# The spread and extreme terms of Jones polynomials

Yongju Bae and H.R.Morton

February 1, 2008

**Abstract**

We adapt Thistlethwaite's alternating tangle decomposition of a knot diagram to identify the potential extreme terms in its bracket polynomial, and give a simple combinatorial calculation for their coefficients, based on the intersection graph of certain chord diagrams.

## Introduction

One of the most striking combinatorial applications of the Jones polynomial has been the result of Murasugi and Thistlethwaite which characterises alternating knots by relating the spread of the Jones polynomial to the number of crossings in the knot diagram. The result follows from the identification of the two potential extreme terms in the bracket polynomial and the calculation that each term occurs with coefficient $\pm 1$.

Lickorish and Thistlethwaite [4] were able to widen the class of knots for which a similar exact bound for the spread of the polynomial could be given. In this wider class of 'adequate' diagrams they were again able to specify extreme terms and show that their coefficients were $\pm 1$. Later Thistlethwaite [6] made use of a natural decomposition of a given link diagram into maximal alternating pieces, and formulated a bound for the spread purely in terms of combinatorial data from the non-alternating part of the diagram.

In this paper we extend his approach, again looking at the structure of a given diagram in terms of its maximal alternating tangles. We incorporate some 'boundary connection' information about each alternating tangle, which depends only on the immediate neighbourhood of the boundary of the tangle, and combine this with the data from the non-alternating part to give a simple combinatorial calculation for the coefficients of two potential extreme terms in the bracket polynomial for the diagram.

In the case of adequate diagrams our calculations immediately give $\pm 1$ for these coefficients, and hence the result on the spread of such diagrams. In more general conditions the extreme terms considered will depend on the diagram chosen. When our extreme coefficients are both non-zero, we can again identify the spread of the Jones polynomial, while if one of the coefficients is zero we then at least get a better upper bound for the spread than the initial one from the given diagram.

# 1 The extreme states bound

We recall the states sum description of the Kauffman bracket polynomial $\langle D \rangle$ for an unoriented link diagram $D$.

Label the four quadrants at each crossing $A$ or $B$, according to the rule that the overcrossing strand sweeps out the $A$ quadrants when turned anticlockwise. The two possible local splittings are termed the $A$-split and the $B$-split.

A *state* $s$ of $D$ is a labelling of each crossing by either $A$ or $B$. Making the corresponding split for each crossing gives a number of disjoint embedded closed curves, called the *state circles* for $s$. We write $|s|$ for the number of state circles, and $a(s)$, $b(s)$ respectively for the number of crossings labelled $A$, $B$ in the state $s$. Following Kauffman [3] we retain information about the original crossings, in the form of chords on the split diagram, which we call $A$-chords or $B$-chords according to the splitting, as in figure 1.

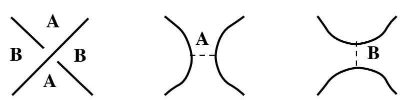



**Figure 1.**

The Kauffman bracket polynomial $\langle D \rangle \in \mathbf{Z}[A^{\pm 1}]$ is defined by

$$\langle D \rangle = \sum_{\text{states } s} \varphi_s,$$

where $\varphi_s = A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|-1}$.

We write $\max(s)$ for the maximum degree of $\varphi_s$, so that

$$\max(s) = a(s) - b(s) + 2|s| - 2,$$

and similarly $\min(s) = a(s) - b(s) - 2|s| + 2$.

Suppose that a state $s'$ is given by changing $k$ of the $A$-splittings of a state $s$ to $B$-splittings. It can be readily shown that

$$\max(s') \leq \max(s),$$

and that equality occurs if and only if $|s'| = |s| + k$. Similarly, $\min(s') \leq \min(s)$, with equality under the same conditions.

Write $s_A$ and $s_B$ for the *extreme states* in which all crossings are labelled $A$ or all are labelled $B$. It follows at once [3] that for any diagram $D$ with $c(D)$ crossings

$$\begin{aligned}\max\deg\langle D\rangle &\leq \max(s_A) = c(D) + 2|s_A| - 2\\ \text{and } \min\deg\langle D\rangle &\geq \min(s_B) = -(c(D) + 2|s_B| - 2).\end{aligned}$$

We will call these bounds the *extreme degrees* for $D$. They may of course be different from the actual minimum and maximum degrees. Indeed both depend on the diagram chosen to represent a given link, and they can be changed substantially by Reidemeister moves, whereas $< D >$ is unaltered by moves II and III and is simply multiplied by a power of $-A^3$ under move I.

