

From Matrix Diagrams to the Associativity of  
Quaternion Multiplication

Matrix Mult

(0)

DK

$$M : M_{ij} \quad \left. \begin{matrix} \\ N : N_{ke} \end{matrix} \right\} \begin{matrix} n \times a \\ \text{matrices} \end{matrix}$$

$$\overset{i}{\overbrace{M}} \underset{j}{\overbrace{N}} = M_{ij}$$

$$\overset{i}{\overbrace{M}} \overset{j}{\overbrace{N}} = \sum_k M_{ik} N_{kj}$$

def   
 $\uparrow$   $\uparrow$   $\uparrow$   
sum over  
all  $k$   
on a closed edge

$$\overset{i}{\overbrace{M}} \overset{j}{\overbrace{N}} = \overset{i}{\overbrace{MN}}$$

$$(MN)_{ij} = \sum_k M_{ik} N_{kj}$$

||  
 $(i^{\text{th Row}}(M)) \cdot (j^{\text{th col}}(N))$

$$\overset{i}{\overbrace{I}} \underset{j}{\overbrace{I}} = I_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

Sometimes use

$$\overset{i}{\overbrace{---}} \equiv I$$

$$u \left[ \begin{array}{c} \text{vector} \\ \text{index} \end{array} \right] \left[ \begin{array}{c} u = u_1 \\ |_1 \\ u = u_2 \\ |_2 \\ u = u_3 \\ |_3 \\ \vdots \\ u = u_i \end{array} \right] \quad \text{①}$$

$$u \cdot v \stackrel{\text{def}}{=} \sum_{i=1}^3 u_i v_i = u \cdot v$$

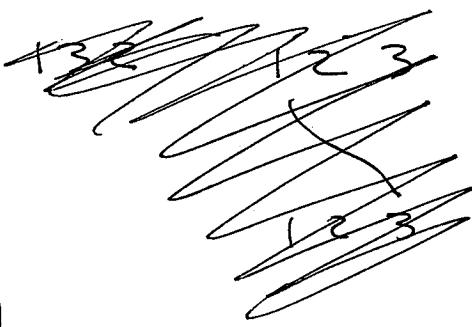
$$\begin{matrix} i \\ \diagdown \\ j \\ \diagup \\ k \end{matrix} \stackrel{\text{def}}{=} \epsilon_{ijk} \text{ where } \begin{cases} \epsilon_{123} = +1 \\ \epsilon_{132} = -1 \\ \epsilon_{213} = +1 \\ \epsilon_{231} = -1 \\ \epsilon_{312} = +1 \\ \epsilon_{321} = -1 \end{cases} \left. \begin{array}{l} \epsilon_\alpha = \text{sgn}(\alpha) \\ \alpha \in S_3 \end{array} \right\}$$

$$\begin{matrix} 1 \\ \diagdown \\ 2 \\ \diagup \\ 3 \end{matrix} = 1$$

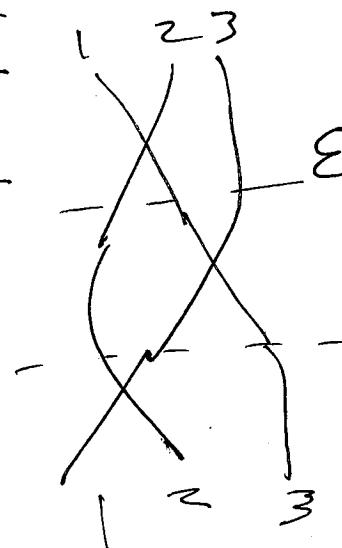
$$\begin{matrix} 2 \\ \diagdown \\ 1 \\ \diagup \\ 3 \end{matrix} = -1$$

etc

$$\epsilon_{123} = 1$$



$$123 \rightarrow 132$$



$$\epsilon_{112} = 0 \text{ e.g.}$$

$$\epsilon_{ijk} = \epsilon_{jik}$$

etc

$$\epsilon_{112} = -\epsilon_{112}$$

$$\therefore \epsilon_{112} = 0$$

$$\begin{aligned} \epsilon_{123\dots n} &= 1 \\ \epsilon_{i_1 i_2 \dots i_n} &= -\epsilon_{i_1 i_2 \dots i_{(n-1)} i_n} \end{aligned}$$

