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On the multiplicative ergodic theorem for uniquely ergodic systems

by

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ABSTRACT. – We consider the question of uniform convergence in the multiplicative ergodic theorem

$$\lim_{n\to\infty}\frac{1}{n}\cdot\log\|A(T^{n-1}x)\cdots A(x)\| = \Lambda(A)$$

for continuous function $A : X \to GL_d(\mathbb{R})$, where (X,T) is a uniquely ergodic system. We show that the inequality $\limsup_{n\to\infty} n^{-1}\cdot\log\|A(T^{n-1}x)\cdots A(x)\| \leq \Lambda(A)$ holds *uniformly* on X, but it may happen that for some exceptional zero measure set $E \subset X$ of the second Baire category: $\liminf_{n\to\infty} n^{-1}\cdot\log\|A(T^{n-1}x)\cdots A(x)\| < \Lambda(A)$. We call such A a *non-uniform* function.

We give sufficient conditions for A to be uniform, which turn out to be necessary in the two-dimensional case. More precisely, A is uniform iff either it has *trivial* Lyapunov exponents, or A is continuously cohomologous to a diagonal function.

For equicontinuous system (X,T), such as irrational rotations, we identify the collection of non-uniform matrix functions as the set of discontinuity of the functional Λ on the space $C(X,\mathrm{GL}_2(\mathbb{R}))$, thereby proving, that the set of all uniform matrix functions forms a dense G_δ-set in $C(X,\mathrm{GL}_2(\mathbb{R}))$.

It follows, that M. Herman's construction of a non-uniform matrix function on an irrational rotation, gives an example of discontinuity of Λ on $C(X,\mathrm{GL}_2(\mathbb{R}))$.

RÉSUMÉ. – Nous considérons la question de la convergence uniforme dans le théorème ergodique multiplicatif

$$\lim_{n\to\infty}\frac{1}{n}\cdot\log\|A(T^{n-1}x)\cdots A(x)\| = \Lambda(A)$$

pour des fonctions continues $A : X \to GL_d(\mathbb{R})$, où (X, T) est un système uniquement ergodique. Nous montrons que l'inégalité $\limsup_{n\to\infty} n^{-1} \cdot \log \|A(T^{n-1}x) \cdots A(x)\| \leq \Lambda(A)$ a lieu *uniformément* sur X, mais il peut arriver que pour des ensembles exceptionnels de mesure nulle $E \subseteq X$ de la seconde catégorie de Baire, nous ayons $\liminf_{n\to\infty} n^{-1} \cdot \log \|A(T^{n-1}x) \cdots A(x)\| < \Lambda(A)$. Une telle fonction A est dite *non-uniforme*.

Nous donnons des conditions suffisantes pour que A soit uniforme; ces conditions sont aussi nécessaires dans le cas bidimensionnel. Plus précisément, A est uniforme ssi son exposant de Lyapunov est trivial, où A est continuement cohomologue à une fonction diagonale.

Pour les systèmes équicontinus (X, T), comme les rotations irrationnelles, nous identifions la collection des fonctions matricielles uniformes à l'ensemble des discontinuités de la fonctionnelle Λ sur l'espace $C(X, \mathrm{GL}_2(\mathbb{R}))$, prouvant ainsi que l'ensemble des fonctions matricielles uniformes forme un ensemble G_δ dense.

Il s'ensuit que la construction de M. Herman d'une fonction matricielle non uniforme sur les rotations non rationnelles, donne un exemple de discontinuité de Λ sur $C(X, \mathrm{GL}_2(\mathbb{R}))$.

1. INTRODUCTION

Let (X, μ, T) be an ergodic system, *i.e.* T is a measure preserving transformation of a probability space (X, μ) without nontrivial invariant measurable sets. The following theorem is a non-commutative generalization of the classical Pointwise Ergodic theorem of Birkhoff:

MULTIPLICATIVE ERGODIC THEOREM (Furstenberg-Kesten, [2]). – *Let* $A : X \to \mathrm{GL_d}(\mathbb{R})$ *be a measurable function, with both* $\log \|A(x)\|$ *and* $\log \|A^{-1}(x)\|$ *in* $L^1(\mu)$. *Then there exists a constant* $\Lambda(A)$, *s.t.*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A(T^{n-1}x) \cdots A(x)\| = \Lambda(A)$$

for μ*-a.e.* $x \in X$ *and in* $L^1(\mu)$.

This result follows from the more general

SUBADDITIVE ERGODIC THEOREM (Kingman, see [6], [5]). – *Let* $\{f_n\}$ *be a sequence in* $L^1(X, \mu)$, *forming a subadditive cocycle,* i.e. *for* μ*-a.e.* $x \in X$: $f_{n+m}(x) \leq f_n(x) + f_m(T^n x)$ *for* $n, m \in \mathbb{N}$. *Then there* *exists a constant* $\Lambda(f) \geq -\infty$, *so that for* μ*-almost all* x *and in* $L_1(\mu)$*:* $\lim_{n\to\infty} n^{-1} \cdot f_n(x) = \Lambda(f)$. *The constant* $\Lambda(f)$ *satisfies:*

$$\Lambda(f) = \lim_{n\to\infty} \frac{1}{n} \int f_n \, d\mu = \inf_n \frac{1}{n} \int f_n \, d\mu.$$

We shall consider the situation, where (X, μ, T) is a *uniquely ergodic* system, *i.e.* X is a metric compact, $T : X \to X$ is a homeomorphism with μ being the unique T-invariant probability measure on X. In this case for any *continuous* function f on X the convergence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} f(T^j x) = \int f d\mu$$

holds *everywhere* and *uniformly* on X, rather than μ-almost everywhere and in $L^p(\mu)$, as it is guaranteed by Birkhoff's ergodic theorem.

In this paper we consider the question of *everywhere* and *uniform* convergence in the Multiplicative and Subadditive ergodic theorems, under the assumption that the system (X, μ, T) is uniquely ergodic and all the functions involved are continuous.

This work was stimulated by the examples of M. Herman [4] and P. Walters [11], who have constructed continuous functions $A : X \to SL_2(\mathbb{R})$ on a uniquely ergodic system (X, μ, T), s.t. for some non-empty $E \subset X$ with $\mu(E) = 0$:

$$\liminf_{n\to\infty} \frac{1}{n} \log \|A(T^{n-1}x) \cdots A(x)\| < \Lambda(A), \quad \forall x \in E.$$

2. PRELIMINARIES

In this section we summarize the assumptions and the notations which are used in the sequel.