The *spread* (or Laurent degree)

$$\beta(L) = \max\deg\langle D\rangle - \min\deg\langle D\rangle,$$

which depends only on the Jones polynomial of the link $L$ represented by $D$, then satisfies the *extreme states bound*

$$\beta(L) \leq 2c(D) + 2(|s_A| + |s_B| - 2).$$

We now give an algorithm for calculating the coefficients of the terms of extreme degree, $\max(s_A)$ and $\min(s_B)$, for a chosen diagram in terms of combinatorial features of the state curves and splitting chords for each extreme state. Thistlethwaite [6] showed that for an *adequate* diagram both extreme coefficients are $\pm 1$. Adequate diagrams include reduced alternating diagrams, and so the bound shown gives an exact count of the spread for any link with such a diagram. Thistlethwaite also found a bound for the spread expressed in terms of simple features of the decomposition of a given diagram into maximal alternating tangles. We show also how the non-alternating skeleton of the diagram can be used to calculate the extreme states bound, which is in general stronger than Thistlethwaite's bound.

# 2 The extreme coefficients

In this section we give a combinatorial formula for the coefficient $a_{s_A}$ of the extreme term of degree $\max(s_A)$.

As noted in section 1, the only contributions to $a_{s_A}$ come from states $s$ with $k$ $B$-splittings, such that $|s| = |s_A| + k$, or equivalently $|s| = |s_A| + b(s)$.

Each such state $s$ then contributes $(-1)^{|s_A|+k-1}$ to the extreme coefficient $a_{s_A}$.

Perform the $A$-splitting at each crossing of $D$ to get the $|s_A|$ state circles for the extreme state $s_A$, marking all the $A$-chords, as in figure 1.

Each state $s$ corresponds to a selection of a subset of $b(s)$ of the $A$-chords at which the $B$-splitting is to be done in place of the $A$-splitting. When a change of splitting takes place the number of circles either increases or decreases by 1, depending on whether the ends of the splitting $A$-chord lie on the same component or not. In order to finish with an extra $b(s)$ circles after performing $b(s)$ splittings we must increase the number of circles at each splitting.

**Theorem 1** *Necessary and sufficient conditions for the splitting along a set of $k$ $A$-chords of the state circles of $s_A$ to yield $k$ extra curves are*

*(i) the ends of each chord lie on the same state circle of $s_A$,*

*(ii) the ends of each pair of chords which lie on the same circle do not alternate in order round the circle.*

*Proof :* These conditions are clearly necessary, to ensure that the number of circles increases after each splitting. They are also sufficient, by induction on the number of chords, since after one splitting the conditions are maintained. □

**Definition.** A subset $C$ of the $A$-chords of the diagram $D$ is *independent* if it satisfies conditions (i) and (ii) above. We include the case $C = \phi$.

**Theorem 2** *The coefficient of the term of degree* $\max(s_A)$ *is given by*

$$a_{s_A} = (-1)^{|s_A|-1} \sum (-1)^{|C|},$$

*where the sum is taken over all independent sets of $A$-chords $C$ in the state circles for $s_A$.*

*Proof :* Each set of $A$-chords corresponds to a state of $D$, and the set $C$ makes a contribution of $(-1)^{|s_A|+|C|-1}$ to the term of degree $\max(s_A)$ if and only if $C$ is independent, as observed above. □

A diagram is $+$ *adequate* in Thistlethwaite's sense when $\max(s_A) > \max(s)$ for all states $s \neq s_A$. This is the case when there are no non-empty independent sets of $A$-chords.

An exactly similar analysis of the term of degree $\min(s_B)$ can be made in terms of the $B$-chords joining the state circles of the state $s_B$. Here the coefficient is $(-1)^{|s_B|-1}\sum(-1)^{|C|}$, with the sum taken over independent sets of $B$-chords. A diagram is $-$ adequate when $\min(s_B) < \min(s)$ for all $s \neq s_B$, or equivalently when there are no non-empty independent sets of $B$-chords.

In these calculations the only chords that need to be considered are those whose ends lie on the same state circle of $s_A$ (or $s_B$). Indeed we can reformulate theorem 2 in a graph theory context as follows.

**Definition.** Let $K$ be a graph and let $C$ be a subset of the vertices of $K$. Say that $C$ is *independent* if no two vertices of $C$ are joined by an edge of $K$. Define an integer-valued function $f$ on graphs by $f(K) = \sum(-1)^{|C|}$, where the sum is taken over all independent subsets $C$ of vertices of $K$, including the empty set.

Theorem 2 can then be restated in terms of the function $f$ for a suitable graph.

**Theorem 3 (graphical version)** *Let $K$ be the 'intersection graph', in the sense of Lando [1], of the $A$-chords with endpoints on the same state circle for $s_A$, namely the graph whose vertices are these chords, with an edge joining each pair of chords whose ends occur alternately on the same circle. Then the required coefficient $a_{s_A}$ is just $(-1)^{|s_A|-1}f(K)$.*

Calculation of $f$ for a general graph $K$ is simplified by the use of some readily established properties.

