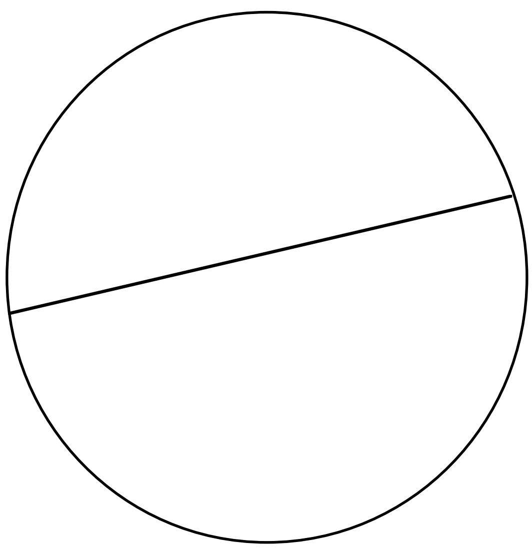
Math 215 - Spring 2013 - Homework Number 3.

1. Eccles, page 55, problems 14,15,19,20.

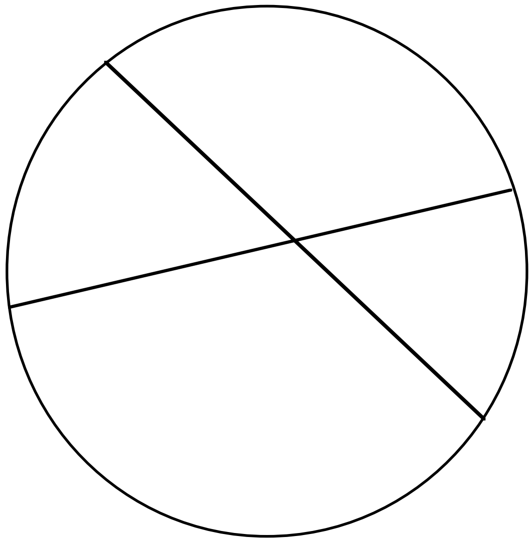
2. Let Un be the n-th Fibonacci number as defined by Eccles in Definition 5.4.2. Show by induction that
$U_1^2 + U_2^2 + \ldots + U_n^2 = U_n U_{n+1}$ **for n = 1,2,3,... .**

3. Prove that if a square is odd then it is one more than a multiple of 8. For example 9 = 1 + 8, 25 = 1 + 8 x 3, 49 = 1 + 8 x 6. (This is not a problem to be proved by induction. Note that if a square is odd, then it must be the square of an odd number.)

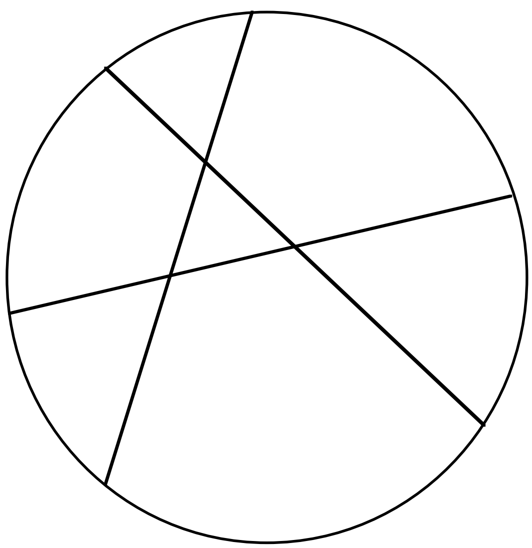
4. Consider straight lines cutting a disk. One line divides the disk into two regions.



Two lines divide the disk into four regions.



But three lines divide the disk into seven regions.



Experiment with this situation. Conjecture a formula for the (largest)number of regions that can be made by n lines cutting a disk. Prove your conjecture.

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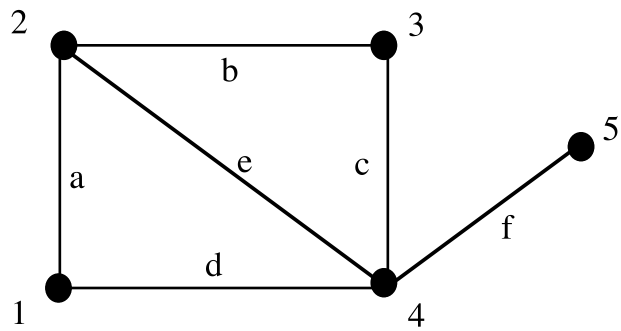
Information Section (Not Homework!)

