) = \delta_{ij}$, $f\colon H_n(F) \times H_n(F, \partial F) \to \mathbb{Z}$ the Poincaré–Lefschetz duality pairing. The dual basis in F can be used to construct a dual basis in $F(a)$.

Definition 5.8. Let $B_i = \bigcup_{0 \le t \le 1} i_t(\hat{a}_i) \subset W_+ \subset D$ and let $\bar{B}_i = T(B_i)(T\colon W_+ \to W_-$ as in (5.1)). Let $\mathscr{T}_i^j = \mathscr{T}^j(\hat{a}_i) = B_i - x^j\bar{B}_i$ for $0 \le j \le a-2$ and $1 \le i \le r$. Note that one may regard $\mathscr{T}^j$ as a map $\mathscr{T}^j\colon C_n(F, \partial F) \to C_{n+1}(F(a), \partial F(a))$.

LEMMA 5.9. *The set $\hat{\mathscr{B}} = \{\mathscr{T}_i^j \mid 0 \le j \le a-2,\ 1 \le i \le r\}$ is a basis for $H_{n+1}(F(a), \partial F(a))$. Poincaré dual to $\mathscr{B}$.*

Proof. Simply note that the $x^i\Sigma a_i$ are transverse to the $\mathscr{T}_i{}^j$, intersecting only along F. Also $\mathscr{T}_i{}^j$ is a relative cycle. If $\bar{f}: H_{n+1}(F(a)) \times H_{n+1}(F(a), F(a)) \to \mathbb{Z}$ is the duality pairing then it follows, using the deformation d of (5.5), that $\bar{f}(x^j\Sigma a_i, \mathscr{T}_{i'}{}^{j'}) = \delta_{jj'} \cdot \delta_{ii'}$. This proves the lemma.

§6. CONSTRUCTION OF AN OPEN BOOK WITH LEAF $F(a)$

Now add to the structure by assuming that K and F are the binding and leaf of a simple book decomposition of S^{2n+1}. Let $h: F \to F$, $h|\partial F = 1_{\partial F}$ be the monodromy. Let $E = F \times [0, 1]/[x \times 1 \sim (hx) \times 0]$ so that $S^{2n+1} \simeq E \cup (K \times D^2)$.

In this section a map $H(a): F(a) \to F(a)$ will be defined and we show that the open book with monodromy $H(a)$ is a homotopy sphere.

Note that under the above assumptions one may define $h: E \to E$ by $h[x, t] = [hx, t]$ ($[x, t]$ = equivalence class of $x \times t$ in E). Since $h|\partial F = 1_{\partial F}$, this extends to a diffeomorphism $h: S^{2n+1} \to S^{2n+1}$.

LEMMA 6.1. *The diffeomorphism $h: S^{2n+1} \to S^{2n+1}$ is isotopic to the identity map.*

Proof. Regard $E = F \times \mathbb{R}/\sim$ where $(f, t) \sim (h^{-1}f, t+1)$, $[f, t] = \sim$ class of (f, t). Let $h_\varepsilon([f, t]) = [f, t+\varepsilon]$, $0 \le \varepsilon \le 1$. Thus $h_0 = 1_E$ and $h_1[f, t] = [f, t+1] = [hf, t] = h[f, t]$. Hence $f|E$ is isotopic to 1_E. To extend to S^{2n+1}, define $h_\varepsilon: K \times S^1 \to K \times S^1$, $h(x, [t]) = (x, [t+\varepsilon])$, $S^1 = \mathbb{R}/\mathbb{Z}$. This agrees with $h_\varepsilon|\partial E$. Viewing $S^1 \subset \mathbb{C}$, $h_\varepsilon(x, \lambda) = (x, e^{2\pi i\varepsilon} \cdot \lambda)$. Thus we may define $h_\varepsilon: K \times D^2 \to K \times D^2$ by $h_\varepsilon(x, z) = (x, e^{2\pi i\varepsilon} \cdot z)$. This gives the extension $h_\varepsilon: S^{2n+1} \to S^{2n+1}$ such that h_0 = identity and $h_1 = h$.