**Property 1 (Recursion)** *Let $K-v$ be the subgraph of $K$ given by deleting a vertex $v$ and its incident edges, and let $K-N(v)$ be given by deleting the immediate neighbours of $v$ along with their incident edges. Then*

$$f(K) = f(K-v) - f(K-N(v)).$$

*Proof :* This follows by grouping the subsets $C$ into those which do and those which do not contain the vertex $v$. □

**Property 2 (Multiplication under disjoint union)** *Let $K = K_1 \cup K_2$ with $K_1 \cap K_2 = \phi$ then $f(K) = f(K_1)f(K_2)$.*

*Proof :* This follows by induction on the number of vertices in $K_2$, and recursion. □

**Property 3 (Duplication)** *If an extra vertex $v$ is inserted which is* not *joined to one vertex $w$ but which is joined to* all *the neighbours of $w$ (and possibly to other vertices as well) then $f$ is unchanged.*

*Proof :* This follows from the first two properties, noting that here $K - N(v)$ has an isolated vertex $w$ and that $f = 0$ on a graph with a single vertex. □

**Remark.** The intersection graphs which arise here are naturally bipartite graphs, as the planar diagram of chords has one set of non-intersecting chords inside each circle, and another set of non-intersecting chords outside the curvecircle. Intersections in the graph can only take place between the two different types of chord, and can be realised by redrawing so that both sets of chords lie inside the circle, when the condition that the endpoints occur alternately will correspond to an intersection of a pair of chords. Such graphs are sometimes known as *circle graphs*.

While the calculation of $f(K)$ clearly depends only on the intersection matrix of the graph $K$, we do not have a simple formula for $f$ in terms of this matrix. Clearly if there is just a single chord then $f = 0$, and equally if there is no intersection among any of the chords then again $f = 0$ by the multiplicative property. On the other hand there are plenty of examples where the graph is non-empty and the value of $f$ is non-zero, giving us exact bounds on the spread of the bracket polynomial beyond the cases of $\pm$ adequate diagrams.

# 3 Alternating tangle decompositions

In using theorem 2 to calculate $a_{s_A}$ it is enough to consider the state circles individually, because of the multiplicative property 2 of $f$.

Where a diagram has a substantial number of alternating edges there will in general be many of the extreme state circles with no cross-chords. These can therefore be ignored completely in the calculation.

Consider the projection of the diagram $D$ as a 4-valent planar graph, which we call the *projection graph* of $D$. Each edge is either *alternating* or *non-alternating* according to the crossings at its ends. The non-alternating edges are of two types, *over* and *under*, indicated by $+$ and $-$ respectively.

The state circles of $s_A$ and the $A$-chords are constructed by separating the vertices of the projection graph slightly and inserting the appropriate chord. They can be generated dynamically as circuits in the projection graph by turning right at each undercrossing of $D$, and left at each overcrossing. We assume throughout that $D$ is not a split diagram, and is *reduced* in the sense that it has no cut-vertex. The closure of each complementary region of its graph in $S^2$ is then a disc. Any state circle for $s_A$ consisting entirely of alternating edges forms the boundary of one of these discs, as the remainder of the graph lies entirely on the same side of the state circle which consequently has no cross-chords. In using theorem 2 we need then only consider chords with ends on those state circles which include some non-alternating edges.

To find these systematically we draw the graph $G$ which is dual to the non-alternating edges in the projection graph of $D$; there is one vertex of $G$ in each complementary disc of the projection graph whose boundary contains non-alternating edges.

When we superimpose $G$ on the original knot diagram $D$ we find that the complementary regions of $G$ are discs if $G$ is connected, or more generally discs with holes, which meet $D$ in alternating tangles. The intersections of $D$ with these complementary regions are the maximal alternating pieces as defined by Thistlethwaite [6]; when $G$ is not connected some of them will be tangles in a disc with holes, rather than a classical tangle in a disc. In our setting, Thistlethwaite's 'channels' are the planar neighbourhoods of the components of $G$.

In any event, the major part of each tangle can be ignored in making our calculations, and we concentrate on the graph $G$, which we call the *non-alternating skeleton* of $D$.

Each state circle for $s_A$ made up of alternating edges bounds a disc lying entirely in one of the alternating tangles. Only the state circles with non-alternating edges intersect the skeleton $G$; we show how to recover them up to isotopy by making a standard splitting of the graph $G$. We then add the 'boundary information' about the $A$-chords with ends on these circles to complete the data needed in the calculations of theorem 2.