(2)

A  $3 \times 3$  matrix

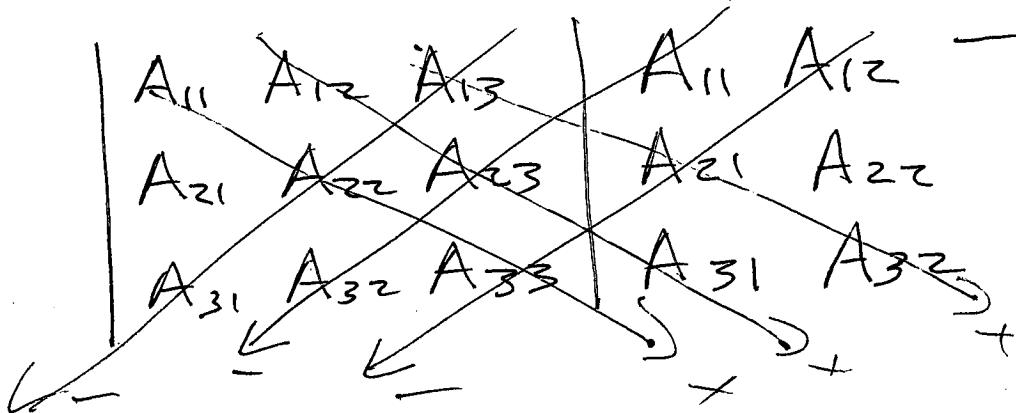
$$|A| = \sum_{\sigma \in S_3} \text{sgn}(\sigma) A_{1\sigma_1} A_{2\sigma_2} A_{3\sigma_3}$$

$\sigma \in S_3$

$(\sigma = ijk)$   
odd diff  
 $\sigma = \sigma_1 \sigma_2 \sigma_3$

+ 123  
+ 231  
+ 312

— 321  
— 132  
— 213



II

$$A_{11} A_{22} A_{33} + A_{12} A_{23} A_{31} + A_{13} A_{21} A_{32}$$

$$- A_{13} A_{22} A_{31} - A_{11} A_{23} A_{32} - A_{12} A_{21} A_{33}$$

123  
321

II

$$\sum_{\sigma \in S_3} \text{sgn}(\sigma) A_{1\sigma_1} A_{2\sigma_2} A_{3\sigma_3}$$

$\sigma \in S_3$

$$321 \rightarrow 312 \rightarrow 132 \rightarrow 123$$

(3)

 $A_{2 \times 2}$ 

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

$\uparrow$        $\uparrow$   
 $(12)$        $(z_1)$

$$= \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) A_{1\sigma_1} A_{2\sigma_2}$$

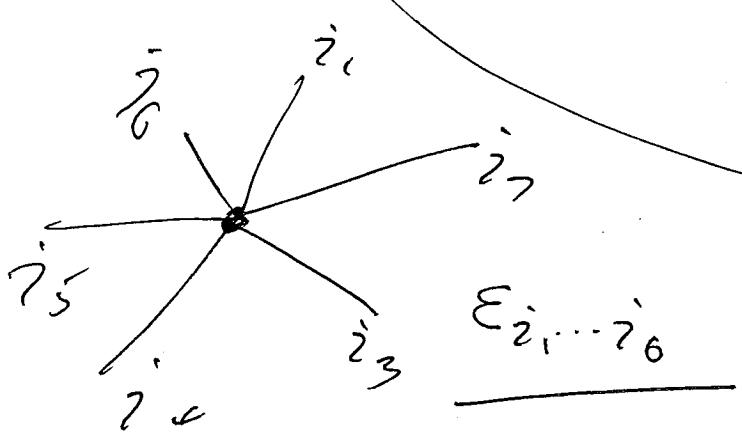
Genl def  $A_{nxn}$ 

$$\rightarrow |A| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma_1} \cdots A_{n\sigma_n}$$

Use to prove  $|AB| = |A||B|$ .

$$\Rightarrow |A| = \sum_{\substack{i_1, \dots, i_n=1 \\ i_1 \dots i_n}}^n \epsilon_{i_1 \dots i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$

vanishes  
when  
 $i_1 \dots i_n$  not a perm.



