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Integrable measure equivalence and rigidity of hyperbolic lattices

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Abstract We study rigidity properties of lattices in $\mathrm{Isom}(\mathbf{H}^n) \simeq \mathrm{SO}_{n,1}(\mathbb{R})$, $n \geq 3$, and of surface groups in $\mathrm{Isom}(\mathbf{H}^2) \simeq \mathrm{SL}_2(\mathbb{R})$ in the context of *integrable measure equivalence*. The results for lattices in $\mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$, are generalizations of Mostow rigidity; they include a *cocycle* version of strong rigidity and an integrable measure equivalence classification. Despite the lack of Mostow rigidity for $n = 2$ we show that cocompact lattices in $\mathrm{Isom}(\mathbf{H}^2)$ allow a similar integrable measure equivalence classification.

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1 Introduction and statement of the main results

1.1 Introduction

Measure equivalence is an equivalence relation on groups, introduced by Gromov [25] as a measure-theoretic counterpart to quasi-isometry of finitely generated groups. It is intimately related to *orbit equivalence* in ergodic theory, to the theory of von Neumann algebras, and to questions in descriptive set theory. The study of rigidity in measure equivalence or orbit equivalence goes back to Zimmer's paper [61], which extended Margulis' superrigidity of higher rank lattices [39] to the context of *measurable cocycles* and applied it to prove *strong rigidity* phenomena in *orbit equivalence* setting. In the same paper [61, §6] Zimmer poses the question of whether these strong rigidity for orbit equivalence results extend to lattices in rank one groups $G \not\simeq \mathrm{PSL}_2(\mathbb{R})$; and in [62] and later in a joint paper with Pansu [45] obtained some results under some restrictive geometric condition.

The study of measure equivalence and related problems has recently experienced a rapid growth, with [14, 15, 21, 22, 26, 28, 29, 33, 35, 43, 46–49, 51] being only a partial list of important advances. We refer to [17, 50, 56] for surveys and further references. One particularly fruitful direction of research in this area has been in obtaining the complete description of groups that are measure equivalent to a given one from a well understood class of groups. This has been achieved for lattices in simple Lie groups of higher rank [15], products of hyperbolic-like groups [43], mapping class groups [33–35], and certain amalgams of groups as above [36]. In all these results, the measure equivalence class of one of such groups turns out to be small and to consist of a list of "obvious" examples obtained by simple modifications of the original group. This phenomenon is referred to as *measure equivalence rigidity*. On the other hand, the class of groups measure equivalent to lattices in $\mathrm{SL}_2(\mathbb{R})$ is very rich: it is uncountable, includes wide classes of groups and does not seem to have an explicit description (cf. [2, 23]).

In the present paper we obtain measure equivalence rigidity results for lattices in the least rigid family of simple Lie groups $\mathrm{Isom}(\mathbf{H}^n) \simeq \mathrm{SO}_{n,1}(\mathbb{R})$ for $n \geq 2$, including surface groups, albeit within a more restricted category of *integrable measure equivalence*, hereafter also called L^1*-measure equivalence* or just L^1*-ME*. Let us briefly state the classification result, before giving the precise definitions and stating more detailed results.

Theorem A *Let Γ be a lattice in $G = \mathrm{Isom}(\mathbf{H}^n)$, $n \geq 2$; in the case $n = 2$ assume that Γ is cocompact. Then the class of all finitely generated groups that are L^1-measure equivalent to Γ consists of those Λ, which admit a short exact sequence $\{1\} \to F \to \Lambda \to \bar{\Lambda} \to \{1\}$ where F is finite and $\bar{\Lambda}$ is a lattice in G; in the case $n = 2$, $\bar{\Lambda}$ is also cocompact in $G = \mathrm{Isom}(\mathbf{H}^2)$.*

The integrability assumption is necessary for the validity of the rigidity results for cocompact lattices in $\mathrm{Isom}(\mathbf{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$. It remains possible, however, that the L^1-integrability assumption is superfluous for lattices in $\mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$. We also note that a result of Fisher and Hitchman [12] can be used to obtain L^2-ME rigidity results similar to Theorem A for the family of rank one Lie groups $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{H}}) \simeq \mathrm{Sp}_{n,1}(\mathbb{R})$ and $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{O}}) \simeq F_{4(-20)}$;[1] it is possible that this L^2-integrability assumption can be relaxed or removed altogether.

[1] Here $\simeq$ means *locally isomorphic*.

The proof of Theorem A for the case $n \geq 3$ proceeds through a cocycle version of Mostow's strong rigidity theorem stated in Theorems B and 1.8. This cocycle version relates to the original Mostow's strong rigidity theorem in the same way in which Zimmer's cocycle superrigidity theorem relates to the original Margulis' superrigidity for higher rank lattices. Our proof of the cocycle version of Mostow rigidity, which is inspired by Gromov-Thurston's proof of Mostow rigidity using simplicial volume [59] and Burger-Iozzi's proof for dimension 3 [5], heavily uses bounded cohomology and other homological methods. A major part of the relevant homological technique, which applies to general Gromov hyperbolic groups, is developed in the companion paper [1]; *in fact*, Theorem 4.11 *taken from* [1] *is the only place in this paper where we require the integrability assumption.*

Theorem A and the more detailed Theorem D are deduced from the strong rigidity for integrable cocycles (Theorem B) using a general method described in Theorem 2.1. The latter extends and streamlines the approach developed in [15], and further used in [43] and in [35].

The proof of Theorem A for surfaces uses a cocycle version of the fact that an abstract isomorphism between uniform lattices in $\mathrm{PGL}_2(\mathbb{R})$ is realized by conjugation in $\mathrm{Homeo}(S^1)$. The proof of this generalization uses homological methods mentioned above and a cocycle version of the Milnor-Wood-Ghys phenomenon (Theorem C), in which an integrable ME-cocycle between surface groups is conjugate to the identity map in $\mathrm{Homeo}(S^1)$. In the case of surfaces in Theorem A, this result is used together with Theorem 2.1 to construct a representation $\rho : \Lambda \to \mathrm{Homeo}(S^1)$. Additional arguments (Lemma 2.5 and Theorem 5.2) are then needed to deduce that $\rho(\Lambda)$ is a uniform lattice in $\mathrm{PGL}_2(\mathbb{R})$.