# 4 The non-alternating extreme state circles

The non-alternating edges of the diagram $D$ and hence the edges of its non-alternating skeleton $G$ come in two types, *over*, labelled $+$ and *under*, labelled $-$. As we trace out a state circle of $s_A$ which contains some non-alternating edges we will come to a non-alternating over edge, labelled $+$, where the circle will cross $G$. It will then continue past some crossings, turning left each time, until it reaches the next non-alternating edge, necessarily an under edge, with sign $-$, where it again crosses $G$. These two edges of $G$ have a common vertex, lying in the same complementary region of the projection graph, and so the local picture of the projection graph and $G$ will look like figure 2.

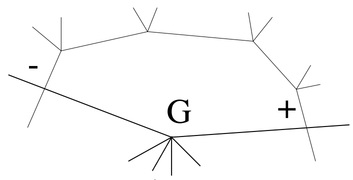



**Figure 2**

The two edges of $G$ are then isotopic, in the complement of $G \cup D$ to this segment of the $s_A$ state circle.

The $+$ and $-$ edges must alternate around each vertex of $G$, and all pairs will correspond in this way to pieces of states circles. So when we break the graph $G$ apart at each vertex by pairing adjacent $+$ and $-$ edges, matching each $+$ edge with the next $-$ edge in the anticlockwise sense, as in figure 3,

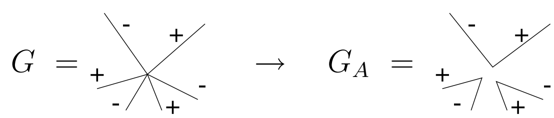



**Figure 3**

the resulting curves, which we denote by $G_A$, are isotopic to the $s_A$ curves with non-alternating edges. These are the only $s_A$ circles which can appear in our extreme term calculation.

Separating all the vertices of $G$ in the opposite sense will similarly yield curves $G_B$ isotopic to the non-alternating $s_B$ circles.

## 4.1 Boundary information

Having found all the $s_A$ circles needed for theorem 2 we can identify those $A$-chords which may be involved in the formula. Recalling that only $A$-chords with both ends on the same circle will contribute, we may treat the components of $G$ separately, and combine the results by use of the multiplicative property of $f$.

From our picture of the construction of the non-alternating $s_A$ circles we see that the possible $A$-chords occur in a complementary region of $G$ where there is an arc across the region which passes through just one crossing of the projection graph, as in figure 4.

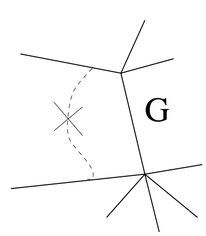



**Figure 4**

To give an $A$-chord the crossing must be approached through the $B$-quadrants, in terms of the original diagram $D$. The $A$-chord which arises from the crossing is then isotopic to this arc when the $s_A$ circle is isotoped to $G_A$. The union of such arcs drawn across the complementary regions of $G$ is the 'boundary information' which we need for the $A$-chords, with a similar union of arcs drawn for the $B$-chords.

### 4.2 Algorithm for finding the extreme coefficients

Given a diagram $D$ we can assemble the data needed to apply theorem 2 as follows.
**Step 1**. Construct the non-alternating skeleton $G$.
**Step 2**. Add the boundary information.

For each component of $G$ consider separately each tangle defined by $D$ in the complementary regions of this component. Draw any arcs across each tangle which meet $D$ in just one crossing approached through the $B$-quadrants. In general there will be a relatively small number of these, as they can only involve the complementary regions of $D$ which are adjacent to the boundary of the tangle, hence the term 'boundary information'. The result is to decorate $G$ with a number of non-intersecting chords drawn across the complementary regions.
**Step 3**. Separate $G$ into the non-alternating $s_A$ circles $G_A$ by splitting apart at the vertices, as above, while retaining the decorating chords.
**Step 4**. For each circle separately calculate the value of $f$ on the intersection graph given by the cross-chords. Multiply the values, to give the eventual value of the coefficient $a_{s_A}$ of top degree.

Repeat steps 2-4 with the $B$-chords across the tangles, and the splitting of $G$ into $s_B$ circles, to find the lowest degree coefficient similarly.

*Proof :* Separating $G$ gives the circles $G_A$ which are isotopic to the non-alternating $s_A$ state circles. The boundary information recovers all possible $A$-chords with ends on the same circle. The calculations then follow from theorem 2 and property 2 of $f$. □

## 5 States surfaces

One of the neatest techniques in the proof of the original results about alternating diagrams is the use of 'states surfaces'. Each state $s$ of a diagram $D$ has a dual state $\hat{s}$ defined by changing all the markers of $s$. In particular the extreme states $s_A$ and $s_B$ are dual to each other. The states surface for $s$ is a closed orientable surface with Euler characteristic $|s| + |\hat{s}| - c(D)$. The *extreme states surface* $F$ for the states $s_A$ or $s_B$ then has Euler characteristic $\chi(F) = |s_A| + |s_B| - c(D)$ and the extreme states bound for the spread of $\langle D\rangle$, which is $2(c(D) + |s_A| + |s_B| - 2)$, can be written as $4c(D) - 4g(F)$ in terms of the genus $g(F)$ of $F$.