Graph Theory and Euler's Formula.

A *graph* is a collection of *nodes* or *vertices*, usually depicted as dark spots or points, and a collection of *edges* that can connect two nodes or connect a node with itself. For example, the graph below has five nodes and six edges. It is a *connected* graph in the sense that there is a pathway along the edges between any two nodes.



A connected graph G.

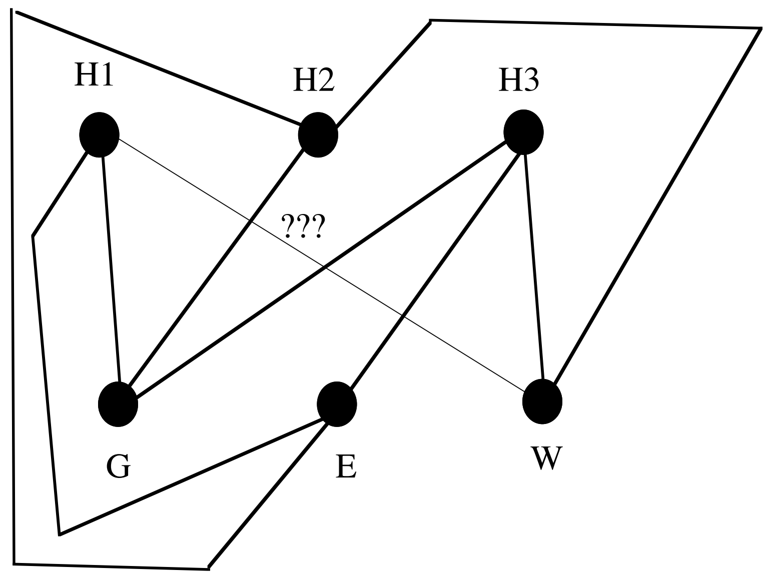
Graphs are fundamental mathematical structures and they have lots of applications. We are all familiar with the graphical notation for electrical circuits. Subway system maps are graphs with special decorations. In general, when we want to desribe engineering systems, economic systems, and other systems of relationship, we can start with a collection of definite entities (the nodes) and the information about how they are connected with one another (the edges).

As you can see, the game of sprouts is a game that is played by constructing a graph. The graph constructed in sprouts is special in that it cannot have more than three edges touching any node, and the sprouts graph is drawn in the plane in such a way that no two edges of it touch except at the nodes of th graph. We say that a graph that can be drawn in the plane in this way is a **plane graph**. The graph in the figure above is also a plane graph.

Not every graph is planar! That is if you specify a set of nodes and a set of connections to be made among these nodes, it may not be possible to accomplish these connections in the plane without having some edges cross over one another. Here is a problem that will show you how that can happen.

Problem. (The Gas-Electricity-Water Problem)
Three companies, the gas company, the electricity compay and the water company want to make connections from the gas main (G), the electrical source (E) and the water main (W) to three houses (H1, H2 and H3). They wish to lay their lines so that no two lines meet except at the sources (G,E and W) and at the houses

(H1, H2 and H3). Can you find a solution to this design problem? If not, then why not?



In the illustration above the city planners have drawn a graph to help them design the connections but they have run into a difficulty with making a water line from W to H1. Everything went fine with the design up to that point, but then there does not seem to be any way to conncet from W to H1 without crossing previously created lines. It will cost the city a great deal to dig tunnels to make lines cross over one another. So these designers really need to know whether the job can be done with no crossovers, and if it cannot be done that way, then they want to know the least number of crossovers that are needed to do the job.
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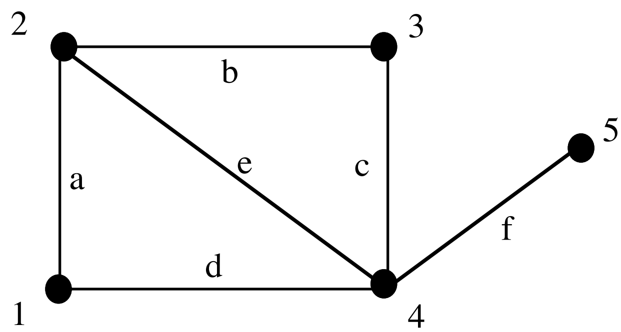
We will now discuss a formula about plane graphs that was discoverd around 1750 by the Swiss mathematician Leonhard Euler.
<http://en.wikipedia.org/wiki/Leonhard_Euler>
Euler was one of the greatest mathematicians of all time, and his formula about plane graphs is the beginning of the subjects of graph theory and topology (topology is the study of mathematical spaces and includes and generalizes classical geometry). Here is Euler's result:

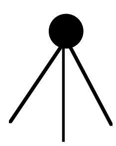
Theorem. Let **G** be a connected finite plane graph with **V** nodes, **E** edges and **F** faces (a face is a region in the plane that is delineated by the graph in the plane). Then **V - E + F = 2.**

Here is an example of Euler's formula for a specific graph in the plane.



Here we have V = 5, E = 6 and F = 3. The regions we count are the interiors of the two triangles and the outer region consisting in the rest of the plane. Note that V - E + F = 5 - 6 + 3 = 2 as promised by Euler's Theorem.

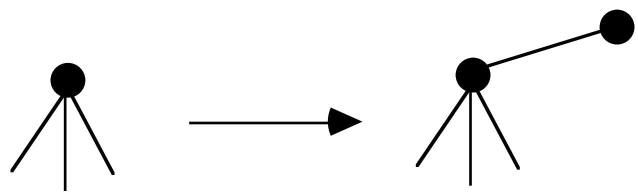
Problem. Construct a proof of Euler's formula by induction on the total number of edges and vertices in the graph **G**. You should consider how the graph can be built up from simpler graphs by adding edges to them. In fact, any connected graph can be built from a single vertex graph by adding new edges in two ways that I will now explain, but first we introduce an abbreviation: The diagram below stands for s*ome vertex in a larger graph.*



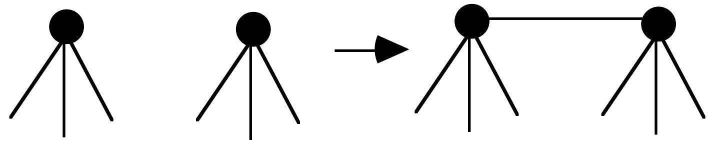
You can tell when I am using this abbreviation because the edges that go out of this vertex are not meeting any other vertices in the picuture. The picture is a shorthand for a possibly larger and more complete picture. In the abbreviation we show three edges touching the vertex. In a real situation some edges touch the vertex, but the number is not necessarily equal to three. Ok?

Now lets use this and illustrate two ways to make a larger graph.

I.

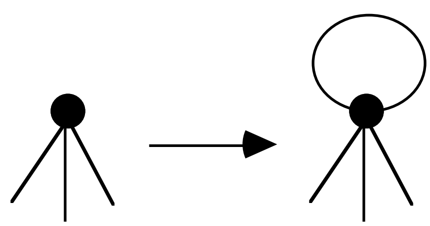


II.



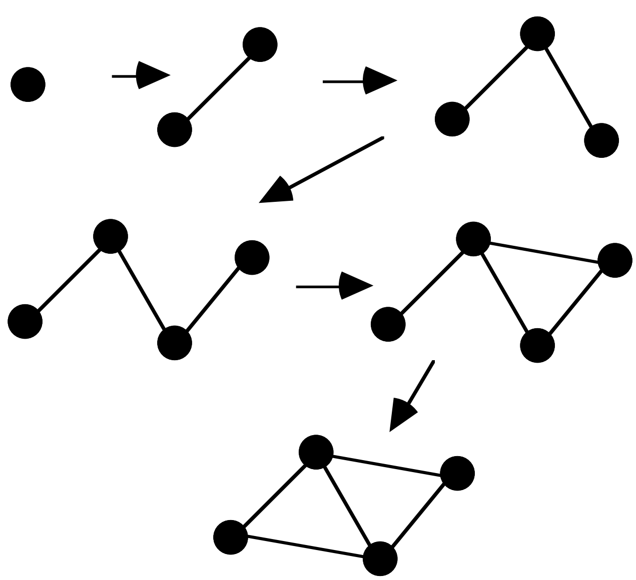
In method number I we add a new edge and a new vertex by attaching the new edge to an already existing vertex. In method number II we connect two vertices with a new edge.