P^{d-1} denotes the $(d-1)$-dimensional real projective space. The projective point defined by $u \in \mathbb{R}^d \setminus \{0\}$ is denoted by $\bar{u}$. Given $A \in GL_d(\mathbb{R})$, we write $\bar{A}$ for the corresponding projective transformation. For $\bar{u} \in P^{d-1}$, $\hat{u} \in \mathbb{R}^d$ denotes either of the unit vectors in direction $\bar{u}$; although $\hat{u}$ is not unique, the norm $\|A\hat{u}\|$ is well defined for any $A \in GL_d(\mathbb{R})$. The projective space P^{d-1} is endowed with the angle metric θ, given by

$$\theta(\bar{u}, \bar{w}) = \cos^{-1} |\langle \hat{u}, \hat{w} \rangle|, \quad \bar{u}, \bar{w} \in P^{d-1}.$$

Any function $A : X \to \mathrm{GL_d}(\mathbb{R})$ uniquely defines a cocycle $A(n,x)$, which is given by:

$$A(n,x) = \begin{cases} A(T^{n-1}x)\cdots A(x) & n>0 \\ I & n=0 \\ A^{-1}(T^n x)\cdots A^{-1}(T^{-1}x) & n<0 \end{cases}$$

This formula gives a 1-1 correspondence between functions $A : X \to \mathrm{GL_d}(\mathbb{R})$ and $\mathrm{GL_d}(\mathbb{R})$-valued cocycles, *i.e.* functions $A : \mathbb{Z} \times X \to \mathrm{GL_d}(\mathbb{R})$ satisfying

$$A(n+m,x) = A(m,T^n x)\cdot A(n,x), \quad n,m \in \mathbb{Z}\ x \in X.$$

Our main tool will be Oseledec theorem [10], which describes the asymptotics of matrix products applied to vectors. The reader is referred to [10] and [1] for the complete formulation and proofs. We shall be mostly interested in the two-dimensional case:

OSELEDEC THEOREM ([10]). – *Let (X,μ,T) be an ergodic system, and $A : X \to \mathrm{GL_2}(\mathbb{R})$ be a measurable function with both $\log\|A(x)\|$ and $\log\|A^{-1}(x)\|$ in $L^1(\mu)$. Then there exists a T-invariant set $X_0 \subset X$, with $\mu(X_0)=1$, and constants $\lambda_1 \geq \lambda_2$ with the properties:*

If $\lambda_1 = \lambda_2 = \lambda$ then for any $x \in X_0$ and any $u \in \mathbb{R}^2 \setminus \{0\}$: $\lim_{n\to\pm\infty} n^{-1}\cdot\log\|A(n,x)u\| = \lambda$.

If $\lambda_1 > \lambda_2$ then there exist measurable functions $\bar{u}_1, \bar{u}_2 : X_0 \to P^1$, so that for $u \in \mathbb{R}^2 \setminus \{0\}$:

$$\begin{aligned} \bar{u} \neq \bar{u}_2(x) \quad &\Rightarrow \quad \lim_{n\to+\infty} n^{-1}\cdot\log\|A(n,x)u\| = \lambda_1, \\ \bar{u} \neq \bar{u}_1(x) \quad &\Rightarrow \quad \lim_{n\to-\infty} n^{-1}\cdot\log\|A(n,x)u\| = \lambda_2. \end{aligned}$$

The functions $\bar{u}_i(x)$ satisfy $\bar{A}(x)\bar{u}_i(x) = \bar{u}_i(Tx)$ for $x \in X_0$, and the constants λ_i satisfy:

$$\lambda_1 = \Lambda(A) \quad \text{and} \quad \lambda_1+\lambda_2 = \int_X \log|\det A(x)|\,d\mu.$$

REMARK 1. – It follows from the proof of the theorem (cf. [1]), that at each $x \in X$, for which both $n^{-1}\cdot\log\|A(n,x)\|$ and $n^{-1}\cdot\log|\det A(n,x)|$ converge, the limit $\lim_{n\to\infty} n^{-1}\cdot\log\|A(n,x)u\|$ exists for *every* $u \in \mathbb{R}^2 \setminus \{0\}$.

From this point on (X, μ, T) is assumed to be a **uniquely ergodic system**, *i.e.* T is a homeomorphism of a compact metric space X, and μ is the unique T-invariant probability measure on X. In some cases we shall assume also, that (X, T) is minimal.

The space of continuous real valued functions with the max-norm is denoted by $C(X)$. The space of all continuous functions $X \to \mathrm{GL_d}(\mathbb{R})$ is denoted by $C(X, \mathrm{GL_d}(\mathbb{R}))$. For matrices $M_1, M_2 \in \mathrm{GL_d}(\mathbb{R})$ we use the metric $\rho(M_1, M_2) = \|M_1 - M_2\| + \|M_1^{-1} - M_2^{-1}\|$. For functions $A, B \in C(X, \mathrm{GL_d}(\mathbb{R}))$ we use (with some abuse of notation) the metric $\rho(A, B) = \max_{x \in X}\{\rho(A(x), B(x))\}$, which makes it a complete metric space.

Given a function $A : X \to \mathrm{GL_d}(\mathbb{R})$ we consider an A-defined skew-product $(X \times P^{d-1}, T_A)$, given by

$$T_A(x, \bar{u}) = (Tx, \bar{A}(x)\bar{u}), \quad x \in X, \ \bar{u} \in P^{d-1}.$$

Note, that $T_A^n(x, \bar{u}) = (T^n x, \bar{A}(n, x)\bar{u})$ for $n \in \mathbb{Z}$.

Two functions $A, B \in C(X, \mathrm{GL_d}(\mathbb{R}))$ and the corresponding cocycles $A(n, x), B(n, x)$ are said to be continuously (measurably) *cohomologous*, if there exists a continuous (measurable) function $C : X \to \mathrm{GL_d}(\mathbb{R})$, so that

$$A(x) = C^{-1}(Tx) \cdot B(x) \cdot C(x)$$

and thus

$$A(n, x) = C^{-1}(T^n x) \cdot B(n, x) \cdot C(x).$$

Obviously, continuously cohomologous functions (cocycles) A, B have the same growth $\Lambda(A) = \Lambda(B)$, and the same pointwise asymptotics for every $x \in X$. Note also, that considering everywhere and/or uniform convergence in the Multiplicative Ergodic Theorem, we can always reduce the discussion to the case $|\det A(x)| \equiv 1$, replacing $A \in C(X, \mathrm{GL_d}(\mathbb{R}))$ by $A'(x) = |\det A(x)|^{-1/d} \cdot A(x)$. In this case $T_{A'} = T_A$. If A and B are continuously (measurably) cohomologous, then the systems $(X \times P^{d-1}, T_A)$ and $(X \times P^{d-1}, T_B)$ are continuously (measurably) isomorphic.

REMARK 2. – All the statements and proofs in the sequel hold when the space $C(X, \mathrm{GL_d}(\mathbb{R}))$ of continuous functions $A : X \to \mathrm{GL_d}(\mathbb{R})$ is replaced by continuous $\mathrm{SL_d}(\mathbb{R})$-valued functions, or continuous functions satisfying $|\det A(x)| \equiv 1$.

3. ON THE SUBADDITIVE ERGODIC THEOREM

THEOREM 1. – *Let* $\{f_n\}$ *be a* continuous *subadditive cocycle on a uniquely ergodic system* (X, μ, T), i.e. $f_n \in C(X)$ *and* $f_{n+m}(x) \leq f_n(x) + f_m(T^n x)$ *for all* $x \in X$. *Then for* every $x \in X$ *and* uniformly *on* X*:*

$$\limsup_{n\to\infty} \frac{1}{n} f_n(x) \leq \Lambda(f). \tag{1}$$

However, for any F_σ *set* E *with* $\mu(E) = 0$, *there exists a continuous subadditive cocycle* $\{f_n\}$, *such that*

$$\limsup_{n\to\infty} \frac{1}{n} f_n(x) < \Lambda(f), \quad x \in E. \tag{2}$$

Proof. – We follow the elegant proof of Kingman's theorem, given by Katznelson and Weiss [5]. Let us fix some $\epsilon > 0$. For $x \in X$, define $n(x) = \inf\{\ n \in \mathbb{N} \mid f_n(x) < n \cdot (\Lambda(f) + \epsilon)\ \}$. For a fixed N, consider the open set

$$A_N = \{x \in X |\ n(x) \leq N\} = \bigcup_{n=1}^{N} \{\ x \in X \mid f_n(x) < n \cdot (\Lambda(f) + \epsilon)\ \}.$$

By Kingman's theorem $\lim_{n\to\infty}\mu(A_n) = 1$. Choose N so that $\mu(A_N) > 1 - \epsilon$.