In [7] Turaev gives a construction for the extreme states surface $F$ in which discs round each crossing of $D$ are connected by an untwisted band for each alternating edge, and a half-twisted band for each non-alternating edge. This gives a surface with $|s_A| + |s_B|$ boundary components, and yields $F$ when they are capped off by discs.

Make this construction with the non-alternating skeleton $G$ in place, inserting first only the bands for the alternating edges. The boundary of the resulting planar surface includes all state circles for $s_A$ and $s_B$ made of alternating edges only. These all lie in complementary regions of $G$, along with circles parallel to the boundary of each complementary region. Capping off the alternating state circles then gives the complement of a neighbourhood of $G$.

The surface $F$ is completed by adding a twisted band across each edge of $G$ and capping off the boundary of the resulting surface. The boundary curves of this surface can be readily identified with the non-alternating state circles given by separating the vertices of $G$ to yield the curves $G_A$ and $G_B$. Then

$$\begin{aligned} \chi(F) &= 2 - \chi(G) - e(G) + |G_A| + |G_B| \\ &= 2 - v(G) + |G_A| + |G_B|, \end{aligned}$$

where $G$ has $v(G)$ vertices and $e(G)$ edges. The extreme states bound is then

$$\begin{aligned} 4c(D) - 4g(F) &= 4c(D) + 2\chi(F) - 4 \\ &= 4c(D) + 2(|G_A| + |G_B|) - 2v(G). \end{aligned}$$

The extreme states bound for the spread can thus be found readily in terms of the non-alternating skeleton $G$, as an embedded graph (so as to find $G_A$ and $G_B$).

**Theorem 4** *The extreme states bound is lower than Thistlethwaite's bound in general.*

*Proof :* Thistlethwaite's bound for the spread $\max\deg\langle D\rangle - \min\deg\langle D\rangle$ is given in terms of the number of alternating tangles $n$, (the number of complementary regions of $G$), and the number of non-alternating edges $\nu$ ($=e(G)$). Explicitly, his bound is $4c(D) + 4(n-1) - 2\nu$.

Suppose that $G$ has $r$ components, so that $n = r + 1 - v(G) + e(G)$. Now construct a surface from $G$ by putting a disc at each vertex, and joining them by a twisted band for each edge. The boundary can again be regarded as the curves $G_A$ and $G_B$. Cap these off to give a closed surface $F^*$ with $r$ components, so that $\chi(F^*) \leq 2r$. Then

$$0 \leq 4r - 2\chi(F^*) = 4r - 2(v(G) - e(G) + |G_A| + |G_B|),$$

and so the extreme states bound of $4c(D) - 4g(F)$ satisfies

$$\begin{aligned} 4c(D) - 4g(F) &= 4c(D) + 2(|G_A| + |G_B|) - 2v(G) \\ &\leq 4c(D) + 4r - 4v(G) + 4e(G) - 2e(G) \\ &= 4c(D) + 4(n-1) - 2\nu. \end{aligned}$$

□

# 6 Some examples

In figure 5 we show a diagram with its non-alternating skeleton $G$, and the two sets of curves $G_A$ and $G_B$ resulting from splitting $G$. In this case $e(G) = 10$ and $\chi(G) = -2$, while $|G_A| = |G_B| = 2$. The extreme states bound for the spread of the bracket polynomial is then $4c + 2 \times 4 - 2(\chi(G) + e(G)) = 4c - 8$, giving a bound of $c - 2 = 19$ for the spread of its Jones polynomial.

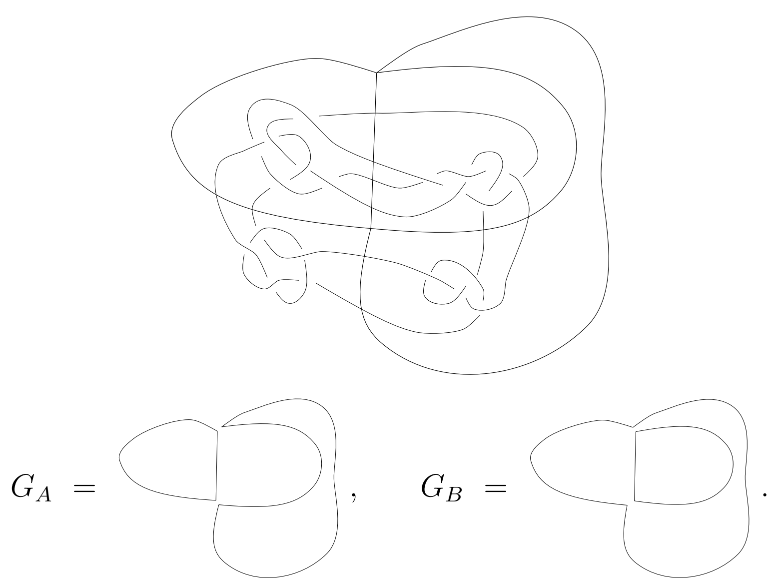



**Figure 5**

In figure 6 we show separately the reducing $A$-chords and $B$-chords on $G$. After splitting $G$ and retaining only the cross-chords with both ends on the same circle we get the essential boundary information shown in figure 7.