Remark. We regard the move



as a special case of II.

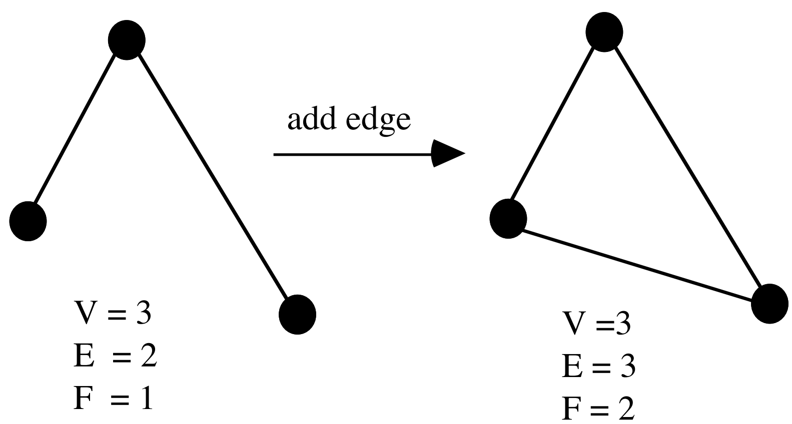
I claim that *any connected graph can be built up by performing a sequence of operations of these two types*. Here is an example.



You can use this claim in your proof, and if you want, you can also make a proof of the claim. We will discuss why the claim is true in class.

Now, to prove the Euler Theorem, you can proceed by induction, showing that V - E + F does not change its value when you perform a move of type I or type II. You will find that it is very easy to see this for type I, and that in order to see it for type II you need to start with a connected graph. If the graph is connected, then a move of type II will create a new region in the graph. Look at the example above and see how this works. You can use this fact also in your proof (that a move of type II will create a new region). You should then be able to construct an inductive proof of the Euler formula.

Here is an example:
We create a triangle graph by adding an edge to a tree.



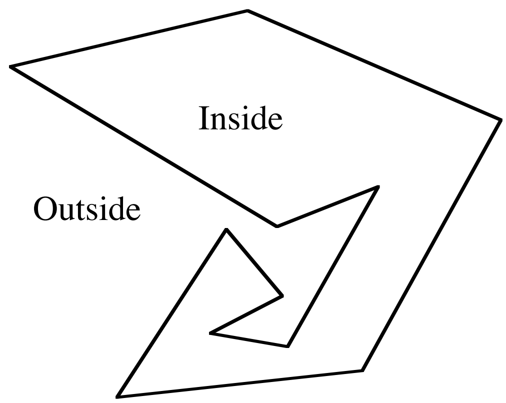
Note that adding the edge creates a new region, and V-E + F does not change from before to after the addition of the new edge.

(c) Discuss your proof of the Euler formula with another student in the class. Do you both feel that the proof is complete? What might be missing? In this problem, it worth having the discussion.

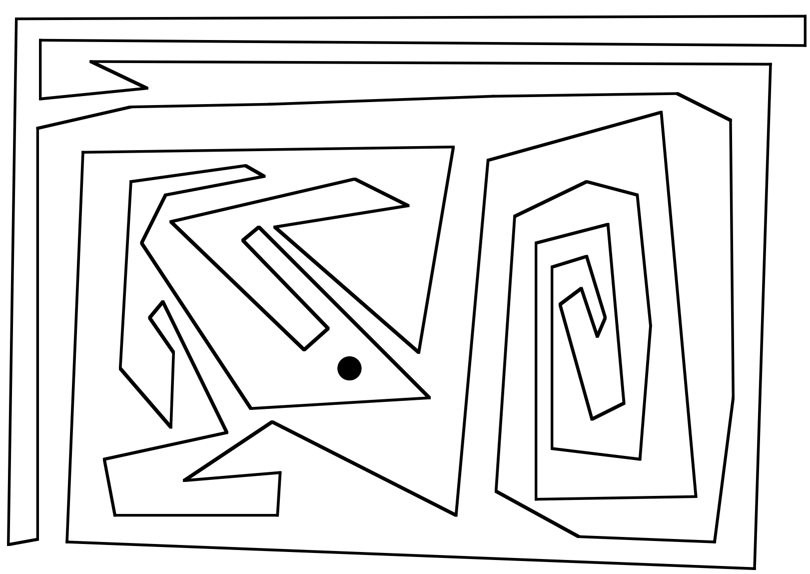
Supplement.
There is a fact about curves in the plane that you can use in thinking about regions that are created when graphs are drawn in the plane. This fact is called the