Let us now make precise definitions and describe in more detail the main results.

1.2 Basic notions

1.2.1 Measure equivalence of locally compact groups

We recall the central notion of *measure equivalence* which was suggested by Gromov [25, 0.5.E]. It will be convenient to work with general unimodular, locally compact second countable (lcsc) groups rather than just countable ones.

Definition 1.1 Let G, H be unimodular lcsc groups with Haar measures m_G and m_H. A (G, H)-*coupling* is a Lebesgue measure space (Ω, m) with a measurable, measure-preserving action of $G \times H$ such that there exist finite measure spaces (X,μ), (Y,ν) and measure space isomorphisms

$$i:(G,m_G)\times(Y,\nu)\xrightarrow{\cong}(\Omega,m)\quad\text{and}\quad j:(H,m_H)\times(X,\mu)\xrightarrow{\cong}(\Omega,m) \tag{1.1}$$

such that i is G equivariant and j is H equivariant, that is

$$i\big(gg',y\big)=gi\big(g',y\big)\quad\text{and}\quad j\big(hh',x\big)=hj\big(h',x\big)$$

for every $g\in G$ and $h\in H$ and almost every $g'\in G$, $h'\in H$, $y\in Y$ and $x\in X$. Groups which admit such a coupling are said to be *measure equivalent* (abbreviated ME).

In the case where G and H are countable groups, the condition on the commuting actions $G\curvearrowright(\Omega,m)$ and $H\curvearrowright(\Omega,m)$ is that they admit finite m-measure Borel fundamental domains $X,Y\subset\Omega$ with $\mu=m|_X$ and $\nu=m|_Y$ being the restrictions.

As the name suggest, measure equivalence is an equivalence relation between unimodular lcsc groups. For reflexivity, consider the $G\times G$-action on (G,m_G), $(g_1,g_2):g\mapsto g_1gg_2^{-1}$. We refer to this as the *tautological self coupling* of G. The symmetry of the equivalence relation is obvious. For transitivity and more details we refer to Appendix A.1.

Example 1.2 Let Γ_1,Γ_2 be lattices in a lcsc group G.[2] Then Γ_1 and Γ_2 are measure equivalent, with (G,m_G) serving as a natural (Γ_1,Γ_2)-coupling when equipped with the action $(\gamma_1,\gamma_2):g\mapsto\gamma_1g\gamma_2^{-1}$ for $\gamma_i\in\Gamma_i$. In fact, any lattice $\Gamma<G$ is measure equivalent to G, with (G,m_G) serving as a natural (G,Γ)-coupling when equipped with the action $(g,\gamma):g'\mapsto gg'\gamma^{-1}$.

1.2.2 Taut groups

We now introduce the following key notion of taut couplings and taut groups.

Definition 1.3 (Taut couplings, taut groups) A (G,G)-coupling (Ω,m) is *taut* if it has the tautological coupling as a factor uniquely; in other words if it admits a unique, up to null sets, measurable map $\Phi:\Omega\to G$ so that for m-a.e. $\omega\in\Omega$ and all $g_1,g_2\in G$[3]

$$\Phi\big((g_1,g_2)\omega\big)=g_1\Phi(\omega)g_2^{-1}.$$

Such a $G\times G$-equivariant map $\Omega\to G$ will be called a *tautening map*. A unimodular lcsc group G is *taut* if every (G,G)-coupling is taut.

[2] Any lcsc group containing a lattice is necessarily unimodular.

[3] If one only requires equivariance for almost all $g_1,g_2\in G$ one can always modify Φ on a null set to get an everywhere equivariant map [63, Appendix B].

The requirement of uniqueness for tautening maps in the definition of taut groups is equivalent to the *strongly ICC* property for the group in question (see Definition 2.2 and Lemmas A.7 and A.8(1) in the Appendix A.4 for a proof of this claim). This property is rather common; in particular it is satisfied by all center-free semi-simple Lie groups and all ICC countable groups, i.e. countable groups with infinite conjugacy classes. On the other hand the existence of tautening maps for (G, G)-coupling is hard to obtain; in particular taut groups necessarily satisfy Mostow's strong rigidity property.

Lemma 1.4 (Taut groups satisfy Mostow rigidity) *Let G be a taut unimodular lcsc group. If $\tau : \Gamma_1 \xrightarrow{\cong} \Gamma_2$ is an isomorphism of two lattices Γ_1 and Γ_2 in G, then there exists a unique $g \in G$ so that $\Gamma_2 = g^{-1}\Gamma_1 g$ and $\tau(\gamma_1) = g^{-1}\gamma g$ for $\gamma \in \Gamma_1$.*

The lemma follows from considering the tautness of the measure equivalence (G, G)-coupling given by the $(G \times G)$-homogeneous space $(G \times G)/\Delta_\tau$, where Δ_τ is the graph of the isomorphism $\tau : \Gamma_1 \to \Gamma_2$; see Lemma A.3 for details.

The phenomenon, that any isomorphism between lattices in G is realized by an inner conjugation in G, known as *strong rigidity* or *Mostow rigidity*, holds for all simple Lie groups[4] $G \not\simeq \mathrm{SL}_2(\mathbb{R})$. More precisely, if X is an irreducible non-compact, non-Euclidean symmetric space with the exception of the hyperbolic plane $\mathbf{H}^2$, then $G = \mathrm{Isom}(X)$ is Mostow rigid. Mostow proved this remarkable rigidity property for uniform lattices [44]. It was then extended to the non-uniform cases by Prasad [52] ($\mathrm{rk}(X) = 1$) and by Margulis [38] ($\mathrm{rk}(X) \geq 2$).