Remarks. (1) Under the above conditions we may define a diffeomorphism $h: D^{2n+2} \to D^{2n+2}$ which agrees with h above on the boundary by using the isotopy on $S^{2n+1} \times I$ and filling in the identity map on a smaller concentric disk.

(2) For $\lambda \in S^1$ let $[0, \lambda] = \{t\lambda \in D^2 | 0 \le t \le 1\}$. In S^{2n+1}, let $\bar{F} = F \cup (K \times [0, 1])$. Thus $h_\varepsilon(\bar{F}) = h_\varepsilon(F) \cup (K \times [0, e^{2\pi i\varepsilon}])$ and $S^{2n+1} = \bigcup_{-1/2 \le \varepsilon \le +1/2} h_\varepsilon(\bar{F})$. Thus we may take for $W \subset S^{2n+1}$ (as in (5.1) and (5.2)) $W = \bigcup_{-1/4 \le \varepsilon \le +1/4} h_\varepsilon(\bar{F})$. Letting $i_* = h_{1/4}$ and $i^* = h_{-1/4}$ one has $\partial W = i_*\bar{F} \cup i^*\bar{F}$ and W has an involution $T: W \to W$ with $T(x) = h_{-2\varepsilon}(x)$ for $x \in h_\varepsilon(\bar{F})$. Proposition 5.2 now holds using this choice of W.

The next lemma observes that under the assumption that K is a binding of a simple book structure on S^{2n+1} it follows that $K(a)$ is also an open book with the same binding and leaf.

LEMMA 6.2. *Let $E(a) = F \times \mathbb{R}/\sim$, $(f, t) \sim (h^{-a}f, t+a)$. Let $\langle f, t\rangle$ = the equivalence class of (f, t) in $E(a)$ and $[f, t]$ = the equivalence class of (f, t) in E ($E = E(1)$). Define $\pi: E(a) \to E$ by $\pi\langle f, t\rangle = [f, t]$ and extend this to $\pi: E(a) \cup (K \times D^2) \to E \cup (K \times D^2) \simeq S^{2n+1}$ via $K \times D^2 \xrightarrow{1 \times \lambda_a} K \times D^2$.*

Then there is a diffeomorphism $\psi: K(a) \to E(a) \cup (K \times D^2)$ such that the following diagram commutes:

$$\begin{tikzcd} K(a) \arrow[r, "\psi"] \arrow[d, "\pi'"] & E(a) \cup (K \times D^2) \arrow[dl, "\pi"] \\ S^{2n+1} & \end{tikzcd}$$

Here π' is the branched covering map.

Proof. This is easily seen from Remark 2 above and the proof of Proposition 5.1.

Remark. In $K(a) = \partial F(a)$ we have denoted the action of $\mathbb{Z}/a\mathbb{Z}$ by $x : K(a) \to K(a)$. In $E(a)$ this corresponds to $\langle f, t\rangle \mapsto \langle h^{-1}f, t+1\rangle$ and extends over $K \times D^2$. This justifies writing $x\langle f, t\rangle = \langle h^{-1}f, t+1\rangle$.

Definition 6.3. Define a diffeomorphism $\hat{h} : F(a) \to F(a)$ via $F(a) = \bigcup_{i=0}^{a-1} x^i D^{2n+2}$ and $\hat{h}(x^i p) = x^i h(p)$ where $p \in D^{2n+2}$ and $h : D^{2n+2} \to D^{2n+2}$ is the extension of the monodromy defined in Remark 1 after (6.1). One sees that this is well-defined by examining Proposition 5.2 in the light of Remark 2.

LEMMA 6.4. *Let $H : F(a) \to F(a)$ denote the composite $H = x \circ \hat{h}$. Then $H|K(a)$ is isotopic to $1_{K(a)}$.*

Proof. Let $g = H|K(a)$. Identifying $K(a)$ and $E(a) \cup (K \times D^2)$ by (6.2) we find

$$g\langle f, t\rangle = x\langle hf, t\rangle = \langle h^{-1}hf, t+1\rangle = \langle f, t+1\rangle.$$

This is isotopic to the identity in such a way that it extends over $K \times D^2$. Define $g_\varepsilon\langle f, t\rangle = \langle f, t+\varepsilon\rangle$ and proceed as in (6.1).