Jordan Curve Theorem: *A closed curve in the plane without any self-intersections divides the plane into exactly two regions.*

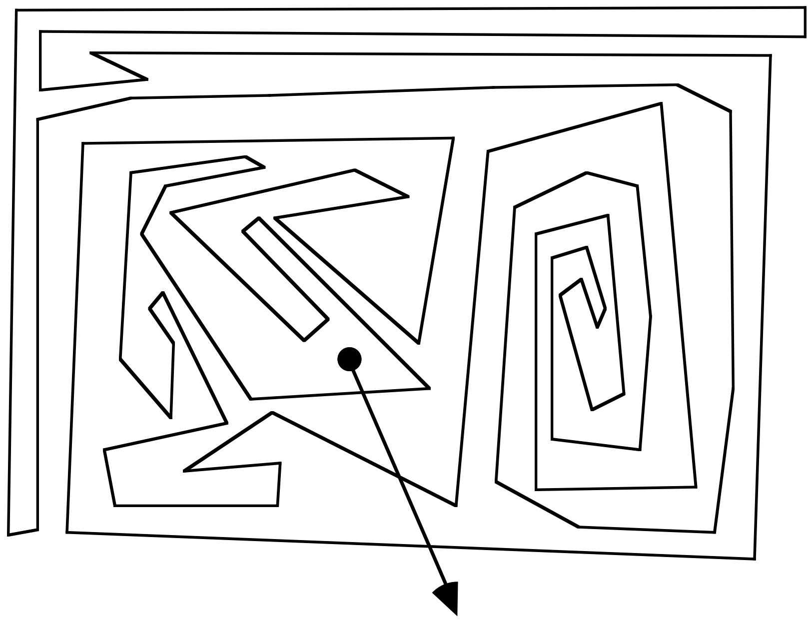
Here is an example:



You are not required to prove this result, but you can use it and it is interesting to see how complex examples can look!



Is the black dot inside or outside this curve?
Of course you can solve this like solving a maze, but look!



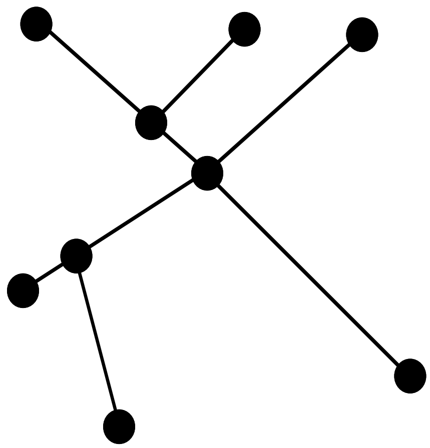
An arrow from the dot intesects the curve in an ODD number of points. I claim that this tells you that the point must be INSIDE. If the intersection number were EVEN, then the point would be outside. Can you explain why this works? (I say explain, and of course I am hoping that your explanation will turn into a mathematical proof. But lets explore.)

We will discuss in class why and how the Jordan Curve Theorem is relevant to proving Euler's Formula.

Remark. Another approach to the Euler formula uses the concept of a tree: A graph is said to be a *tree* if it does *not* contain any cycles (a cycle is a sequence of distinct edges such that the each edge shares its endpoints with the edges before and after it in the sequence. For example in the graph above, bce is a cycle and abcd is a cycle. When a plane graph has no cycles then the only region it can delineate is the rest of the plane other than itself, and so a tree has **F = 1**.

Show that for a connected tree, V - E = 1.

From this is follows that for connected plane trees **V - E + F = 2,** and so we know the Euler formula already for trees.



V = 9
E = 8
V - E = 1

The picture above illustrates this result for trees. You can prove that V - E = 1 for a connected tree by induction on the number of edges in the tree.

You can then prove the Euler formula for an arbitrary connected plane graph by just making that graph by adding edges by our type II move to a tree. Think about this and try some examples.