Now let us fix some $x \in X$, and define a sequence of indexes $\{n_j\}$ and points $\{x_j\}$ by the following rule. Let $x_1 = x$, $n_1 = n(x)$, and for $j > 1$ let $n_j = n(x_j)$ if $x \in A_N$, and set $n_j = 1$ otherwise. Always set $x_{j+1} = T^{n_j} x_j$. Note that $1 \leq n_j \leq N$.

Consider index $M > N \cdot \|f_1\|_\infty / \epsilon$, and let $p \geq 1$ satisfy $n_1 + \ldots + n_{p-1} \leq M < n_1 + \ldots n_p$. Denote $K = M - (n_1 + \ldots + n_{p-1}) \leq N$. Now, using subadditivity, we have

$$f_M(x) \leq \sum_{j=1}^{p} f_{n_j(x)}(x_j) + f_K(x_p) \leq \sum_{j=1}^{p} f_{n_j(x)}(x_j) + N \cdot \|f_1\|_\infty.$$

By the definition of n_j:

$$f_{n_j}(x_j) \leq n_j \cdot (\Lambda(f) + \epsilon) \cdot 1_{A_N}(x_j) + \|f_1\|_\infty \cdot 1_{X \setminus A_N}(x_j).$$

Thus, estimating from above, we obtain

$$\frac{1}{M} f_M(x) \leq (\Lambda(f) + \epsilon) + \|f_1\|_\infty \cdot \frac{1}{M} \sum_{1}^{M} 1_{X \setminus A_N}(T^i x). + \|f_1\|_\infty \cdot \frac{N}{M}.$$

We claim that for M large, the second summand is *uniformly* bounded by $O(\epsilon)$. Indeed, the set $X \setminus A_N$ is closed and has small measure: $\mu(X \setminus A_N) < \epsilon$. By Urison's Lemma, there exists a *continuous* function $g : X \to [0,1]$ with $g|_{X\setminus A_N} \equiv 1$ and

$$\mu(g) \leq \mu(X \setminus A_N) + \epsilon \leq 2\epsilon.$$

Therefor for M sufficiently large, *uniformly* on X:

$$\frac{1}{M}\sum_{1}^{M} 1_{X\setminus A_N}(T^i x) \leq \frac{1}{M}\sum_{1}^{M} g(T^i x) \leq \int g\, d\mu + \epsilon \leq 3\epsilon.$$

Thus for sufficiently large M, for all $x \in X$: $1/n \cdot f_n(x) \leq \Lambda(f) + O(\epsilon)$ for all $n > M$. This proves (1).

For (2), let $E = \bigcup E_k \subseteq X$, where each E_k is closed and $\mu(E_k) = 0$. There exist continuous functions $g_k : X \to [0,1]$ with $g_k|_{E_k} \equiv 1$ and $\mu(g_k) \leq 2^{-k-2}$. Define continuous functions $\{f_n\}$, by $f_n(x) = -\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} g_k(T^j x)$. One can check that $\{f_n\}$ is a subadditive cocycle, with

$$\Lambda(f) = \lim_{n\to\infty}\frac{1}{n}\int f_n\, d\mu \geq -\sum_{0}^{\infty}\frac{1}{2^{n+2}} = -\frac{1}{2}.$$

But for any $x \in E$, $\limsup_{n\to\infty} n^{-1} \cdot f_n(x) < -1$. This completes the proof of the Theorem. □

COROLLARY 2. – *Let (X, μ, T) be a uniquely ergodic system, and let $A : X \to \mathrm{GL_d}(\mathbb{R})$ be a continuous function, then for* every $x \in X$ *and* uniformly *on X:*

$$\limsup_{n\to\infty}\frac{1}{n}\log\|A(n,x)\| \leq \Lambda(A).$$

Proof. – Take $f_n(x) = \log\|A(n,x)\|$, and apply Theorem 1. □

4. ON THE MULTIPLICATIVE ERGODIC THEOREM

DEFINITION. – A function $A \in C(X, \mathrm{GL_d}(\mathbb{R}))$ (and the corresponding cocycle $A(n,x)$) is said to be:

- **uniform** if $\lim_{n\to\infty} n^{-1} \cdot \log\|A(n,x)\| = \Lambda(A)$ holds for *every* $x \in X$ and *uniformly* on X.

- **positive** if for all all the entries of $A(x)$ are positive: $A_{i,j}(x) > 0$ for all $x \in X$.
- **eventually positive** if for some $p \in \mathbb{N}$ the function $A(p, x)$ is positive.
- **continuously diagonalizible** if it is continuously cohomologous to a diagonal function: $A(x) = C^{-1}(Tx) \cdot \mathrm{diag}(e^{b_1(x)}, \ldots, e^{b_d(x)}) \cdot C(x)$ for some $C \in C(X, \mathrm{GL_d}(\mathbb{R}))$ and $b_1, \ldots, b_d \in C(X)$.

Continuously diagonalizable cocycles with $\lambda_1 > \lambda_d$ are usually referred to as *uniformly hyperbolic*. We do not use this term.

THEOREM 3. – *Let (X, μ, T) be a uniquely ergodic system, then each one of the following conditions implies that $A \in C(X, \mathrm{GL_d}(\mathbb{R}))$ is uniform:*

1. *A is continuously diagonalizable.*
2. *A has trivial Lyapunov filtration,* i.e. $\lambda_1 = \ldots = \lambda_d$.
3. *A is continuously cohomologous to an eventually positive function.*

In dimension $d = 2$ these conditions are necessary, as the following Theorem shows:

THEOREM 4. – *Let (X, μ, T) be a uniquely ergodic and minimal system. If $A \in C(X, \mathrm{GL_2}(\mathbb{R}))$ does not satisfy* 1-3 *of Theorem* 3, *then there exists a dense set $E \subset X$ of second Baire category, s.t. for all $x \in E$:*

$$\liminf_{n \to \infty} \frac{1}{n} \log \|A(n, x)\| < \limsup_{n \to \infty} \frac{1}{n} \log \|A(n, x)\| \leq \Lambda(A). \tag{3}$$

*Moreover, if A has a non-trivial Lyapunov filtration (*i.e. $\lambda_1 > \lambda_2$*), then A is continuously diagonalizable iff it is continuously cohomologous to an eventually positive function.*

In the proof of Theorem 4 we shall need the following Lemma, which is essentially due to M. R. Herman (see [4]):

LEMMA 3. – *Let (Y, S) be a minimal system and let $\phi \in C(Y)$ be a continuous function. Then the ergodic averages $n^{-1} \cdot \sum_0^{n-1} \phi(S^i y)$ converge for* every $y \in Y$ *iff all S-invariant probability measures ν on Y assign the same value to ϕ. More precisely:*

1. *If $\nu(\phi) = c$ for all S-invariant probability measures ν, then*

$$\lim_{n \to \infty} \|\frac{1}{n} \cdot \sum_0^{n-1} \phi(T^i y) - c\|_\infty = 0. \tag{4}$$