In the higher rank case, more precisely, if X is a symmetric space without compact and Euclidean factors with $\mathrm{rk}(X) \geq 2$, Margulis proved a stronger rigidity property, which became known as *superrigidity* [39]. Margulis' superrigidity for lattices in higher rank, was extended by Zimmer in the cocycle superrigidity theorem [61]. Zimmer's cocycle superrigidity was used in [15] to show that higher rank simple Lie groups G are *taut* (albeit the use of term tautness in this context is new). In [43] Monod and Shalom proved another case of cocycle superrigidity and proved a version of tautness property for certain products $G = \Gamma_1 \times \cdots \times \Gamma_n$ with $n \geq 2$. In [33, 35] Kida proved that mapping class groups are taut. Kida's result may be viewed as a cocycle generalization of Ivanov's theorem [31].

1.2.3 Measurable cocycles

Let us elaborate on this connection between tautness and *rigidity* of *measurable cocycles*. Recall that a *cocycle* over a group action $G \curvearrowright X$ to another

[4]For the formulation of Mostow rigidity above we have to assume that G has trivial center.

group H is a map $c : G \times X \to H$ such that for all $g_1, g_2 \in G$

$$c(g_2 g_1, x) = c(g_2, g_1 x) \cdot c(g_1, x).$$

Cocycles that are independent of the space variable are precisely homomorphisms $G \to H$. One can conjugate a cocycle $c : G \times X \to H$ by a map $f : X \to H$ to produce a new cocycle $c^f : G \times X \to H$ given by

$$c^f(g, x) = f(g . x)^{-1} c(g, x) f(x).$$

In our context, G is a lcsc group, H is lcsc or, more generally, a Polish group, and $G \curvearrowright (X, \mu)$ is a measurable measure-preserving action on a Lebesgue finite measure space. In this context all maps, including the cocycle c, are assumed to be μ-measurable, and all equations should hold μ-a.e.; we then say that c is a *measurable cocycle*.

Let (Ω, m) be a (G, H)-coupling and $H \times X \xrightarrow{j} \Omega \xrightarrow{i^{-1}} G \times Y$ be as in (1.1). Since the actions $G \curvearrowright \Omega$ and $H \curvearrowright \Omega$ commute, G acts on the space of H-orbits in Ω, which is naturally identified with X. This G-action preserves the finite measure μ. Similarly, we get the measure preserving H-action on (Y, ν). These actions will be denoted by a dot, $g : x \mapsto g . x$, $h : y \mapsto h.y$, to distinguish them from the $G \times H$ action on Ω. Observe that in Ω one has for $g \in G$ and almost every $h \in H$ and $x \in X$,

$$g j(h, x) = j\left(h h_1^{-1}, g . x\right)$$

for some $h_1 \in H$ which depends only on $g \in G$ and $x \in X$, and therefore may be denoted by $\alpha(g, x)$. One easily checks that the map

$$\alpha : G \times X \to H$$

that was just defined, is a measurable cocycle. Similarly, one obtains a measurable cocycle $\beta : H \times Y \to G$. These cocycles depend on the choice of the measure isomorphisms in (1.1), but different measure isomorphisms produce conjugate cocycles. Identifying (Ω, m) with $(H, m_H) \times (X, \mu)$, the action $G \times H$ takes the form

$$(g, h) j\left(h', x\right) = j\left(h h' \alpha(g, x)^{-1}, g . x\right). \tag{1.2}$$

Similarly, cocycle $\beta : H \times Y \to G$ describes the $G \times H$-action on (Ω, m) when identified with $(G, m_G) \times (Y, \nu)$. In general, we call a measurable cocycle $G \times X \to H$ that arises from a (G, H)-coupling as above an *ME-cocycle*.

The connection between tautness and cocycle rigidity is in the observation (see Lemma A.4) that a (G, G)-coupling (Ω, m) is taut iff the ME-cocycle $\alpha : G \times X \to G$ is conjugate to the identity isomorphism, $\alpha(g, x) =$

$f(g\,.\,x)^{-1}gf(x)$ by a unique measurable $f : X \to G$. Hence one might say that

G is *taut* if and only if it satisfies a *cocycle version* of Mostow rigidity.

1.2.4 Integrability conditions

Our first main result—Theorem B below—shows that $G = \mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$, are 1-*taut* groups, i.e. all *integrable* (G, G)-couplings are taut. We shall now define these terms more precisely.

A *norm* on a group G is a map $|\cdot| : G \to [0, \infty)$ so that $|gh| \leq |g| + |h|$ and $|g^{-1}| = |g|$ for all $g, h \in G$. A norm on a lcsc group is *proper* if it is measurable and the balls with respect to this norm are pre-compact. Two norms $|\cdot|$ and $|\cdot|'$ are equivalent if there are $a, b > 0$ such that $|g|' \leq a \cdot |g| + b$ and $|g| \leq a \cdot |g|' + b$ for every $g \in G$. On a compactly generated group[5] G with compact generating symmetric set K the function $|g|_K = \min\{n \in \mathbb{N} \mid g \in K^n\}$ defines a proper norm, whose equivalence class does not depend on the chosen K. Unless stated otherwise, we mean a norm in this equivalence class when referring to a proper norm on a compactly generated group.

Definition 1.5 (Integrability of cocycles) Let H be a compactly generated group with a proper norm $|\cdot|$ and G be a lcsc group. Let $p \in [1, \infty]$. A measurable cocycle $c : G \times X \to H$ is L^p*-integrable* if for a.e. $g \in G$

$$\int_X \left|c(g, x)\right|^p d\mu(x) < \infty.$$

For $p = \infty$ we require that the essential supremum of $|c(g, -)|$ is finite for a.e. $g \in G$. If $p = 1$, we also say that c is *integrable*. If $p = \infty$, we say that c is *bounded*.

The integrability condition is independent of the choice of a norm within a class of equivalent norms. L^p-integrability implies L^q-integrability whenever $1 \leq q \leq p$. In the Appendix A.2 we show that, if G is also compactly generated, the L^p-integrability of c implies that the above integral is uniformly bounded on compact subsets of G.