Thus, by collaring $F(a)$ and carrying the isotopy g_ε on the collar $K(a) \times [0, 1]$ we obtain from H a map $H(a)$: $F(a) \to F(a)$ such that $H(a)|_{\partial F(a)} = 1_{\partial F(a)}$.

Definition 6.5. Let $M^{2n+3}(a)$ be the manifold with open book structure: leaf $F(a)$, binding $K(a)$, monodromy $H(a)$.

THEOREM 6.6. *The manifold $M(a)$ is a homotopy sphere for $n \geq 1$. Furthermore, one can modify the monodromy $H(a)$ so that its action on $H_{n+1}(F(a))$ and its variation $\Delta(a) : H_{n+1}(F(a), \partial F(a)) \to H_{n+1}(F(a))$ are unchanged but $M^{2n+3}(a)$ is diffeomorphic to S^{2n+3}. Thus S^{2n+3} inherits an open book structure with binding $K(a)$ and leaf $F(a)$.*

Proof. By (3.3) it suffices to show that $\Delta(a)$ is an isomorphism. Let $\mathcal{B}$ and $\hat{\mathcal{B}}$ be the bases for $H_{n+1}(F(a))$ and $H_{n+1}(F(a), \partial F(a))$ discussed in §5.

Claim. With respect to the bases $\hat{\mathcal{B}}$ and $\mathcal{B}$, $\Delta(a)$ has matrix (also written $\Delta(a)$) $\Delta(a) = \mu \cdot \Delta \otimes S_a$ where Δ is the matrix of $\Delta : H_n(F, \partial F) \to H_n(F)$ and

$$S_a = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \cdots & 1 \\ & & & \ddots & \vdots \\ & \bigcirc & & & 1 \end{bmatrix} ((a-1) \times (a-1)).$$

Here $\mu = \pm 1$ to be determined later.

Note that, given this claim, it follows that $\Delta(a)$ is an isomorphism. For, Δ is unimodular and $\Delta^{-1} = (-1)^{n+1}V$ where V is the Seifert matrix (see Lemma 2.7). Let $V(a)$ be as in (5.7). Then $\Delta(a) \cdot V(a) = \mu \cdot (\Delta \otimes S_a) \cdot (V \otimes L_a) = \mu \cdot (-1)^{n+1} I$. Hence, as a matrix, $\Delta(a)$ is unimodular and thus as a map it is an isomorphism.

The modification of the monodromy follows a remark of Milnor: Taking connected sum of M with an exotic sphere may be described by splitting M along a ball $D^{2n+2} \subset M^{2n+3}$ and repasting via a diffeomorphism $\varphi: D^{2n+2} \to D^{2n+2}$ such that $\varphi | \partial D^{2n+2} =$ the identity. Thus, if $M \simeq \Sigma$, Σ an exotic sphere, let $\varphi: D^{2n+2} \to D^{2n+2}$ be a diffeomorphism corresponding to $\#(-\Sigma)$. Modify $H(a)$ by composing with φ along a tiny ball in the interior of $F(a)$. The new open book will be diffeomorphic to $M \#(-\Sigma) = \Sigma \# (-\Sigma) \simeq S^{2n+3}$. This certainly leaves the homological properties of the monodromy unaffected.