2. *If there exist S-invariant probability measures ν_1, ν_2 with $\nu_1(\phi) = c_1 < c_2 = \nu_2(\phi)$, then there exists a dense G_δ-set $E \subset Y$, s.t. for any $y \in E$:*

$$\liminf_{n \to \infty} \frac{1}{n} \sum_0^{n-1} \phi(S^i y) \ \leq c_1 < c_2 \ \leq \ \limsup_{n \to \infty} \frac{1}{n} \sum_0^{n-1} \phi(S^i y).$$

Proof. – Case 1. We claim that $(\phi - c)$ belongs to the $\|\cdot\|_\infty$-closure of the space $V = \{\psi - \psi \circ S \mid \psi \in C(Y)\}$. Indeed, otherwise by Hahn-Banach theorem there would exist a functional $\nu \in C(Y)^*$, with $V \subseteq Ker(\nu)$ and $\nu(\phi - c) \neq 0$. But S-invariant probability measures span all ν, annihilating V, and we get a contradiction to the assumption in 1. Therefore, given $\epsilon > 0$, there exists $\psi \in C(Y)$ with $\|(\phi - c) - (\psi - \psi \circ S)\|_\infty < \epsilon$, and for large n:

$$\left\| \frac{1}{n} \sum_0^{n-1} \phi \circ S^i - c \right\|_\infty \leq \frac{1}{n} \|\psi - \psi \circ S^n\| + \epsilon < 2\epsilon.$$

This proves (4).

Case 2. Replacing, if necessary, ν_i by extremal S-invariant points μ_i, we obtain S-*ergodic* measures μ_1, μ_2 with $\mu_1(\phi) \leq c_1 < c_2 \leq \mu_2(\phi)$. Given $\epsilon > 0$ and $N \geq 1$, let:

$$W_1(N, \epsilon) = \left\{ y \in Y \,\middle|\, \frac{1}{n} \sum_0^{n-1} \phi(S^i y) \geq c_1 + \epsilon, \quad \forall n \geq N \right\}$$

$$W_2(N, \epsilon) = \left\{ y \in Y \,\middle|\, \frac{1}{n} \sum_0^{n-1} \phi(S^i y) \leq c_2 - \epsilon, \quad \forall n \geq N \right\}$$

These sets are closed, and we claim that $W_i(N, \epsilon)$ have empty interior. Indeed, assume $W_1(N, \epsilon)$ contains an open non-empty set U, and take a μ_1-generic point y_1. Then for sufficiently large M the ergodic averages satisfy: $M^{-1} \cdot \sum_0^{M-1} \phi(S^i y_1) < c_1 + \epsilon$. By minimality, some iterate $S^m y_1 \in U$ and, for sufficiently large M:

$$(M - m)^{-1} \cdot \sum_m^{M-1} \phi(S^i y_1) = (M - m)^{-1} \cdot \sum_0^{M-m-1} \phi(S^i S^m y_1)$$

is less than $c_1 + \epsilon$, contradicting the assumption $U \subset W_1(N, \epsilon)$. The same argument applies to $W_2(N, \epsilon)$. We conclude that $E = Y \setminus \bigcup_n W_1(n, n^{-1}) \cup W_2(n, n^{-1})$ is a dense G_δ-set in Y, as required.

LEMMA 4. – *Let (X, μ, T) be a uniquely ergodic invertible system, and suppose $A : X \to \mathrm{GL}_2(\mathbb{R})$ satisfies $\lambda_1(A) > \lambda_2(A)$. Then the system $(Z, S) = (X \times P^1, T_A)$ has exactly two ergodic probability measures μ_1, μ_2 of the form:*

$$\int_{X \times P^1} F(x, \bar{u})\, d\mu_i(x, \bar{u}) = \int_X F(x, \bar{u}_i(x))\, d\mu(x), \qquad F \in C(X \times P^1),$$

where $\bar{u}_i : X \to P^1$, $i = 1, 2$ is the Oseledec filtration (1).

Proof. – Since μ-a.e. $\bar{u}_i(Tx) = \bar{A}(x)\bar{u}_i(x)$, the measures μ_i are S-invariant, and thus are S-ergodic. Suppose now that $\nu \neq \mu_2$ is an S-ergodic measure. We claim that $\nu = \mu_1$. The projection of ν on X is T-invariant, and hence coincides with μ. Let $\{\nu_x\}$, $x \in X$ be the disintegration of ν with respect to μ, *i.e.* $\nu = \int \nu_x \, d\mu(x)$. Then $\nu_{Tx} = \bar{A}(x)\nu_x$ and, since $\nu \perp \nu_2$, $\nu_x(\bar{u}_2(x)) = 0$ for μ-a.e. $x \in X$. We claim that for any $x \in X_0$, the graph of any function $\bar{u} \neq \bar{u}_2(x)$ "converges to" the graph of $\bar{u}_1(x)$ under T_A^n, namely:

$$\begin{aligned}
&\lim_{n\to\infty} \sin\, \theta(\bar{A}(n,x)\,\bar{u},\ \bar{A}(n,x)\,\bar{u}_1(x)) \\
&\quad = \lim_{n\to\infty} \frac{|\det A(x)| \cdot \sin\ \theta(\bar{u},\ \bar{u}_1(x))}{\|A(n,x)\hat{u}\| \cdot \|A(n,x)\hat{u}_2(x)\|} = 0. \qquad (5)
\end{aligned}$$

Define the sets $V_{i,\delta} = \{(x,\bar{u}) \mid \theta(\bar{u},\bar{u}_i(x)) < \delta\}$ and $V_{i,\delta}^c = Z \setminus V_{i,\delta}$ for $i = 1,2$. Then, using (5) and the fact that $\lim_{\delta\to 0}\nu(V_{2,\delta}^c) = 1$, we get $\lim_{\delta\to 0}\lim_{n\to\infty}\nu\big(T_A^n(V_{2,\delta}^c)\cap V_{1,\delta}\big) = 1$ and, therefore, $\nu\{(x,\bar{u}_1(x)) \mid x \in X\} = 1$, so that $\nu = \nu_1$ $\square$.

LEMMA 5. – *Let $A_n \in \mathrm{GL_d}(\mathbb{R})$ be a sequence of positive matrices, bounded in the sense that there exists some $\delta > 0$, s.t. $\delta < (A_n)_{i,j} < \delta^{-1}$ for all $n \geq 1$ and $1 \leq i,j \leq d$. Let $\Delta \subset \mathbb{R}^d$ be the simplex*

$$\Delta = \left\{ u \in \mathbb{R}^d \ \mid\ \sum_1^d u_i = 1,\ u_i \geq 0 \right\},$$

and $\bar{\Delta}$ the corresponding set in P^{d-1}. Then there exists a unique point $\bar{u} \in P^{d-1}$:

$$\{\bar{u}\} = \bigcap_{n=1}^{\infty} \bar{A}_1 \cdots \bar{A}_n \bar{\Delta}.$$

Proof. – The sets Δ and $\bar{\Delta}$ are naturally identified. With this identification $\bar{A}_n$ are projective transformations of the affine space $\{u \in \mathbb{R}^d \mid \sum_1^d u_i = 1\}$, which preserve the four points cross ratios

$$[u;v;w;z] = \frac{\|u-w\| \cdot \|v-z\|}{\|u-z\| \cdot \|v-w\|}$$

provided that u,v,w,z lie on the same line. Now let $K = \bigcap K_n$, where $K_n = \bar{A}_1 \cdots \bar{A}_n \bar{\Delta}$ form a descending sequence of convex compacts. Assume that K is not a single point, and let $u \neq v$ be two extremal points of