Definition 1.6 (Integrability of couplings) A (G, H)-coupling (Ω, m) of compactly generated, unimodular, lcsc groups is L^p*-integrable*, if there exist measure isomorphisms as in (1.1) so that the corresponding ME-cocycles $G \times X \to H$ and $H \times Y \to G$ are L^p-integrable. If $p = 1$ we just say that (Ω, m) is *integrable*. Groups G and H that admit an L^p-integrable (G, H)-coupling are said to be L^p*-measure equivalent*.

[5] Every connected lcsc group is compactly generated [57, Corollary 6.12 on p. 58].

For each $p \in [1, \infty]$, being L^p-measure equivalent is an equivalence relation on compactly generated, unimodular, lcsc groups (see Lemma A.1). Furthermore, L^p-measure equivalence implies L^q-measure equivalence if $1 \leq q \leq p$. So among the L^p-measure equivalence relations, L^∞-measure equivalence is the strongest and L^1-measure equivalence is the weakest one; all being subrelations of the (unrestricted) measure equivalence.

Let $\Gamma < G$ be a lattice, and assume that G is compactly generated and Γ is finitely generated; as is the case for semi-simple Lie groups G. Then the (Γ, G)-coupling (G, m_G) is L^p-integrable iff Γ is an L^p*-integrable lattices* in G; if there exists a Borel cross-section $s : G/\Gamma \to G$ of the projection, so that the cocycle $c : G \times G/\Gamma \to \Gamma$, $c(g, x) = s(g \,.\, x)^{-1} g s(x)$ is L^p-integrable. In particular L^∞-integrable lattices are precisely the uniform, i.e. cocompact ones. Integrability conditions for lattices appeared for example in Margulis's proof of superrigidity (cf. [40, V. §4]), and in Shalom's [55].

Definition 1.7 A lcsc group G is *p-taut* if every L^p-integrable (G, G)-coupling is taut.

1.3 Statement of the main results

Theorem B *The groups* $G = \mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$, *are* 1*-taut.*

This result has an equivalent formulation in terms of cocycles.

Theorem 1.8 (Integrable cocycle strong rigidity) *Let* $G = \mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$, $G \curvearrowright (X, \mu)$ *be a probability measure preserving action, and* $c : G \times X \to G$ *be an integrable ME-cocycle. Then there is a measurable map* $f : X \to G$, *which is unique up to null sets, such that for* μ*-a.e.* $x \in X$ *and every* $g \in G$ *we have*

$$c(g, x) = f(g \,.\, x)^{-1}\, g\, f(x).$$

Note that this result generalizes Mostow-Prasad rigidity for lattices in these groups. This follows from the fact that any 1-taut group satisfies Mostow rigidity for L^1-integrable lattices, and the fact, due to Shalom, that all lattices in groups $G = \mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$, are L^1-integrable.

Theorem 1.9 ([55, Theorem 3.6]) *All lattices in simple Lie groups not locally isomorphic to* $\mathrm{Isom}(\mathbf{H}^2) \simeq \mathrm{PSL}_2(\mathbb{R})$, $\mathrm{Isom}(\mathbf{H}^3) \simeq \mathrm{PSL}_2(\mathbb{C})$, *are* L^2*-integrable, hence also* L^1*-integrable. Further, lattices in* $\mathrm{Isom}(\mathbf{H}^3)$ *are* L^1*-integrable.*

The second assertion is not stated in this form in [55, Theorem 3.6] but the proof therein shows exactly that. In fact, for lattices in $\mathrm{Isom}(\mathbf{H}^n)$ Shalom shows $\mathrm{L}^{n-1-\epsilon}$-integrability.

Lattices in $G = \mathrm{Isom}(\mathbf{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$, such as surface groups, admit a rich space of deformations—the Teichmüller space. In particular, these groups do not satisfy Mostow rigidity, and therefore are not taut (they are not even ∞-taut). However, it is well known viewing $G = \mathrm{Isom}(\mathbf{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$ as acting on the circle $S^1 \cong \partial\mathbf{H}^2 \cong \mathbb{R}\,\mathrm{P}^1$, any abstract isomorphism $\tau : \Gamma \to \Gamma'$ between cocompact lattices $\Gamma, \Gamma' < G$ can be realized by a conjugation in $\mathrm{Homeo}(S^1)$, that is,

$$\exists_{f\in\mathrm{Homeo}(S^1)}\ \forall_{\gamma\in\Gamma} \quad \pi\circ\tau(\gamma) = f^{-1}\circ\pi(\gamma)\circ f,$$

where $\pi : G \to \mathrm{Homeo}(S^1)$ is the imbedding as above. (Such f is the "boundary map" constructed in Mostow's proof of strong rigidity: the isomorphism $\tau : \Gamma \to \Gamma'$ gives rise to a quasi-isometry of $\mathbf{H}^2$, and Morse-Mostow lemma is used to extend this quasi-isometry to a (quasi-symmetric) homeomorphism f of the boundary $S^1 = \partial\mathbf{H}^2$.) Motivated by this observation we generalize the notion of tautness as follows.

Definition 1.10 Let G be a unimodular lcsc group, $\mathcal{G}$ a Polish group, $\pi : G \to \mathcal{G}$ a continuous homomorphism. A (G, G)-coupling is *taut relative* to $\pi : G \to \mathcal{G}$ if there exists a up to null sets unique measurable map $\Phi : \Omega \to \mathcal{G}$ such that for m-a.e. $\omega \in \Omega$ and all $g_1, g_2 \in G$

$$\Phi\big((g_1, g_2)\omega\big) = \pi(g_1)\Phi(\omega)\pi(g_2)^{-1}.$$

We say that G is *taut* (resp. *p-taut*) *relative to* $\pi : G \to \mathcal{G}$ if all (resp. all L^p-integrable) (G, G)-couplings are taut relative to $\pi : G \to \mathcal{G}$.

Observe that G is taut iff it is taut relative to itself. Note also that if $\Gamma < G$ is a lattice, then G is taut iff Γ is taut relative to the inclusion $\Gamma < G$; and G is taut relative to $\pi : G \to \mathcal{G}$ iff Γ is taut relative to $\pi|_\Gamma : \Gamma \to \mathcal{G}$ (Proposition 2.9). If $\Gamma < G$ is L^p-integrable, then these equivalences apply to p-tautness.