$k=1, 2, 3$

(4)

$$(u \times v)_{ik} = \sum_{i,j=1}^3 \epsilon_{ijk} u_i v_j$$

$$\begin{aligned}
 (u \times v)_1 &= \sum_{i,j=1}^3 \epsilon_{ij1} u_i v_j \\
 &= \epsilon_{231} u_2 v_3 + \epsilon_{321} u_3 v_2 \\
 &= +u_2 v_3 - u_3 v_2
 \end{aligned}$$

$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

||

$$\vec{j} \begin{vmatrix} u_2 v_3 \\ v_2 v_3 \end{vmatrix} \quad u_2 v_3 - u_3 v_2$$

$$-\vec{i} \begin{vmatrix} u_1 u_3 \\ v_1 v_3 \end{vmatrix} \quad \left\{ \begin{array}{l} -u_1 v_3 \\ +u_3 v_1 \end{array} \right.$$

$$+\vec{k} \begin{vmatrix} u_1 u_2 \\ v_1 v_2 \end{vmatrix} \quad \left\{ \begin{array}{l} u_1 v_2 \\ -u_2 v_1 \end{array} \right.$$

$$\begin{aligned}
 (u \times v)_2 &= \sum_{i,j=1}^3 \epsilon_{ij2} u_i v_j \\
 &= \epsilon_{132} u_1 v_3 + \epsilon_{312} u_3 v_1 \\
 &= -u_1 v_3 + u_3 v_1
 \end{aligned}$$

$$\begin{aligned}
 (u \times v)_3 &= \sum_{i,j=1}^3 \epsilon_{ij3} u_i v_j \\
 &= \epsilon_{123} u_1 v_2 + \epsilon_{213} u_2 v_1 \\
 &= u_1 v_2 - u_2 v_1
 \end{aligned}$$

✓

(5)

$$u \swarrow \begin{matrix} i \\ \kappa \end{matrix} \searrow v = \underset{\pi}{\cancel{u \times v}} \quad | \quad (u \times v)_k = \sum_{i,j} \epsilon_{ijk} u_i v_j$$

$$u \swarrow \begin{matrix} \vee \\ \pi \end{matrix} = u \times v \quad | \quad \text{Diagram with three lines labeled } a, b, c \text{ meeting at a point, labeled } \epsilon_{abc}$$

$$\epsilon_{abc} = \epsilon_{bca} = \epsilon_{cab}$$

Lemma.

$\text{G}(\text{C})$	$\text{G}(\text{R})^+$
$\text{G}(\text{C})$	$\text{G}(\text{R})^+$
$\text{G}(\text{C})$	$\text{G}(\text{R})^-$

$$u \swarrow \begin{matrix} i \\ \rho \end{matrix} \searrow v = - \left( \begin{matrix} i \\ m \end{matrix} \right) \left( \begin{matrix} j \\ l \end{matrix} \right) + \left( \begin{matrix} i \\ m \end{matrix} \right) \left( \begin{matrix} j \\ l \end{matrix} \right)$$

$$u \swarrow \begin{matrix} i \\ \sigma \end{matrix} \searrow v = + \quad | \quad \text{Diagram with three lines labeled } 1, 2, 3 \text{ meeting at a point, labeled } \epsilon_{123}$$

$$u \swarrow \begin{matrix} i \\ \delta \end{matrix} \searrow v = - \quad | \quad \text{Diagram with three lines labeled } 1, 2, 3 \text{ meeting at a point, labeled } \epsilon_{132}$$

$$\sum_k \epsilon_{ijk} \epsilon_{kem} = - \sum_{im} \delta_{ijl} + \sum_l \delta_{im}$$