K. Let w_n, z_n be the intersection of the line (u, v) with the boundary ∂K_n. Let $w'_n, z'_n, u'_n, v'_n \in \bar{\Delta}$ be the preimages of w_n, z_n, u, v under $\bar{A}_1 \cdots \bar{A}_n$. Then $w'_n, z'_n \in \partial\bar{\Delta}$, but $u'_n, v'_n \in \bar{A}_{n+1}\bar{\Delta}$. The δ-boundness of A_{n+1} implies that $\bar{A}_{n+1}\bar{\Delta}$ are uniformly separated from $\partial\bar{\Delta}$. Thus the cross ratio $[u'_n; v'_n; w'_n; z'_n]$ is bounded from 0 (and ∞). On the other hand $w_n \to u$ and $z_n \to v$ implies $[u; v; w_n; z_n] \to 0$, causing the contradiction. □

Proof of Theorem 3. – *Case 1* follows from the classical one-dimensional (commutative) case.

Case 2. Reducing to the case $|\det A(x)| \equiv 1$, we observe that the assumption is $\Lambda(A) = \lambda_1 = \ldots = \lambda_d = 0$. Obviously for every $x \in X$ and $n \in \mathbb{Z}$: $\log \|A(n, x)\| \geq 0$, so Corollary 2 implies that A is uniform. In this case the result can also be deduced from Case 1 of Lemma 3. Indeed, considering the function $\phi \in C(X \times P^1)$ defined by

$$\phi(x, \bar{u}) = \log \|A(x)\hat{u}\| \tag{6}$$

we observe that for any T_A-invariant probability measure ν: $\nu(\phi) = 0$ (indeed, such ν projects onto μ, hence the projection of the set of ν-generic points intersects the set X_0 of regular points in Oseledec theorem). Therefore we deduce that $n^{-1} \cdot \log \|A(n, x)\hat{u}\| \to 0$ uniformly on $X \times P^1$.

Case 3. Obviously, it is enough to consider the case, that A is actually positive, *i.e.* $A(x)_{i,j} > 0$ for all $x \in X$. Let $\Delta \subset \mathbb{R}^d$ be as in Lemma 5. Then $T_A(X \times \Delta) \subset (X \times \Delta)$, and we claim that the compact set $Q = \bigcap_{n=1}^{\infty} T_A^n(X \times \Delta)$ is a graph of some continuous function $\bar{u} : X \to \Delta \subset P^{d-1}$, which is called in the sequel the *positive core* of $A(x)$. Indeed, for any fixed $x \in X$ the fiber Q_x of Q above x is given by

$$Q_x = \bigcap_{n=1}^{\infty} \bar{A}(n, T^{-n}x)\,\bar{\Delta} = \bigcap_{n=1}^{\infty} \bar{A}(T^{-1}x) \cdots \bar{A}(T^{-n}x)\,\bar{\Delta}$$

and, by Lemma 5, Q_x consists of a single point $\bar{u}(x)$. Since $Q = \{(x, \bar{u}(x)) \mid x \in X\}$ is closed, the function $\bar{u}(x)$ is continuous, and T_A-invariance of Q implies $\bar{u}(Tx) = \bar{A}(x)\bar{u}(x)$. The measure $\tilde{\mu} = \int \delta_{\bar{u}(x)}\, d\mu(x)$ is the unique T_A-invariant on Q, hence the sequence

$$\frac{1}{n} \cdot \log \|A(n, x)\hat{u}(x)\| = \frac{1}{n} \cdot \sum_{k=0}^{n-1} \phi(T_A^k(x, \bar{u}(x)))$$

converges uniformly on X. But the uniform positivity of $A(x)$ and $u(x)$ implies that for some $c > 0$: $\|A(n, x)\| \leq c \cdot \|A(n, x)\hat{u}(x)\| \leq c \cdot \|A(n, x)\|$, and therefore A is uniform. □

Proof of Theorem 4. – Assume $\lambda_1(A) > \lambda_2(A)$. By Lemma 4, there exist two T_A-invariant measures μ_1, μ_2 on $Z = X \times P^1$. Consider the topological structure of (Z, T_A). We claim that there are two alternatives:

(i) There is a unique T_A-minimal set $Y \subset Z$, supporting both μ_1 and μ_2, or

(ii) There exist two T_A-minimal sets $Y_1, Y_2 \subset Z$, supporting μ_1, μ_2 respectively. Moreover, the measurable functions $\{\bar{u}_i(x)\}$, defining the Oseledec filtration, are continuous in this case, and Y_i have the form $Y_i = \{ (x, \bar{u}_i(x)) \mid x \in X \}$, $i = 1, 2$.

Let Y be a T_A-minimal subset in Z. If (Y, T_A) is not uniquely ergodic, then by Lemma 4, (Y, T_A) supports both μ_1 and μ_2. The function $\phi \in C(Y)$, defined by (6), satisfies $\mu_1(\phi) = \lambda_1 > \lambda_2 = \mu_2(\phi)$. Thus, by Lemma 3, there exists a dense G_δ-set $E \subset Y$ of points, where $1/n \cdot \sum \phi(T_A^k(x, \bar{u})) = 1/n \cdot \log \|A(n, x)\hat{u}\|$ diverges. By Remark 1, for any $x \in X$ in the projection of E to X, the sequence $1/n \cdot \log \|A(n, x)\|$ diverges and, using Corollary 2 we deduce (3).

Now assume, that (Y, T_A) is uniquely ergodic, and therefore supports either μ_1 or μ_2. We can assume $|\det A| \equiv 1$, and thus $\lambda = \lambda_1 > \lambda_2 = -\lambda$. Considering, if necessary T^{-1} instead of T, we can assume that (Y, T_A) supports μ_1.

We claim that Y is a graph of a continuous function $X \to P^1$. Suppose $y_1, y_2 \in Y$ have the same X-coordinate x_0, *i.e.* $y_i = (x_0, \bar{v}_i)$. Then for any $n > 0$:

$$\begin{aligned} |\sin \theta(\bar{v}_1, \bar{v}_2)| &\leq \frac{|\det A(n, x_0)|}{\|A(n, x_0)\hat{v}_1\| \cdot \|A(n, x_0)\hat{v}_2\|} \\ &= \exp\left(- \sum_{k=0}^{n-1} \phi(T_A^k y_1) - \sum_{k=0}^{n-1} \phi(T_A^k y_2) \right) \end{aligned}$$

Since $\mu_1(\phi) = \lambda_1 > 0$ and (Y, T_A) is uniquely ergodic, we deduce that the right hand side converges *uniformly* to 0, and therefore $\bar{v}_1 = \bar{v}_2$. This shows that Y is a graph of a function $X \to P^1$ (note that by minimality of (X, T), Y projects *onto* X). This function has to be continuous for its graph - Y - is a closed set. Since the graph of the function $\bar{u}_1(x)$ (defined by (1)) is contained in Y, we conclude that $\bar{u}_1 : X \to P^1$ is continuous, and thus $Y = \{ (x, \bar{u}_1(x)) \mid x \in X \}$.