Theorem C *The group* $G = \mathrm{Isom}(\mathbf{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$ *is* 1*-taut relative to the natural embedding* $G < \mathrm{Homeo}(S^1)$. *Cocompact lattices* $\Gamma < G$ *are* 1*-taut relative to the embedding* $\Gamma < G < \mathrm{Homeo}(S^1)$.

We skip the obvious equivalent cocycle reformulation of this result.

Remark 1.11

(1) The L^1-assumption cannot be dropped from Theorem C. Indeed, the free group $\mathbf{F}_2$ can be realized as a lattice in $\mathrm{PSL}_2(\mathbb{R})$, but most automorphisms of $\mathbf{F}_2$ cannot be realized by homeomorphisms of the circle.

(2) Realizing isomorphisms between surface groups in $\mathrm{Homeo}(S^1)$, one obtains somewhat regular maps: they are Hölder continuous and quasi-symmetric. We do not know (and do not expect) Theorem C to hold with $\mathrm{Homeo}(S^1)$ being replaced by the corresponding subgroups.

We now state the L^1-ME rigidity result which is deduced from Theorem B, focusing on the case of countable, finitely generated groups.

Theorem D (L^1-Measure equivalence rigidity) *Let $G = \mathrm{Isom}(\mathbf{H}^n)$ with $n \geq 3$, and $\Gamma < G$ be a lattice. Let Λ be a finitely generated group, and let (Ω, m) be an integrable (Γ, Λ)-coupling. Then*

(1) *there exists a short exact sequence*

$$1 \to F \to \Lambda \to \bar{\Lambda} \to 1$$

where F is finite and $\bar{\Lambda}$ is a lattice in G,

(2) *and a measurable map $\Phi : \Omega \to G$ so that for m-a.e. $\omega \in \Omega$ and every $\gamma \in \Gamma$ and every $\lambda \in \Lambda$*

$$\Phi\big((\gamma, \lambda)\omega\big) = \gamma \Phi(\omega) \bar{\lambda}^{-1}.$$

Moreover, if $\Gamma \times \Lambda \curvearrowright (\Omega, m)$ is ergodic, then

(2a) *either the push-forward measure $\Phi_* m$ is a positive multiple of the Haar measure m_G or m_{G^0};*

(3a) *or, one may assume that Γ and $\bar{\Lambda}$ share a subgroup of finite index and $\Phi_* m$ is a positive multiple of the counting measure on the double coset $\Gamma e \bar{\Lambda} \subset G$.*

This result is completely analogous to the higher rank case considered in [15], except for the L^1-assumption. We do not know whether Theorem D remains valid in the broader ME category, that is, without the L^1-condition, but should point out that if the L^1 condition can be removed from Theorem B then it can also be removed from Theorem D.

Theorem D can also be stated in the broader context of unimodular lcsc groups, in which case the L^1-measure equivalence rigidity states that a compactly generated unimodular lcsc group H that is L^1-measure equivalent to $G = \mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$, admits a short exact sequence $1 \to K \to H \to \bar{H} \to 1$ where K is compact and $\bar{H}$ is either G, or its index two subgroup G^0, or is a lattice in G.

Measure equivalence rigidity results have natural consequences for *(stable, or weak) orbit equivalence* of essentially free probability measure-preserving group actions (cf. [14, 34, 43, 49]). Two probability measure preserving actions $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$ are *weakly*, or *stably*, *orbit equivalent* if there exist measurable maps $p : X \to Y$, $q : Y \to X$ with $p_*\mu \ll \nu$, $q_*\nu \ll \mu$ so that a.e.

$$p(\Gamma . x) \subset \Lambda . p(x), \qquad q(\Lambda . y) \subset \Gamma . q(y),$$
$$q \circ p(x) \in \Gamma . x, \qquad p \circ q(y) \in \Lambda . y$$

(see [14, §2] for other equivalent definitions). If Γ_1, Γ_2 are lattices in some lcsc group G, then $\Gamma_1 \curvearrowright G/\Gamma_2$ and $\Gamma_2 \curvearrowright G/\Gamma_1$ are stably orbit equivalent via $p(x) = s_1(x)^{-1}$, $q(y) = s_2(y)^{-1}$, where $s_i : G/\Gamma_i \to G$ are measurable cross-sections. Moreover, given any (essentially) free, ergodic, probability measure preserving (p.m.p.) action $\Gamma_1 \curvearrowright (X_1, \mu_1)$ and Γ_1-equivariant quotient map $\pi_1 : X_1 \to G/\Gamma_2$, there exists a canonically defined free, ergodic p.m.p. action $\Gamma_2 \curvearrowright (X_2, \mu_2)$ with equivariant quotient $\pi_2 : X_2 \to G/\Gamma_1$ so that $\Gamma_i \curvearrowright (X_i, \mu_i)$ are stably orbit equivalent in a way compatible to $\pi_i : X_i \to G/\Gamma_{3-i}$ [14, Theorem C].

We shall now introduce integrability conditions on weak orbit equivalence, assuming Γ and Λ are finitely generated groups. Let $|\cdot|_\Gamma$, $|\cdot|_\Lambda$ denote some word metrics on Γ, Λ respectively, and let $\Gamma \curvearrowright (X, \mu)$ be an essentially free action. Define an extended metric $d_\Gamma : X \times X \to [0, \infty]$ on X by setting $d_\Gamma(x_1, x_2) = |\gamma|_\Gamma$ if $\gamma . x_1 = x_2$ and set $d_\Gamma(x_1, x_2) = \infty$ otherwise. Let d_Λ denote the extended metric on Y, defined in a similar fashion. We say that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are L^s-*weakly/stably orbit equivalent*, if there exists maps $p : X \to Y$, $q : Y \to X$ as above, and such that for every $\gamma \in \Gamma$, $\lambda \in \Lambda$

$$\big(x \mapsto d_\Lambda\big(p(\gamma . x), p(x)\big)\big) \in \mathrm{L}^s(X, \mu),$$
$$\big(x \mapsto d_\Gamma\big(q(\lambda . y), q(y)\big)\big) \in \mathrm{L}^s(Y, \nu).$$

Note that the last condition is independent of the choice of word metrics.