Proof of Claim. Recall that $\hat{\mathscr{B}} = \{\mathscr{T}_i^j\}$. In order to compute $\Delta(a)(\mathscr{T}_i^j)$ it suffices to know the intersection numbers $\bar{f}(\Delta(a)(\mathscr{T}_i^j), \mathscr{T}_l^k)$ since these are the coefficients of $\Delta(a)(\mathscr{T}_i^j)$ expressed as a combination of the elements of the dual basis $\mathscr{B}$. (Here $\bar{f}$ is as in (5.9).) As in (5.2), regard $F(a) = \cup\, x^i D$ where $x^i D \cap x^{i+1} D = x^{i+1} W_+$ and so on. Then one can deform $\Delta(a)(\mathscr{T}_i^j)$ so that $\Delta(a)(\mathscr{T}_i^j) \subset D \cup x^{j+1} D$ and $\Delta(a)(\mathscr{T}_i^j) \cap [\bigcup_{k=0}^{a-1} x^k W_+] = \Delta(\hat{a}_i) \subset F$ (see Fig. 3). It then follows that

$$\bar{f}(\Delta(a)(\mathscr{T}_i^j), \mathscr{T}_l^k) = \begin{cases} \mu \cdot f(\Delta\hat{a}_i, \hat{a}_l) & \text{for } k \leq j \\ 0 & \text{otherwise} \end{cases}$$

since $\mathscr{T}_l^k \subset \bigcup_{k=0}^{a-1} x^k W_+$ and $\mathscr{T}_l^k \cap F = \hat{a}_l$. It is another deformation argument to see that the intersections occur in the range $k \leq j$.

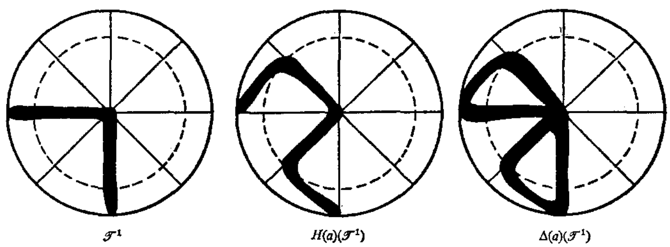

FIG. 3

Reformulation of this in matrix terms gives the claim as stated.

This completes the proof of the theorem.

Discussion. Since $M^{2n+3}(a)$ is a sphere, it follows that $\Delta(a)^{-1} = (-1)^n \bar{V}$ where $\bar{V}$ is the Seifert matrix for $F(a) \subset M(a)$ (Lemma 2.7). Now the matrix forms of Lemmas 2.4 and 2.5 read $\bar{V}H(a) = (-1)^n \bar{V}^t$ and $(-1)^n N(a) = \bar{V}(I - H(a))$. Here $H(a)$ is the matrix of the monodromy with respect to the basis $\mathscr{B}$. Now $H(a)(x^i \Sigma a_j) = x^{i+1} \Sigma h a_j$ and, since $1 + x + x^2 + \cdots + x^{a-1} = 0$ on homology, $H(a) = h \otimes C_a$ where

$$C_a = \begin{bmatrix} 0 & & & \bigcirc & -1 \\ 1 & 0 & & & \vdots \\ & 1 & \ddots & & \\ & & \ddots & 0 & -1 \\ & \bigcirc & & 1 & -1 \end{bmatrix} ((a-1) \times (a-1)).$$

In the proof of (6.6) we found that $\Delta(a)^{-1} = \mu \cdot (-1)^{n+1}V(a)$. Thus $\bar{V} = -\mu V(a)$. Also, one may check directly that $V(a) \cdot H(a) = (-1)^n V(a)^t$. Thus $N(a) = V(a) + (-1)^{n+1}V(a)^t = V(a) - V(a) \cdot H(a) = V(a)(I - H(a))$. Thus $(-1)^n V(a)(I - H(a)) = -\mu V(a)(I - H(a))$ and hence $\mu = (-1)^{n+1}$. Thus $\Delta(a)^{-1} = V(a)$ and therefore $\bar{V} = (-1)^n V(a)$. We have shown:

LEMMA 6.7. *The Seifert matrix for $F(a) \subset M(a)$ is given by $\bar{V} = (-1)^n V(a)$ with respect to the basis $\mathscr{B}$ for $H_{n+1}(F(a))$.*