We claim now that μ_2 is also supported on a graph of a continuous function. Let $\bar{v} : X \to P^1$ be any continuous function with $\bar{v}(x) \neq \bar{u}_1(x)$ for all $x \in X$. Then there exists continuous $C : X \to \mathrm{GL}_2(\mathbb{R})$ with

$|\det C(x)| \equiv 1$, s.t. $\bar{C}(x)$ takes $\{\bar{u}_1(x), \bar{v}(x)\}$ to the directions of the standard basis $\{\bar{e}_1, \bar{e}_2\}$, so that

$$B(x) = C(Tx) \cdot A(x) \cdot C^{-1}(x) = e^{-a(x)/2} \cdot \begin{pmatrix} \pm e^{a(x)} & b(x) \\ 0 & 1 \end{pmatrix}$$

where $a, b \in C(X)$ with $\mu(a) = \lambda_1 > 0$. We claim that the T_A^{-n}-image of the graph of $\bar{v}(x)$, namely the set

$$T_A^{-n} \{(x, \bar{v}(x)) \mid x \in X\} = \{(x, \bar{v}_n(x)) \mid x \in X\},$$

converges uniformly to the graph of $\bar{u}_2(x)$, which is thereby continuous. Indeed

$$\bar{v}_n(x) = \bar{A}(-n, T^n x)\, \bar{v}(T^n x) = \bar{C}^{-1}(x) \bar{B}(-n, T^n x)\, \bar{e}_2.$$

and, more precisely, $\bar{v}_n(x)$ is spanned by the vector $C^{-1}(x) \begin{pmatrix} w_n(x) \\ 1 \end{pmatrix}$, where

$$w_n(x) = \pm b(x) \pm b(Tx) \cdot e^{-a(x)} \pm \ldots \pm b(T^{n-1}x) \cdot e^{-a(x) - \ldots - a(T^{n-2}x)}$$

Since (X, μ, T) is uniquely ergodic, and $a(x)$ is continuous with $\mu(a) > 0$, we deduce that $w_n(x)$ converges uniformly to a continuous function $w : X \to \mathbb{R}$, and the continuous function $u(x) = C^{-1}(x) \begin{pmatrix} w(x) \\ 1 \end{pmatrix}$ satisfies $\bar{A}(x)\bar{u}(x) = \bar{u}(Tx)$ and $\bar{u}(x) \neq \bar{u}_1(x)$. The graph of $\bar{u}(x)$ is T_A-invariant, and has to support $\bar{u}_2$, so $\bar{u}_2(x) = \bar{u}(x)$ is continuous. Therefore alternative (ii) holds, in which case A is continuously diagonalizable and, hence, is uniform.

We are left with the last assertion. If A is eventually positive and has positive growth (*i.e.* $\lambda_1 > \lambda_2$), then by Theorem 3, A is uniform, and thereby continuously diagonalizable. We shall prove now the other implication. We can assume $|\det A(x)| \equiv 1$, and $A(x) = \mathrm{diag}(e^{a(x)}, e^{-a(x)})$ with $a \in C(X)$ and $\mu(a) = \Lambda(A) > 0$. Let

$$\begin{aligned} v_0(x) &\equiv e_1 + e_2 \quad \text{and} \quad v_n(x) = A(n, T^{-n}x) u_0(x) \\ w_0(x) &\equiv e_1 - e_2 \quad \text{and} \quad w_n(x) = A(n, T^{-n}x) w_0(x) \end{aligned}$$

then $\bar{v}_n(x) \to \bar{e}_1$ and $\bar{w}_n(x) \to \bar{e}_1$ uniformly on X, and therefore, for sufficiently large p and for all $x \in X$: $\theta(\bar{v}_p(x), \bar{e}_1) < \theta(\bar{v}_0(x), \bar{e}_1)$ and $\theta(\bar{w}_p(x), \bar{e}_1) < \theta(\bar{w}_0(x), \bar{e}_1)$. Changing the coordinates $C : (e_1 + e_2) \mapsto e_1$ and $C : (e_1 - e_2) \mapsto e_2$, one easily checks that $C \cdot A(p, x) \cdot C^{-1}$ becomes positive. □

5. CONTINUITY OF THE UPPER LYAPUNOV EXPONENT

In this section we consider the question of continuity of the functional $\Lambda : C(X, \mathrm{GL_d}(\mathbb{R})) \to \mathbb{R}$ and connect it with uniform functions in $C(X, \mathrm{GL_d}(\mathbb{R}))$. More precisely:

THEOREM 5. – *Let (X, μ, T) be a uniquely ergodic system. The functional Λ is continuous at each uniform $A \in C(X, \mathrm{GL_2}(\mathbb{R}))$.*

If $\{T^n\}$ are equicontinuous on X, then the functional Λ is discontinuous at each non-uniform $A \in C(X, \mathrm{GL_d}(\mathbb{R}))$, $d \geq 2$. Moreover, if such non-uniform A takes values in a locally closed submanifold $L \subseteq \mathrm{GL_d}(\mathbb{R})$ then the restriction of Λ to $C(X, L)$ is discontinuous at A.

Therefore, the example of non-uniform function $A \in C(X, \mathrm{GL_2}(\mathbb{R}))$ on an irrational rotation, constructed by M. Herman [4], gives the following negative answer to the question on continuity of Λ on $C(X, \mathrm{GL_d}(\mathbb{R}))$ arised in [7]:

COROLLARY 6. – *There exists an irrational rotation (X, T), s.t. the functional Λ is discontinuous on $C(X, \mathrm{GL_2}(\mathbb{R}))$.*

COROLLARY 7. – *For equicontinuous uniquely ergodic system (X, T), the set of all uniform functions in $C(X, \mathrm{GL_d}(\mathbb{R}))$, $d \geq 2$ is a dense G_δ-set in $C(X, \mathrm{GL_d}(\mathbb{R}))$ and in $C(X, L)$ for any locally closed submanifold $L \subseteq \mathrm{GL_d}(\mathbb{R})$.*

Proof. – The functional Λ is a pointwise limit of continuous functionals Λ_n on $C(X, L)$, defined by

$$\Lambda_n(A) = \frac{1}{n} \int \log \|A(n,x)\| \, d\mu(x), \quad A \in C(X, L).$$

Since Λ_n are continuous on $C(X, L)$ with respect to the metric ρ, the non-uniform functions, which are points of discontinuity for Λ, form a set of the first Baire category. □

Proof of Theorem 5. – Let A be a uniform function in $C(X, \mathrm{GL_2}(\mathbb{R}))$, and take $A_k \to A$. By Theorem 4, either A has trivial Lyapunov filtration ($\lambda_1 = \lambda_2$), or A is continuously cohomologous to an eventually positive function.

Suppose A satisfies $\lambda_1 = \lambda_2$, and assume that $|\det A| \equiv |\det A_k| \equiv 1$. Then $\Lambda(A) = 0$, and $|\det A_k| \equiv 1$ gives $\Lambda_n(A_k) \geq 0$. On the other hand, since $\Lambda(A_k) = \inf_n \Lambda_n(A_k)$ and Λ_n are continuous, Λ is always lower semi-continuous, *i.e.*

$$\lim_{k \to \infty} \rho(A_k, A) = 0 \quad \Rightarrow \quad \limsup_{k \to \infty} \Lambda(A_k) \leq \Lambda(A).$$

Therefore Λ is continuous at A.