The following result[6] is deduced from Theorem D in essentially the same way Theorems A and C in [14] are deduced from the corresponding measure equivalence rigidity theorem in [15]. The only additional observation is that the constructions respect the integrability conditions.

Theorem E (L^1-Orbit equivalence rigidity) *Let $G = \mathrm{Isom}(\mathbf{H}^n)$ where $n \geq 3$, and $\Gamma < G$ be a lattice. Assume that there is a finitely generated group Λ and essentially free, ergodic, p.m.p actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$, which admit a stable L^1-orbit equivalence $p : X, \to Y$, $q : Y \to X$ as above. Then either one the following two cases occurs*:

[6]The formulation of the virtual isomorphism case in terms of induced actions is due to Kida [34].

Virtual isomorphism: *There exists a short exact sequence* $1 \to F \to \Lambda \to \bar{\Lambda} \to 1$, *where* F *is a finite group and* $\bar{\Lambda} < G$ *is a lattice with* $\Delta = \Gamma \cap \bar{\Lambda}$ *having finite index in both* Γ *and* $\bar{\Lambda}$, *and an essentially free ergodic p.m.p. action* $\Delta \curvearrowright (Z, \zeta)$ *so that* $\Gamma \curvearrowright (X, \mu)$ *is isomorphic to the induced action* $\Gamma \curvearrowright \Gamma \times_\Delta (Z, \zeta)$, *and the quotient action* $\bar{\Lambda} \curvearrowright (\bar{Y}, \bar{\nu}) = (Y, \nu)/F$ *is isomorphic to the induced action* $\bar{\Lambda} \curvearrowright \bar{\Lambda} \times_\Delta (Z, \zeta)$, *or*

Standard quotients: *There exists a short exact sequence* $1 \to F \to \Lambda \to \bar{\Lambda} \to 1$, *where* F *is a finite group and* $\bar{\Lambda} < G$ *is a lattice, and for* $G' = G$ *or* $G' = G^0$ (*only if* $\bar{\Lambda}, \Gamma < G^0$), *and equivariant measure space quotient maps*

$$\pi : (X, \mu) \to \left(G'/\bar{\Lambda}, m_{G'/\bar{\Lambda}}\right), \qquad \sigma : (Y, \nu) \to \left(G'/\Gamma, m_{G'/\Gamma}\right)$$

with $\pi(\gamma \,.\, x) = \gamma \,.\, \pi(x)$, $\sigma(\lambda \,.\, y) = \bar{\lambda} \,.\, \sigma(y)$. *Moreover, the action* $\bar{\Lambda} \curvearrowright (\bar{Y}, \bar{\nu}) = (Y, \nu)/F$ *is isomorphic to the canonical action associated to* $\Gamma \curvearrowright (X, \mu)$ *and the quotient map* $\pi : X \to G'/\bar{\Lambda}$.

The family of rank one simple real Lie groups $\mathrm{Isom}(\mathbf{H}^n)$ is the least rigid one among simple Lie groups. As higher rank simple Lie groups are rigid with respect to measure equivalence, one wonders about the remaining families of simple real Lie groups: $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{C}}) \simeq \mathrm{SU}_{n,1}(\mathbb{R})$, $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{H}}) \simeq \mathrm{Sp}_{n,1}(\mathbb{R})$, and the exceptional group $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{O}}) \simeq F_{4(-20)}$. The question of measure equivalence rigidity (or L^p-measure equivalence rigidity) for the former family remains open, but the latter groups are rigid with regard to L^2-measure equivalence. Indeed, recently, using harmonic maps techniques (after Corlette [9] and Corlette-Zimmer [10]), Fisher and Hitchman [12] proved an L^2-cocycle superrigidity result for isometries of quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$ and the Cayley plane $\mathbf{H}^2_{\mathbb{O}}$. This theorem can be used to deduce the following.

Theorem 1.12 (Corollary of [12]) *The rank one Lie groups* $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{H}}) \simeq \mathrm{Sp}_{n,1}(\mathbb{R})$ *and* $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{O}}) \simeq F_{4(-20)}$ *are* 2*-taut.*

Using this result as an input to the general machinery described above one obtains:

Corollary 1.13 *The conclusions of Theorems D and E hold for all lattices in* $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{H}})$ *and* $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{O}})$ *provided the* L^1*-conditions are replaced by* L^2*-ones.*

1.4 Organization of the paper

In Sect. 2 we show that all taut groups are ME-rigid; this is stated in Theorems 2.1 and the more detailed version in Theorem 2.6. In Sects. 3 and 4 we develop the tools for proving tautness of $G = \mathrm{Isom}(\mathbf{H}^n)$—the statement that generalizes Mostow rigidity, Theorem B ($n \geq 3$), and a generalization of Milnor-Wood-Ghys phenomenon, Theorem C ($n = 2$). More precisely, in Sect. 3 we study the effect of an ME cocycle on boundary actions and boundary maps. This section contains various technical results on the crossroad of ergodic theory and geometry. In Sect. 4 these results are used to analyze the effect of the boundary map on cohomology and bounded cohomology, and specifically on the volume form and the Euler class. At a crucial point, when estimating the norm of the Euler class, Corollary 4.13, we use a result from our companion paper [1], which relies on the integrability of the coupling. This is the only place where the integrability assumption is used. The main theorems stated in the introduction are then proved in Sect. 5. General facts about measure equivalence which are used throughout are collected in Appendix A. In order to improve the readability of Sect. 4 we also added Appendix B which contains a brief discussion of bounded cohomology.