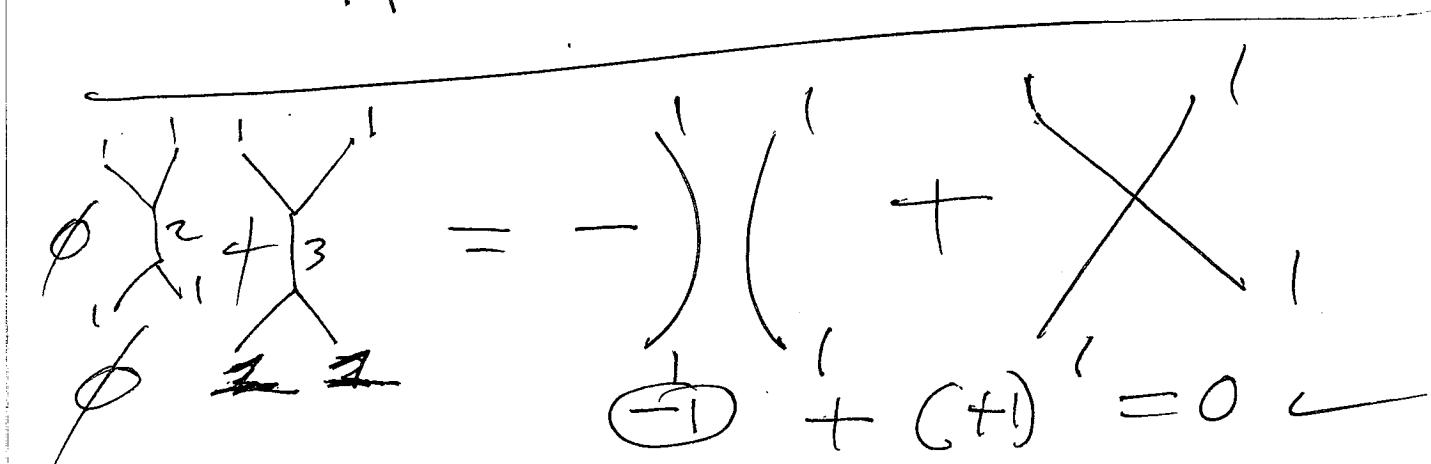
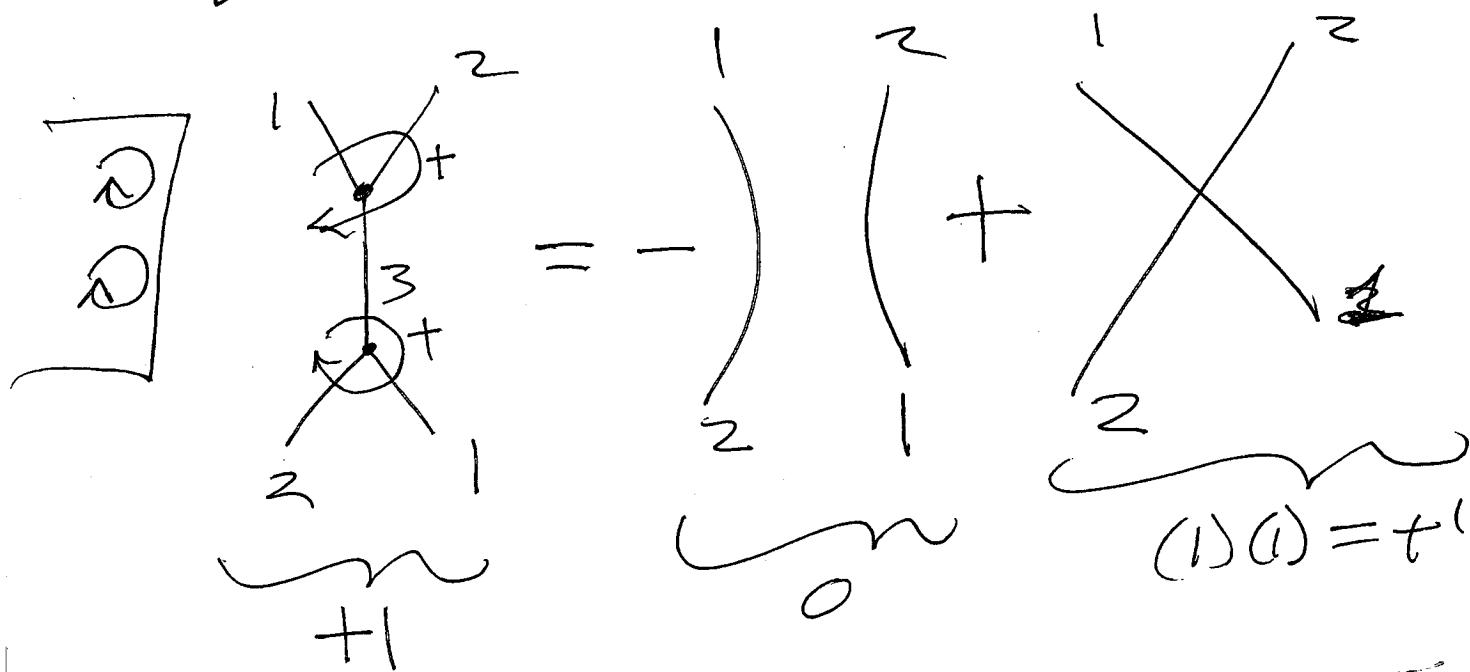