Since we can assume $M^{2n+3} \simeq S^{2n+3}$, the construction may be iterated. Let S^{2n+1} have any simple open book decomposition with binding K. Given a sequence of positive integers $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ define $K(\alpha_1, \ldots, \alpha_n)$ inductively by $K(\alpha_1, \ldots, \alpha_{k+1}) = K(\alpha_1, \ldots, \alpha_k)(\alpha_{k+1})$. Thus one has a tower:

$$\begin{tikzcd}
 & & & \vdots \arrow[d] \\
 & & K(\alpha_1, \alpha_2) \arrow[r] \arrow[d] & S^{2n+5} \\
 & K(\alpha_1) \arrow[r] \arrow[d] & S^{2n+3} & \\
K \arrow[r] & S^{2n+1} & &
\end{tikzcd}$$

The horizontal maps are placements of bindings for open book structures; the vertical maps are cyclic branched coverings.

Note that these "generalized Brieskorn manifolds" have many symmetries and they are themselves open books as well as being bindings for book structures on spheres.

Iterated application of (6.7) allows calculation of the Seifert pairing of $F(\alpha_1, \ldots, \alpha_k) \to S^{2n+2k+1}$. The matrix is $(-1)^{nk+(k(k-1))/2} \cdot V \otimes L_{\alpha_1} \otimes L_{\alpha_2} \otimes \cdots \otimes L_{\alpha_k}$ where V is the Seifert matrix for $F \hookrightarrow S^{2n+1}$ and L_{α_i} is as in (5.7).

§7 APPLICATIONS

(a) Hypersurface Singularities and Algebraic Books

The constructions of this paper were motivated by attempts to understand the algebraic case. Here we examine open books determined by isolated hypersurface singularities and show how to give a topological construction of the Brieskorn manifolds.

Definition 7.1. Let $T(F, h)$ denote the open book determined by $h: F \to F$, $h|\partial F = 1_{\partial F}$. One says that two books $T(F, h)$ and $T(F', h')$ are *isomorphic* if there is a diffeomorphism $\psi: T(F, h) \to T(F', h')$ preserving the book structures (i.e. ψ preserves leaves and bindings).

To relativize this definition let M be a fixed manifold. A *book structure on M* is a triple $\{F, h, g\}$ such that $T(F, h) \xrightarrow{g} M$ is a diffeomorphism. Two book structures $\{F, h, g\}$ and

$\{F', h', g'\}$ on M are said to be isomorphic if there is a book isomorphism $\psi: T(F, h) \to T(F', h')$ such that $g' \circ \psi = g$. Let $S(M)$ denote the set of isomorphism classes of simple book structures on M.

Here we restrict ourselves to weighted homogeneous polynomials, although this is probably unnecessary. Recall that a polynomial $f(z) = f(z_0, z_1, \ldots, z_n)$ in $n + 1$ complex variables is said to be weighted homogeneous if there exists a tuple of positive rational numbers $(w_0, w_1, \ldots, w_n)$ such that $f(z)$ is a linear combination of monomials $z_0{}^{i_0}z_1{}^{i_1} \cdots z_n{}^{i_n}$ with $i_0/w_0 + i_1/w_1 + \cdots + i_n/w_n = 1$.

Given f weighted homogeneous of type $(w_0, \ldots, w_n)$ define $\rho * z = (\rho^{1/w_0}z_0, \ldots, \rho^{1/w_n}z)$ for ρ positive real. Thus $f(\rho * z) = \rho f(z)$.

If f has an isolated singularity at $0 \in \mathbb{C}^{n+1}$ then Milnor [12] constructs a fibration $\phi: S^{2n+1} - K \to S^1$ with $\phi(z) = f(z)/|f(z)|$, $K = V(f) \cap S^{2n+1}$. This gives a simple book structure on S^{2n+1} with (open) leaf $F = \{z \in S^{2n+1} | f(z) \in \mathbb{R}^+\}$.

Let $\hat{f}(x, z) = x^a + f(z)$ where x is a new complex variable. Then $\hat{f}(x, z)$ is also weighted homogeneous.