2 Measure equivalence rigidity for taut groups

This section contains general tools related to the notion of taut couplings and taut groups. The results of this section apply to general unimodular lcsc groups, including countable groups, and are not specific to $\mathrm{Isom}(\mathbf{H}^n)$ or semi-simple Lie groups. Whenever we refer to L^p-integrability conditions, we assume that the groups are also compactly generated. We rely on some basic facts about measure equivalence which are collected in Appendix A. The basic tool is the following:

Theorem 2.1 *Let G be a unimodular lcsc group that is taut* (*resp. p-taut*). *Any unimodular lcsc group H that is measure equivalent* (*resp.* L^p*-measure equivalent*) *to G admits a short exact sequence with continuous homomorphisms*

$$1 \to K \to H \to \bar{H} \to 1,$$

where K is compact and $\bar{H}$ is a closed subgroup in G such that $G/\bar{H}$ carries a G-invariant Borel probability measure.

Theorem 2.6 below contains a more technical statement that applies to more general situations.

2.1 The strong ICC property and strongly proximal actions

We need to introduce a notion of *strongly ICC* group G and, more generally, the notion of a group $\mathcal{G}$ being strongly ICC relative to a subgroup $\mathcal{G}_0 < \mathcal{G}$.

Definition 2.2 A Polish group $\mathcal{G}$ is *strongly ICC relative* to $\mathcal{G}_0 < \mathcal{G}$ if $\mathcal{G} \setminus \{e\}$ does not support any Borel probability measure that is invariant under the conjugation action of $\mathcal{G}_0$ on $\mathcal{G} \setminus \{e\}$. A Polish group $\mathcal{G}$ is *strongly ICC* if it is strongly ICC relative to itself.

The key properties of this notion are discussed in Appendix A.4. Recall that a countable discrete group is said to be ICC (short for Infinite Conjugacy Classes) if all its non-trivial conjugacy classes are infinite. Note that for a discrete group ICC condition is equivalent to the above strong ICC condition. We will be concerned also with some other examples, that are given in the following proposition.

Proposition 2.3

(1) *Any connected, center-free, semi-simple Lie group G without compact factors is strongly ICC relative to any unbounded Zariski dense subgroup. In particular, G itself is strongly ICC.*
(2) *For a semi-simple Lie group without compact factors G, and a parabolic subgroup $Q < G$, the Polish group* $\mathrm{Homeo}(G/Q)$ *is strongly ICC relative to G.*
    In particular, $\mathrm{Homeo}(S^1)$ *is strongly ICC relative to* $\mathrm{PGL}_2(\mathbb{R})$, *or any lattice in* $\mathrm{PGL}_2(\mathbb{R})$.

Before proving this proposition let us recall the notion of *minimal and strongly proximal* action. A continuous action $G \curvearrowright M$ of a (lcsc) group G on a compact metrizable space M is called *minimal* if there are no closed G-invariant nontrivial subsets in a compact metrizable space M, and *strongly proximal* if every G-invariant weak*-closed set of probability measures on M contains some Dirac measures. Clearly, the action $G \curvearrowright M$ is both minimal and strongly proximal if every G-invariant weak*-closed set of probability measures on M contains all the Dirac measures. Thus, being *minimal and strongly proximal* is easily seen to be equivalent to each of the following conditions:

(1) For every Borel probability measure $\nu \in \mathrm{Prob}(M)$ and every non-empty open subset $V \subset M$ one has

$$\sup_{g \in G} g_* \nu(V) = 1.$$

(2) For every $\nu \in \mathrm{Prob}(M)$ the convex hull of the G-orbit $g_* \nu$ is dense in $\mathrm{Prob}(M)$ in the weak-* topology.

We need the following general statement.

Lemma 2.4 *Let M be a compact metrizable space and $G < \mathrm{Homeo}(M)$ be a subgroup which acts minimally and strongly proximally on M. Then* $\mathrm{Homeo}(M)$ *is strongly ICC relative to G.*

Proof Let μ be a probability measure on $\mathrm{Homeo}(M)$. The set of μ-*stationary* probability measures on M

$$\mathrm{Prob}_\mu(M) = \left\{ \nu \in \mathrm{Prob}(M) \mid \nu = \mu * \nu = \int f_* \nu \, d\mu(f) \right\}$$

is a non-empty convex closed (hence compact) subset of $\mathrm{Prob}(M)$, with respect to the weak* topology. Suppose μ is invariant under conjugations by $g \in G$. Since

$$g_*(\mu * \nu) = \mu^g * (g_* \nu) = \mu * (g_* \nu)$$

it follows that $\mathrm{Prob}_\mu(M)$ is a G-invariant set. Minimality and strong proximality of the G-action implies that $\mathrm{Prob}_\mu(M) = \mathrm{Prob}(M)$. In particular, every Dirac measure ν_x is μ-stationary; hence $\mu\{f \mid f(x) = x\} = 1$. It follows that $\mu = \delta_e$. □

Proof of Proposition 2.3

(1) See [16, Proof of Theorem 2.3].
(2) This follows from Lemma 2.4, as by [40, Theorem 3.7 on p. 205] the action of G on $M = G/Q$ is minimal and strongly proximal. □

Next consider two measure equivalent (countable) groups Γ_1 and Γ_2, and a continuous action $\Gamma_2 \curvearrowright M$ on some compact metrizable space M. Let (Ω, m) be a (Γ_1, Γ_2)-coupling. Choosing a fundamental domain X for $\Gamma_2 \curvearrowright \Omega$ we obtain a probability measure-preserving action $\Gamma_1 \curvearrowright (X, \mu)$ and a measurable cocycle $\alpha : \Gamma_1 \times X \to \Gamma_2$, that can be used to define a Γ_1-action on $X \times M$ by

$$\gamma \,.\, (x, p) = \big(\gamma \,.\, x, \alpha(\gamma, x) \,.\, p\big) \quad (x \in X,\ p \in M,\ \gamma \in \Gamma_1).$$

The space $X \times M$ and the above action $\Gamma_1 \curvearrowright (X \times M)$ combine ergodic-theoretic base action $\Gamma_1 \curvearrowright (X, \mu)$ and topological dynamics in the fibers $\Gamma_2 \curvearrowright M$. In [19, §3 and 4] Furstenberg defines notions of minimality and (strong) proximality for such actions. We shall only need the former notion: the action $\Gamma_1 \curvearrowright X \times M$ is *minimal* if there are only trivial measurable α-equivariant families $\{U_x \subset M \mid x \in X\}$ of open subsets $U_x \subset M$. More specifically, whenever a measurable family of open subsets $U_x \subset M$ satisfies for all $\gamma \in \Gamma_1$ and μ-a.e. $x \in X$

$$U_{\gamma . x} = \alpha(\gamma, x)\, U_x \tag{2.1}$$

one either has $\mu\{x \in X \mid U_x = \emptyset\} = 1$ or $\mu\{x \in X \mid U_x = M\} = 1$.