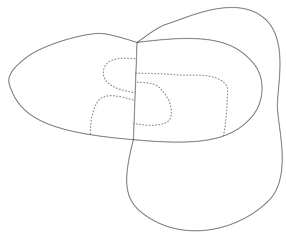
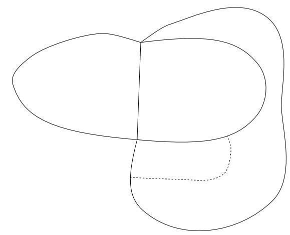
$A$-chords,  , $B$-chords,  .

**Figure 6**

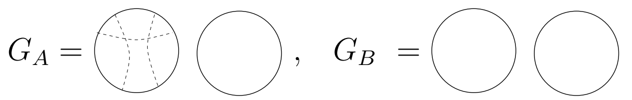


**Figure 7.**

Since there are no chords on $G_B$ the diagram is $-$ adequate, but in view of the three chords on $G_A$ it is not $+$ adequate. Apply the function $f$ to the circle graph of $G_A$ to calculate $\hat{a}_{s_A} = (-1)^{|s_A|-1} a_{s_A} = -1$. Both extreme coefficients are then non-zero and we deduce that the exact spread of the Jones polynomial is $c-2$.

The diagram in figure 8 has the same non-alternating skeleton $G$ and only differs from figure 5 in that the alternating tangle in one of the complementary regions of $G$ has been rotated.

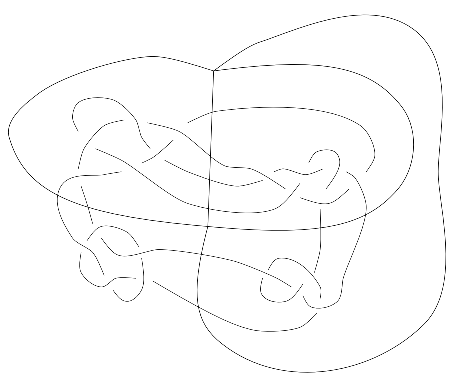


**Figure 8**

Thus $G_A$ and $G_B$, and the bound of $c-2$ for the spread of the Jones polynomial are unaltered. However the two reducing chords in the rotated tangle now lie in a different way relative to $G$, as shown in figure 9, and the new boundary information for $G_A$ is shown in figure 10.

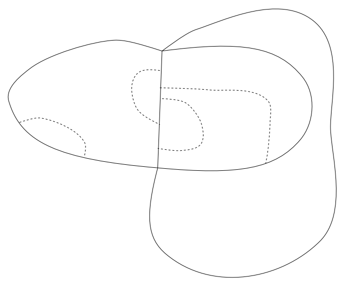


**Figure 9**

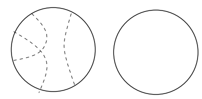
$G_A$ =  .

**Figure 10.**

The diagram is still $-$ adequate, but $f = 0$ for the circle graph of $G_A$ and so $a_{s_A} = 0$. The spread of the Jones polynomial is then at most $c - 3 = 18$.

## 6.1 Calculations on Rolfsen's tables

Table 1 shows the reducing chords, and the corresponding coefficients, for the diagrams of knots up to 10 crossings. The source of the table is Rolfsen's knot diagram table of 10 crossings or less. Since the maximal and minimal terms of the alternating knots are already known, we list data for non-alternating knots only. The non-alternating skeleton for all these diagrams in Rolfsen's table consists of a single circle, so $G_A$ and $G_B$ are a single curve in each case.

In each row of the table we show the diagrams of the $A$-reducing and $B$-reducing chords. The extreme coefficients are given by $a_{s_A} = (-1)^{|s_A|-1}\hat{a}_{s_A}$ and $a_{s_B} = (-1)^{|s_B|-1}\hat{a}_{s_B}$, where $\hat{a}_{s_A}$ and $\hat{a}_{s_B}$ are calculated directly from the intersection graph using the function $f$. The number $\hat{\beta}$ is the extreme states bound for the spread of the Jones polynomial. This is equal to the actual spread$\beta$ when $a_{s_A} \neq 0$ and $a_{s_B} \neq 0$. The value of $\beta$ in other cases is noted in the final column of the table for comparison, calculated directly from the Jones polynomial.