Now assume that $B(p,x) = C(T^p x) \cdot A(p,x) \cdot C^{-1}(x)$ is positive, for some continuous $C : X \to \mathrm{GL_d}(\mathbb{R})$ and $p \geq 1$. Then for large k, the functions $B_k(x) = C(Tx) \cdot A_k(x) \cdot C^{-1}(x)$ are close to $B(x)$, and thus $B_k(p,x)$ are positive. Moreover, the positive core $\bar{u}_1^{(k)}(x)$, corresponding to B_k, become arbitrarily close to the positive core of B, which is $\bar{u}_1(x)$. Therefore, considering the functions $\phi_k(x,\bar{u}) = \log\|B_k(x)\hat{u}\|$ and $\phi(x,\bar{u}) = \log\|B(x)\hat{u}\|$, we have $\phi_k \to \phi$ uniformly as $k \to \infty$, and therefore

$$\Lambda(B_k) = \int \phi_k(x,\, \bar{u}_1^{(k)}(x))\, d\mu(x) \;\to\; \int \phi(x,\, \bar{u}_1(x))\, d\mu(x) = \Lambda(B).$$

This proves the first assertion.

Now assume that $\{T^n\}$ are equicontinuous on X. Let $A \in C(X,L)$, $L \subseteq \mathrm{GL_d}(\mathbb{R})$, $d \geq 2$ be a non-uniform function. Corollary 2 implies that there exists a point x_0, and a constant $\lambda' < \Lambda(A)$, so that

$$\liminf_{n\to\infty} \frac{1}{n} \log\|A(n,x_0)\| < \lambda' < \Lambda(A) \tag{7}$$

Given any $\epsilon > 0$, we shall construct a continuous $B : X \to L$ with $\rho(A,B) < \epsilon$ and $\Lambda(B) < \lambda' < \Lambda(A)$.

The idea of the proof is to construct such B on a large Rohlin-Kakutani tower, using values of A at segments of the x_0 trajectory. This ensures that B is close to A, and at the same time has smaller growth.

A is continuous on X, so there exists $\delta_1 = \delta_1(\epsilon) > 0$ s.t. $\rho(A(x_1),A(x_2)) < \epsilon$ provided $d(x_1,x_2) < \delta_1$. The assumption that $\{T^n\}$ are equicontinuous, implies that there exists $\delta_2 = \delta_2(\delta_1)$, so that

$$d(x_1,x_2) < \delta_2 \quad\Rightarrow\quad \rho(T^n x_1, T^n x_2) < \delta_1/2, \quad n \in \mathbb{Z}. \tag{8}$$

Observe that if x_0 satisfies (7), then so does any point $T^n x_0$ on its orbit, and the minimality of (X,T) implies, that there exists some (finite) set $Q \subset \{T^n x_0\} \subset X$ which is δ_2-dense in X, and such that for each $q \in Q$ there exists an integer $n(q) \geq 1$, satisfying:

$$\frac{1}{n(q)} \log\|A(n(q),\, q)\| < \lambda'.$$

Let $N_0 = \max_{q\in Q}\{n(q)\}$, and $\lambda'' = \max_{q\in Q}\{1/n(q)\cdot\log\|A(n(q),\, q)\|\} < \lambda'$. Denote

$$M = \max_{x\in X} \log(\|A(x)\| + \epsilon).$$

Choose very small $\eta > 0$, and very large integers $N_2 \gg N_1 \gg N_0$, so that

$$\frac{2 \cdot N_0}{N_1} \cdot M + \frac{N_1}{N_2} \cdot M + \eta \cdot M < \lambda' - \lambda''. \tag{9}$$

Finally, construct an (N_2, η)-Rohlin-Kakutani tower in (X, μ, T): $\tilde{K} = \bigcup_0^{N_2-1} T^n K \subset X$, where $\{T^n K\}_{n=0}^{N_2-1}$ are disjoint sets and $\mu(\tilde{K}) > 1 - \eta$. Let us consider a partition of the base K into elements K_i of sufficiently small size, so that $K = \bigcup_1^k K_i$ and for each $0 \leq n < N_2$ and $1 \leq i \leq k$:

$$\operatorname{diam}(T^n K_i) < \delta_1/2. \tag{10}$$

Without loss of generality, we can assume that K and all K_i are closed sets. Let us choose a point p_i in each of K_i.

We shall start by defining the values of B at the points $\{T^n p_i \mid 0 \leq n < N_2,\ 1 \leq i \leq k\}$. We shall choose points $q_{n,i}$ which are $\delta_1/2$-close to $T^n p_i$, and will define $B(T^n p_i) = A(q_{n,i})$. Fix some $1 \leq i \leq k$, choose a point $q_{0,i} \in Q$ which is δ_2-close to p_i, denote $n_1 = n(q_{0,i})$, and set $q_{n,i} = T^n q_{0,i}$ for all $0 \leq n < n_1$. Now choose $q_{n_1,i} \in Q$ to be δ_2-close to $T^{n_1} p_i$, denote $n_2 = n(q_{n_1,i})$ and define $q_{n,i} = T^{n-n_1} q_{n_1,i}$ for all $n_1 \leq n < n_2$. Continue this procedure till $n = N_2 - 1$, and do the same for each of $1 \leq i \leq k$.

We observe, that by (8) and the choice of $q_{n_j,i}$, we have $d(T^n p_i, q_{n,i}) < \delta_1/2$ for $0 \leq n < N_2 - 1$. Moreover with this definition of B, the products of B along each of the segments of length N_1 has sufficiently small norm. More precisely, for each $1 \leq i \leq k$ and $0 \leq n < N_2 - N_1$:

$$\begin{aligned} &\frac{1}{N_1} \log \|B(N_1, T^n p_i)\| \\ &\quad = \frac{1}{N_1} \log \|B(T^{N_1-1} T^n p_i) \cdots B(T^n p_i)\| \leq \lambda'' + \frac{2 \cdot N_0}{N_1} \cdot M. \end{aligned} \tag{11}$$

Indeed, fix i and n, let j and l be s.t. $n_{j-1} < n \leq n_j$ and $n_l \leq n + N_1 < n_l$. Denote $n' = n_j - n < N_0$, $n'' = n + N_1 - n_l < N_0$, then:

$$\begin{aligned} &\log \|B(N_1, T^n p_i)\| \\ &\quad \leq \log \|B(n', T^n p_i)\| + \sum_{m=j}^{l-1} \log \|A(n(q_{n_m,i}), q_{n_m,i})\| + \log \|B(n'', T^{n_l} p_i)\| \\ &\quad < N_0 \cdot M + N_1 \cdot \lambda'' + N_0 \cdot M. \end{aligned}$$