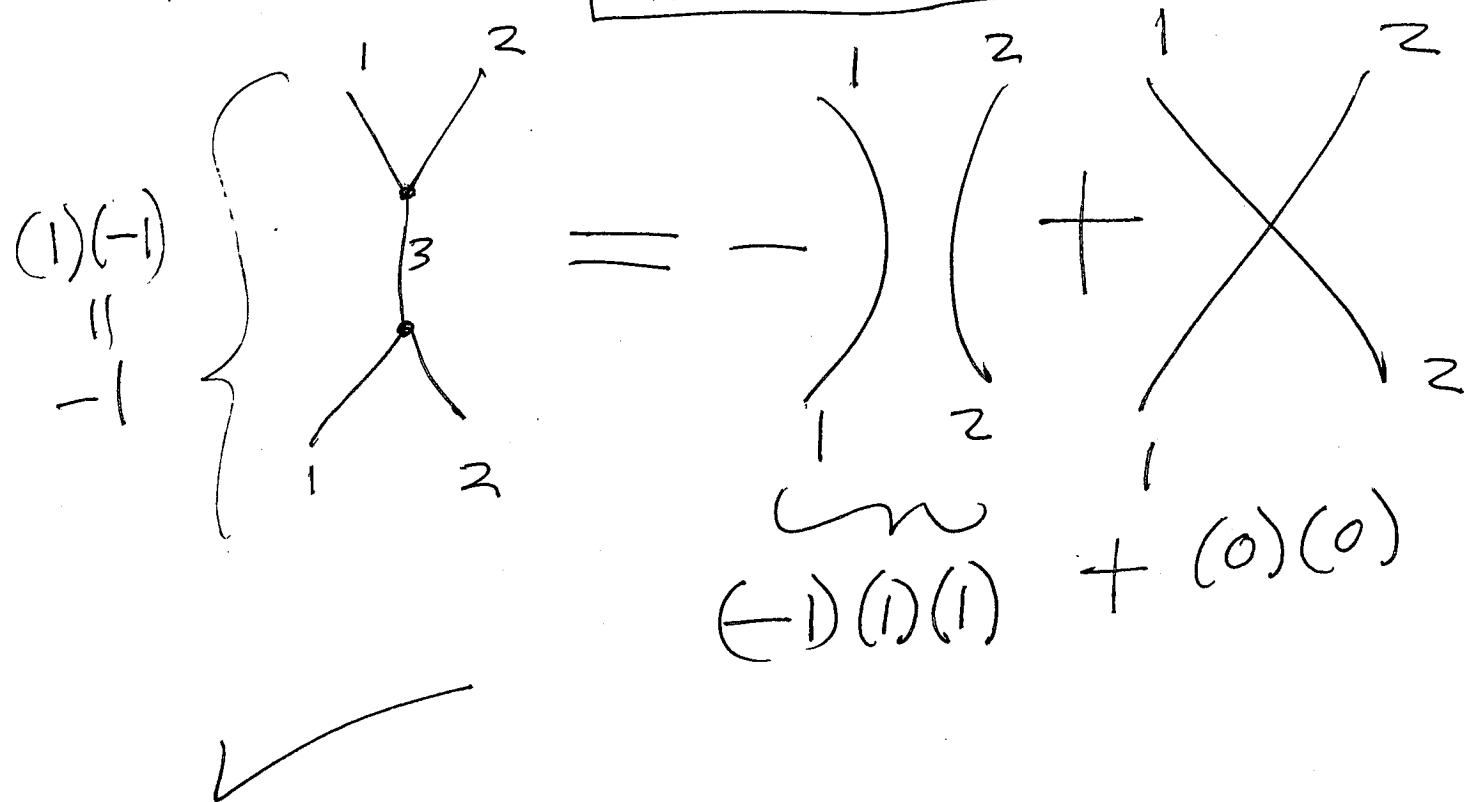
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta$$