Where the diagram of $G_A$ or $G_B$ has no chords the knot diagram is $\pm$ adequate, so we can see that all knots of 10 crossings or less are + adequate or $-$ adequate. In particular, $10_{152}, 10_{153}, 10_{154}$ are the only adequate knots of 10 crossings or less which are non-alternating.

The two different values in $\hat{a}_{s_B}$ for $10_{144}$ reflect a mistake in the diagram in Rolfsen's table. For the knot which is presented by Rolfsen's diagram, $\hat{a}_{s_B} = -1$ while for the genuine knot $10_{144}$, $\hat{a}_{s_B} = 2$.

Table 1. Reducing chords and extreme coefficients up to 10 crossings.

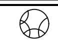
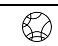
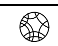
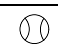
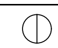
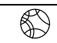
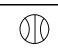
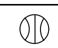
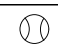
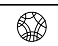
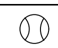
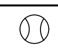
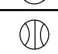
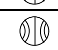
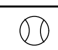
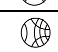
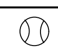
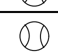
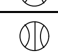
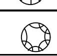
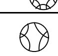
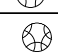
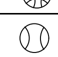
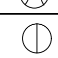
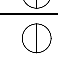
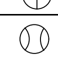
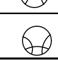
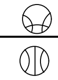
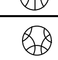
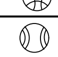
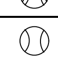
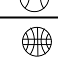
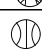
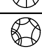
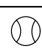
| Knot | $\hat{\beta}$ | $\hat{a}_{s_B}$ | chords on $G_B$ | $\hat{a}_{s_A}$ | chords on $G_A$ | $\beta(\langle L \rangle)$ |
|---|---|---|---|---|---|---|
| $8_{19}$ | 6 | 0 |  | 1 | | 5 |
| $8_{20}$ | 6 | 1 |  | 1 | | |
| $8_{21}$ | 6 | 2 |  | 1 | | |
| $9_{42}$ | 8 | 0 |  | 1 | | 6 |
| $9_{43}$ | 8 | 0 |  | 1 | | 7 |
| $9_{44}$ | 7 | 1 |  | 1 | | |
| $9_{45}$ | 8 | 0 |  | 1 | | 7 |
| $9_{46}$ | 8 | 0 |  | 1 | | 6 |
| $9_{47}$ | 8 | 0 |  | 1 | | 7 |
| $9_{48}$ | 7 | 2 |  | 1 | | |
| $9_{49}$ | 8 | 0 |  | 1 | | 7 |
| $10_{124}$ | 9 | 0 |  | 1 | | 6 |
| $10_{125}$ | 9 | 0 |  | 1 | | 8 |
| $10_{126}$ | 9 | 0 |  | 1 | | 8 |
| $10_{127}$ | 9 | 0 |  | 1 | | 8 |
| $10_{128}$ | 8 | 0 |  | 1 | | 6 |
| $10_{129}$ | 9 | 0 |  | 1 | | 8 |
| $10_{130}$ | 9 | 0 |  | 1 | | 8 |
| $10_{131}$ | 9 | 0 |  | 1 | | 8 |
| $10_{132}$ | 7 | 0 |  | 1 | | 5 |
| $10_{133}$ | 8 | 1 |  | 1 | | |
| $10_{134}$ | 8 | 1 |  | 1 | | |
| $10_{135}$ | 9 | 0 |  | 1 | | 8 |
| $10_{136}$ | 9 | 0 |  | 1 | | 7 |
| $10_{137}$ | 9 | 0 |  | 1 | | 8 |
| $10_{138}$ | 9 | 0 |  | 1 | | 8 |
| $10_{139}$ | 8 | -1 |  | 1 | | |
| $10_{140}$ | 9 | 0 |  | 1 | | 7 |
| $10_{141}$ | 8 | 1 |  | 1 | | |
| $10_{142}$ | 9 | 0 |  | 1 | | 6 |
| $10_{143}$ | 9 | 0 |  | 1 | | 8 |
| $10_{144}$ | 8 | -1(2) |  | 1 | | |

Table 1. Continued.