Now let us extend the definition of B from $\{T^n p_i\}$ to $\tilde{K}$, letting $B(x)$ to be equal to $B(T^n p_i)$ for all $x \in T^n K_i$. Using (10), and the way $q_{n,i}$

were chosen, we observe, that for any $x \in K$ there exists $i = i(x)$, so that $T^n x$ and $q_{n,i}$ are δ_1-close, so that our definition $B(T^n x) = A(q_{n,i})$ implies $\rho(B(T^n x), A(T^n x)) < \epsilon$, for all $x \in K$ and $0 \leq n < N_2$. Hence

$$\max_{x\in\tilde{K}} \rho(A(x), B(x)) < \epsilon. \tag{12}$$

Viewing A and B as two continuous functions from X and $\tilde{K} \subset X$ to a locally closed submanifold $L \subseteq \mathrm{GL_d}(\mathbb{R})$, we note that using Urison's lemma the definition of B can be expanded to the whole space X, so that the inequality

$$\rho(A, B) = \max_{x\in X} \rho(A(x), B(x)) < \epsilon$$

still holds. In particular we will have the bound $\log \|B(x)\| < M$ for all $x \in X$. Now using (9) and (11), we obtain

$$\begin{aligned}
\Lambda(B) &\leq \frac{1}{N_1} \int \log \|B(N_1, x)\| \, d\mu(x) \\
&= \sum_{i=1}^{k} \sum_{n=0}^{N_2-1} \frac{1}{N_1} \int_{T^n K_i} \log \|B(N_1, x)\| \, d\mu(x) \\
&+ \frac{1}{N_1} \int_{X\setminus\tilde{K}} \log \|B(N_1, x)\| \, d\mu(x) \\
&\leq \sum_{i=1}^{k} \mu(K_i) \cdot \sum_{n=0}^{N_2-N_1-1} \frac{1}{N_1} \log \|B(N_1, T^n p_i)\| \\
&+ \ (N_1 \cdot \mu(K) + \mu(X \setminus \tilde{K})) \cdot \max_{x\in X} \log \|B(x)\| \\
&< \lambda'' + \frac{2 \cdot N_0}{N_1} \cdot M + \frac{N_1}{N_2} \cdot M + \eta \cdot M \ < \ \lambda'.
\end{aligned}$$

as required.

6. DISCUSSION

As we have mentioned, examples of non-uniform functions were constructed by M. R. Herman (see [4]) and by P. Walters ([11]). These examples are two dimensional, and in M. Herman's example the base (X, T) is an irrational rotation of the circle. The following question of P. Walters remains open:

QUESTION. – *Does there exist a non-uniform matrix function on every non-atomic uniquely ergodic system* (X, μ, T) *?*

The following remarks summarize some of the (unsuccessful) attempts to answer positively this question:

- An existence of non-uniform functions in $C(X, \mathrm{GL}_2(\mathbb{R}))$ will follow from discontinuity of Λ on $C(X, \mathrm{GL}_2(\mathbb{R}))$ (Theorem 5). It was shown by O. Knill [7], that for aperiodic (X, μ, T) the functional Λ is discontinuous on $L^\infty(X, \mathrm{SL}_2(\mathbb{R}))$. However, this construction does not seem to apply (at least not directly) to $C(X, \mathrm{SL}_2(\mathbb{R}))$.
- It follows from the proof of Theorem 4, that $A \in C(X, \mathrm{SL}_2(\mathbb{R}))$ with $\Lambda(A) > 0$ and such, that T_A is minimal on $X \times P^1$, is non-uniform. E. Glasner and B. Weiss [3] have constructed minimal extensions T_A for any minimal (X, T). They have shown that the set

$$\{A \in C(X, \mathrm{SL}_2(\mathbb{R})) \mid T_A \text{ is minimal}\}$$

forms a dense G_δ-set in the closure of coboundaries:

$$\mathcal{B} = \overline{\{B^{-1}(Tx)B(x) \mid B \in C(X, \mathrm{SL}_2(\mathbb{R}))\}}.$$

However they also proved, that a dense G_δ-set of such functions A gives rise to a uniquely ergodic skew-product T_A, and thus, by Lemma 4, satisfies $\lambda_1 = \lambda_2$. So it remains unclear, whether there always exists a minimal T_A with $\lambda_1 > \lambda_2$.
- It follows from Theorem 4, that if $A \in C(X, \mathrm{SL}_2(\mathbb{R}))$ has the form $A(x) = C^{-1}(Tx) \cdot \mathrm{diag}(e^{\alpha(x)}, e^{-\alpha(x)}) \cdot C(x)$ with measurable $C : X \to \mathrm{SL}_2(\mathbb{R})$ and $\alpha(x)$, $\log \|C(x)\| \in L^1(\mu)$ and $\mu(\alpha) > 0$, but A *cannot* be represented in the above form with $\alpha(x)$ and $C(X)$ being continuous, then A is non-uniform.

We conclude by some remarks and open questions:

1. Motivated by the proof of Theorem 5, we can ask whether every non-uniform function A (on an irrational rotation) is a limit of coboundaries?
2. Another question is, whether every function $A : X \to \mathrm{SL}_2(\mathbb{R})$ with $\Lambda(A) = 0$ is a limit of coboundaries?
3. Does the set of $A : X \to \mathrm{SL}_2(\mathbb{R})$ with $\Lambda(A) > 0$ form a dense G_δ-set in $C(X, \mathrm{SL}_2(\mathbb{R}))$? O. Knill [8] has constructed a dense subset in $L^\infty(X, \mathrm{SL}_2(\mathbb{R}))$ with $\Lambda > 0$. This method seems to apply also to $C(X, \mathrm{GL}_2(\mathbb{R}))$. We have recently learned that N. Nerurkar [9] had proved a sharper statement: positive Lyapunov exponents

occur on a dense set of $C(X, L)$ for all submanifolds $L \subseteq \mathrm{SL}_2(\mathbb{R})$ satisfying certain mild condition. So the question is, whether the set $\{A \in \mathrm{SL}_2(\mathbb{R}) \mid \Lambda(A) > 0\}$ forms a G_δ-set?

4. Note, that an affirmative answer to the previous question for an irrational rotation, will imply that the set of continuously diagonalizable $\mathrm{SL}_2(\mathbb{R})$-cocycles forms a dense G_δ-set (in fact, contains a dense *open* set) in $C(X, \mathrm{SL}_2(\mathbb{R}))$.

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REFERENCES

[1] J. E. Cohen, H. Kesten and M. Newman (editors), Oseledec's multiplicative ergodic theorem: a proof, *Contemporary Mathematics*, Vol. **50**, 1986, pp. 23-30.

[2] H. Furstenberg and H. Kesten, Products of random matrices, *Ann. Math. Stat.*, Vol. **31**, 1960, pp. 457-489.

[3] E. Glasner and B. Weiss, On the construction of minimal skew-products, *Israel J. Math.*, Vol. **34**, 1979, pp. 321-336.

[4] M. R. Herman, Construction d'un difféomorphisme minimal d'entropie topologique non nulle, *Ergod. Th. and Dyn. Sys.*, Vol. **1**, No. 1, 1981, pp. 65-76.

[5] Y. Katznelson and B. Weiss, A simple proof of some ergodic theorems, *Israel J. Math.*, Vol. **42**, No. 4, 1982, pp. 291-296.

[6] J. F. C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Royal Stat. Soc.*, Vol. **B30**, 1968, pp. 499-510.

[7] O. Knill, The upper Lyapunov exponent of $SL_2(R)$ cocycles: discontinuity and the problem of positivity, in Lecture Notes in Math., Vol. **1186**, *Lyapunov exponents*, Berlin-Heidelberg-New York, pp. 86-97.

[8] O. Knill, Positive Lyapunov exponents for a dense set of bounded measurable $SL_2(R)$-cocycles, *Ergod. Th. and Dyn. Syst.*, Vol. **12**, No.2, 1992, pp. 319-331.

[9] M. G. Nerurkar, Typical $SL_2(R)$ valued cocycles arising from strongly accessible linear differential systems have non-zero Lyapunov exponents, *preprint*.

[10] V. I. Oseledec, A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, Vol. **19**, 1968, pp. 197-231.

[11] P. Walters, Unique ergodicity and matrix products, in Lecture Notes in Math., Vol **1186**, *Lyapunov exponents*, Berlin-Heidelberg-New York, pp. 37-56.

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