THEOREM 7.2. *Let $T(\hat{f})$ and $T(f)$ denote the book structures on S^{2+n3} and S^{2n+1} determined by $\hat{f}$ and f respectively. Let $T(F(a), H(a))$ be the book on S^{2n+3} determined by $T(f)$ and (6.6). Then $T(\hat{f})$ is isomorphic to $T(F(a), H(a))$.*

Proof. It will suffice to identify $F(a)$ with the fiber of the Milnor fibration for $x^a + f(z)$, and $H(a)$: $F(a) \to F(a)$ with the monodromy for $x^a + f(z)$. To see this Let

$$E = \{z \in D^{2n+2} | \, |f(z)| = \delta\}$$

where $0 < \delta \ll 1$. Then $\phi': E \to S^1$, $\phi'(z) = f(z)/|f(z)|$ is a C^∞ fiber bundle with fiber $F = \{z | f(z) = \delta, \ |f| \leq 1\}$. Define $\eta: E \to S^{2n+1} - \dot{N}$, $N = \{z \in S^{2n+1} | \, |f(z)| \leq \delta\}$ by $\eta(z) = \rho * z$ for the unique $\rho \geq 1$ such that $|\eta(z)| = 1$. Then, since $f(\rho * z) = \rho f(z)$, η is a bundle isomorphism. Now $N \simeq K \times D^2$ since it is a trivial bundle over $D^2{}_\delta$. The open book structure on S^{2n+1} is given by $\eta(E) \cup N$.

Note that $F = \{z | f(z) = \delta, \ |z| \leq 1\}$ may be taken as the model for the leaf pushed into D^{2n+2} via a normal vector field. Let $\hat{F} = \{(x, z) \in D^{2n+4} | x^a + f(z) = \delta\}$. Define $\pi: \hat{F} \to D^{2n+2}$ by $\pi(x, z) = \rho * z$ for the unique ρ such that $|\rho * z| = |x|^2 + |z|^2$. Then π exhibits $\hat{F}$ as the a-fold cyclic branched cover of D^{2n+2} with branch set F. Hence $\hat{F} \simeq F(a)$.

Since f is weighted homogeneous, its monodromy is given by a map $h: \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ such that $|h(z)| = |z|$. The monodromy of $x^a + f(z)$ is then $\hat{H}(x, z) = (\omega x, h(z))$ where $\omega = \exp(2\pi i/a)$. Here $\hat{H}: \hat{F} \to \hat{F}$ and note that it is the composition of the covering action for π with h acting on D^{2n+2}. This serves to identify $\hat{H}$ with the monodromy $H(a)$ of §6.

Similar analysis shows that an isotopy of $\hat{H} | \partial\hat{F}$ to the identity corresponds to the isotopy used in constructing $H(a)$. Thus $T(\hat{f}) = T(\hat{F}, \hat{H})$ is isomorphic to $T(F(a), H(a))$.

Remark. Since $x^a + f(z)$ is again weighted homogeneous one can iterate the above argument. Thus if $\Sigma(f; a_1, \ldots, a_n) = V(f(z) + x_1{}^{a_1} + \cdots + x_n{}^{a_n}) \cap S^{2n+2k+1}$ then

$$\Sigma(f; a_1, \ldots, a_n) \simeq K(a_1, \ldots, a_n)$$

where $K = V(f) \cap S^{2n+1}$ and the symbol on the right denotes the binding of the iterated book.

In order to give a topological description of Brieskorn manifolds we now need only give a geometric description of the algebraic book structure on S^3 corresponding to $f(z) = z_0^{a_0} + z_1^{a_1}$. This is easily done. Simply let $g(z) = z_0 + z_1$ and take the (trivial) book structure on S^3 determined by g. Here the binding is an unknotted circle and the leaf a disk. Define $P: S^3 \to S^3$ by $P(z_0, z_1) = (\rho z_0^{a_0}, \rho z_1^{a_1})$ for $\rho > 0$ such that $|P(z_0, z_1)| = 1$. Then $P: \Sigma(a_0, a_1) \to \Sigma(1, 1)$ and the *inverse image* of the trivial book structure of S^3 under P is the book corresponding to $z_0^{a_0} + z_1^{a_1}$.