| Knot | $\hat{\beta}$ | $\hat{a}_{s_B}$ | chords on $G_B$ | $\hat{a}_{s_A}$ | chords on $G_A$ | $\beta(\langle L\rangle)$ |
|---|---|---|---|---|---|---|
| $10_{145}$ | 8 | -1 |  | 1 | ○ | |
| $10_{146}$ | 9 | 0 |  | 1 | ○ | 8 |
| $10_{147}$ | 8 | 1 |  | 1 | ○ | |
| $10_{148}$ | 9 | 0 |  | 1 | ○ | 8 |
| $10_{149}$ | 9 | 0 |  | 1 | ○ | 8 |
| $10_{150}$ | 8 | 1 |  | 1 | ○ | |
| $10_{151}$ | 9 | 0 |  | 1 | ○ | 8 |
| $10_{152}$ | 9 | 1 | ○ | 1 | ○ | |
| $10_{153}$ | 9 | 1 | ○ | 1 | ○ | |
| $10_{154}$ | 9 | 1 | ○ | 1 | ○ | |
| $10_{155}$ | 9 | 1 | ○ | 0 |  | 8 |
| $10_{156}$ | 8 | 1 | ○ | 1 |  | |
| $10_{157}$ | 8 | 1 | ○ | 2 |  | |
| $10_{158}$ | 9 | 0 |  | 1 | ○ | 8 |
| $10_{159}$ | 9 | 0 |  | 1 | ○ | 8 |
| $10_{160}$ | 8 | 0 |  | 1 | ○ | 7 |
| $10_{161}$ | 8 | -1 |  | 1 | ○ | |
| $10_{162}$ | 8 | -1 |  | 1 | ○ | |
| $10_{163}$ | 8 | 2 |  | 1 | ○ | |
| $10_{164}$ | 9 | 0 |  | 1 | ○ | 8 |
| $10_{165}$ | 8 | 2 |  | 1 | ○ | |
| $10_{166}$ | 9 | 0 |  | 1 | ○ | 8 |

## 6.2 Realisation of extreme coefficients

We finish with some results about the range of possible values of the extreme coefficients. It is certainly possible to find a graph $K$ with $f(K) = n$ for any chosen integer $n$. Indeed it is easy to see that $f(K_{n+1}) = -n$, where $K_{n+1}$ is the complete graph on $n+1$ vertices. On the other hand the circle graphs which determine the extreme coefficients form a proper subset of all bipartite graphs, and $f(K_{m,n}) = -1$ for the complete bipartite graph $K_{m,n}$.

We initially wondered whether any values of $f$ besides 0 and $\pm 2^k$ were possible for the extreme coefficients. We then managed to find a circle graph with $f = 3$, illustrated in figure 11 along with its realisation by chords, and used it to produce a link with extreme coefficient 3.

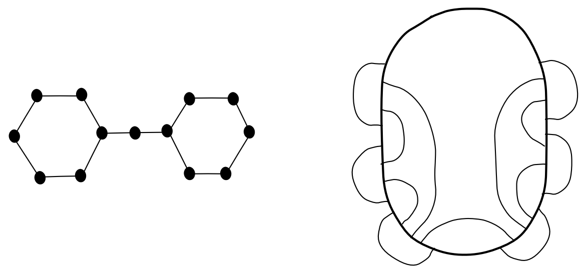


**Figure 11.**

This can be done by replacing each chord, , in the set $E \cup F$ of chords in the realisation, by a single crossing . The circle graph of figure 11 produces in this way the link shown in figure 12, whose bracket polynomial is $3A^{13} - 2A^9 + 4A^5 - A + 4A^{-3} - A^{-7} + A^{-11}$.

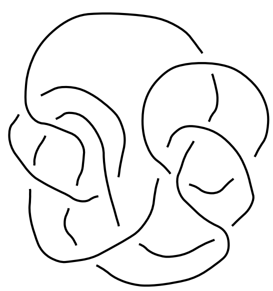


**Figure 12.**

More generally, given any set $E$ of non-intersecting chords inside a circle, and another set $F$ of non-intersecting chords outside the same circle the same procedure will construct a knot having a single curve $G_A$ with $E$ and $F$ as its $A$-reducing chords, hence with one extreme coefficient given by the intersection graph of these chords. As in the example above, the other extreme coefficient may be zero. A more elaborate construction, using for example  in place of some or all of the chords, can be made to ensure that the tangles used in the construction are $B$-reduced, while retaining the same or parallel families of $A$-reducing chords. Starting from a circle graph with value $f$ this will lead to a knot or link with a $-$adequate diagram, whose extreme coefficients are then $f$ and 1, up to sign. It is equally easy to extend this so that both extreme coefficients are any chosen values of $f$ for a circle graph.

The coefficients of the maximal and minimal degree terms in the Jones polynomial may not in general be values of $f$, since the knot may not have a diagram for which they appear as the extreme coefficients. Since the original version of this paper was written there has been a nice construction due to Manchon [5] giving circle graphs (or equivalently the families of chords $E$ and $F$) which realise every integer value of $f$.

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Department of Mathematics,
College of Natural Sciences,
Kyungpook National University,
Taegu, 702-701, Korea.

Department of Mathematical Sciences,
University of Liverpool,
Peach St.
Liverpool, L69 7ZL, England.

email `ybae@knu.ac.kr` `morton@liv.ac.uk`