$\downarrow$

$$\begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

(6)

$$\boxed{I_3^7 = 0, I_7^7 = 1 \text{ etc.}}$$



$$u, v, w = u \cdot v, \quad u \cdot v = u \times v \quad \textcircled{7}$$

$$u \cdot (v \times w) = u \downarrow v \wedge w = u \downarrow v \wedge w$$

$$\boxed{X = -)(+X)}$$

$$= (u \times v) \cdot w$$

$$u \times (v \times w) = u \downarrow v \wedge w = -u \downarrow v \wedge w + u \downarrow v \wedge w$$

$$= -(u \cdot v)w + (u \cdot w)v$$

$$(u \times v) \times w = u \downarrow v \wedge w = -u \downarrow v \wedge w + u \downarrow v \wedge w$$

$$= -(v \cdot w)u + (u \cdot w)v$$

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(8)

$$(u \times v) \cdot (w \times z)$$

$$= u \begin{array}{c} v \\ \diagdown \\ \curvearrowleft \\ \diagup \\ w \end{array} z = - u \begin{array}{c} v \\ \curvearrowleft \\ w \end{array} z + u \begin{array}{c} v \\ \curvearrowright \\ w \end{array} z$$

$$= -(u \cdot z)(v \cdot w) + (u \cdot w)(v \cdot z)$$

$$= \begin{vmatrix} u \cdot w & u \cdot z \\ v \cdot w & v \cdot z \end{vmatrix}$$

$$u \times (v \times w) = -(u \cdot v) \underline{\underline{w}} + (u \cdot w) \underline{\underline{v}}$$

$$= \underline{\underline{-(v \cdot w)u}} + \underline{\underline{+(u \cdot w)v}}$$

$$(u \times v) \times w$$

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$u \cdot (v \times w) = u \cdot (-v \cdot w + v \times w)$$

$$= -(v \cdot w)u + u(v \times w)$$

$$u \cdot (v \times w) = -(v \cdot w)u - u \cdot (v \times w) + \underline{\underline{u \times (v \times w)}}$$

$$(u \cdot v)w = \underline{\underline{u \cdot v + u \times v}}w$$

$$= -(u \cdot v)w + (u \times v)w$$

$$(u \cdot v)w = -(u \cdot v)w - (u \times v) \cdot w + \underline{\underline{(u \times v) \times w}}$$

$$\begin{array}{lcl} u(3 + w) & = & 3u + uw \\ \text{R}^3 \xrightarrow{\text{scalar}} & \xrightarrow{\text{R}^3} & \\ & & = 3u - u \cdot w + u \times w \end{array}$$

etc.

$$(s+t)(v+w) = \cancel{st} + \cancel{sw} + tv + (-v \cdot w + vxw)$$

$$= \underline{\underline{st}} + \cancel{tv} + \cancel{vw} + \underline{\underline{(-v \cdot w + vxw)}}$$

$$\therefore u(vw) - (uv)w$$

Note:  $u \cdot (v \times w) = (u \times v) \cdot w$  (10)

$$= -(v \cdot w)u - u \cdot (v \times w) + u \times (v \times w)$$

$$+ (u \cdot v)w + (u \times v) \cdot w - (u \times v) \times w$$

$$= u \times (v \times w) - (u \times v) \times w$$

$$+ (u \cdot v)w - (v \cdot w)u.$$

$$= \cancel{-(u \cdot v)w} + \cancel{(u \cdot w)v} + \cancel{(v \cdot w)u} - \cancel{(u \cdot w)v}$$

$$+ \cancel{(u \cdot v)w} - \cancel{(v \cdot w)u}$$

$$= 0.$$

Thus quaternion mult  
is associative.