Since $\Sigma(a_0, a_1, \ldots, a_n) \simeq \Sigma(a_0, a_1)(a_2, \ldots, a_n)$ this gives a completely topological construction for Brieskorn manifolds. It follows from §6 and a direct calculation of the Seifert pairing for torus links that the Seifert matrix corresponding to $F(a_0, \ldots, a_n) \subset S^{2n+1}$ (the fiber for the Brieskorn book) is $(-1)^{n(n-1)/2} L_{a_0} \otimes L_{a_1} \otimes \cdots \otimes L_{a_n}$.

(b) Book Structures on Spheres

In [5], Kato discusses the classification of open book structures (spinnable structures in his terminology) on odd dimensional spheres. Given $\mathscr{S} = \{F, h, g\} \in S_n = S(S^{2n+1})$ (Definition 7.1) one has the Seifert matrix $V(\mathscr{S})$ and also the variation $\Delta(\mathscr{S}): H_n(F, \partial F) \to H_n(F)$. Kato proves:

(i) Given a unimodular $n \times n$ matrix A, then there exists a simple book decomposition $\mathscr{S}$ on S^{2n+1} with $V(\mathscr{S}) = A$, for $n \geq 3$.

(ii) Given simple books $\mathscr{S}$, $\mathscr{S}'$ on S^{2n+1}, $n \geq 3$, such that $V(\mathscr{S}) \sim V(\mathscr{S}')$($\sim \equiv$ congruent matrices) then $\mathscr{S} \simeq \mathscr{S}'$ (i.e. the books are isomorphic).

(iii) (Levine) Given simple books $\mathscr{S}$ and $\mathscr{S}'$ on S^{2n+1}, $n \geq 3$, then the leaves F and F' are *ambient isotopic* in S^{2n+1} if $V(\mathscr{S}) \sim V(\mathscr{S}')$.

Given $\mathscr{S} = \{F, h, g\}$ a simple book structure on S^{2n+1} we have shown that $F(2, 2, \ldots, 2)$ is the leaf of a simple structure $\omega_k(\mathscr{S})$ on $S^{2n+2k+1}$(k2's) such that $V(\omega_k(\mathscr{S})) = \pm V(\mathscr{S})$. Hence, as $\mathscr{S}$ runs over all simple books for S^{2n+1}, $n \geq 3$, $\omega_k(\mathscr{S})$ runs over all simple books for $S^{2n+2k+1}$. This implies the following result:

THEOREM 7.3. *Let $S_n = S(S^{2n+1})$ and $\omega_k: S_n \to S_{n+k}$ as above. Then for $n \geq 3$ ω_k is a 1-1 correspondence. In particular $S_{k+3} = \omega_k(S_3)$.*

COROLLARY 7.4. *Let C_m' $m = 2n - 1$ be the subgroup of the Levine knot cobordism group of spherical $(2n - 1)$ knots in S^{2n+1}, $n \geq 3$, which is generated by fibered knots. Then $\omega_2: C_m' \to C_{m+4}'$ is an explicit isomorphism.*

This is an analog of the more general isomorphism constructed explicitly by Bredon [1]. In fact, our construction is essentially the same as his for this case.

(c) Codimension One Foliations of Spheres

Let $f(z) = (z_0 + z_1^2)\ (z_0^2 + z_1^5)$, $K = V(f) \cap S_\varepsilon^3$. Then $K(2, 2, \ldots, 2) \to S^{2n+1}$ $((n-1) - 2\text{'s})$. Calculation of the Seifert pairing for f shows that $K(2, \ldots, 2) \simeq \pm \Sigma \#$

$(S^{n-1} \times S^n)$ where n is odd ≥ 3, Σ = Milnor sphere. This book structure is the basic ingredient for producing codimension one foliations à la Durfee [3] or Tamura [19]. This is Durfee's method placed in an open book context.

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