We shall need the following lemma (generalizing [19, Proposition 4.4]):

Lemma 2.5 *Let Ω be an ergodic (Γ_1, Γ_2)-coupling, and $\Gamma_2 \curvearrowright M$ a minimal and strongly proximal action. Then the induced action $\Gamma_1 \curvearrowright X \times M$ is minimal in the above sense.*

Proof Let $i : \Gamma_2 \times X \cong \Omega$ be a measure space isomorphism as in (1.1). Given a family (U_x) as in (2.1), consider the measurable family $\{O_\omega\}$ of open subsets of M indexed by $\omega \in \Omega$, defined by $O_{i(\gamma,x)} = \gamma U_x$. Then for $\omega = i(\gamma, x)$ and $\gamma_i \in \Gamma_i$ we have

$$O_{(\gamma_1,\gamma_2)\omega} = \gamma_2 \gamma \alpha(\gamma_1, x)^{-1} U_{\gamma_1.x} = \gamma_2 \gamma U_x = \gamma_2 O_\omega.$$

Note that $\omega \to O_\omega$ is invariant under the action of Γ_1. Therefore it descends to a measurable family of open sets $\{V_y\}$ indexed by $y \in Y \cong \Omega/\Gamma_1$, and satisfying a.e. on Y

$$V_{\gamma_2.y} = \gamma_2 V_y \quad (\gamma_2 \in \Gamma_2).$$

The claim about $\{U_x \mid x \in X\}$ is clearly equivalent to the similar claim about $\{V_y \mid y \in Y\}$. By ergodicity, it suffices to reach a contradiction from the assumption that $V_y \neq \emptyset, M$ for ν-a.e. $y \in Y$, where ν is the probability measure associated to (Ω, m).

The assumption that (Ω, m) is $\Gamma_1 \times \Gamma_2$-ergodic is equivalent to ergodicity of the probability measure preserving action $\Gamma_2 \curvearrowright (Y, \nu)$. Since M has a countable base for its topology, while $\mu(\{y \mid V_y \neq \emptyset\}) = 1$, it follows that there exists a non-empty open set $W \subset M$ for which

$$A = \{y \in Y \mid W \subset V_y\}$$

has $\nu(A) > 0$. Since $M \setminus V_y \neq \emptyset$ for ν-a.e. $y \in Y$, there exists a measurable map $s : Y \to M$ with $s(y) \notin V_y$ for ν-a.e. $y \in Y$. Let $\sigma \in \mathrm{Prob}(M)$ denote the distribution of $s(y)$, i.e., $\sigma(E) = \nu\{y \in Y \mid s(y) \in E\}$. Then for any $\gamma_2 \in \Gamma_2$

$$\begin{aligned}
\sigma\left(\gamma_2^{-1} W\right) &= \nu\left\{y \in Y \mid s(y) \in \gamma_2^{-1} W\right\} \\
&\le \nu\left(Y \setminus \gamma_2^{-1} A\right) + \nu\left\{y \in \gamma_2^{-1} A \mid s(y) \in \gamma_2^{-1} V_{\gamma_2.y} = V_y\right\} \\
&= 1 - \nu\left(\gamma_2^{-1} A\right) = 1 - \nu(A).
\end{aligned}$$

So $\sigma(\gamma_2^{-1} W) \le 1 - \nu(A) < 1$ for all $\gamma_2 \in \Gamma_2$, contradicting the assumption that the action $\Gamma_2 \curvearrowright M$ is minimal and strongly proximal. $\square$

2.2 Tautness and the passage to self couplings

Theorem 2.1 is a direct consequence of the following, more technical statement that constructs a representation for arbitrary groups measure equivalent to a given group G, provided some specific (G, G)-coupling is taut.

Theorem 2.6 *Let G, H be unimodular lcsc groups, (Ω, m) a (G, H)-coupling, $\mathcal{G}$ a Polish group, and $\pi : G \to \mathcal{G}$ a continuous homomorphism. Assume that $\mathcal{G}$ is strongly ICC relative to $\pi(G)$ and the (G, G)-coupling $\Omega \times_H \Omega^*$ is taut relative to $\pi : G \to \mathcal{G}$.*

Then there exists a continuous homomorphism $\rho : H \to \mathcal{G}$ and a measurable map $\Psi : \Omega \to \mathcal{G}$ so that a.e.:

$$\Psi\big((g, h)\omega\big) = \pi(g)\Psi(\omega)\rho(h)^{-1} \quad (g \in G,\ h \in H)$$

and the unique tautening map $\Phi : \Omega \times_H \Omega^* \to \mathcal{G}$ is given by

$$\Phi\big([\omega_1, \omega_2]\big) = \Psi(\omega_1) \cdot \Psi(\omega_2)^{-1}.$$

The pair (Ψ, ρ) is unique up to conjugations (Ψ^x, ρ^x) by $x \in \mathcal{G}$, where

$$\Psi^x(\omega) = \Psi(\omega)x^{-1}, \qquad \rho^x(h) = x\rho(h)x^{-1}.$$

If, in addition, $\pi : G \to \mathcal{G}$ has compact kernel and closed image $\bar{G} = \pi(G)$, then the same applies to $\rho : H \to \mathcal{G}$, and there exists a Borel measure $\bar{m}$ on $\mathcal{G}$, which is invariant under

$$(g, h) :\ x \ \mapsto\ \pi(g)\, x\, \rho(h)$$

and descends to finite measures on $\pi(G)\backslash\mathcal{G}$ and $\mathcal{G}/\rho(H)$. In other words, $(\mathcal{G}, \bar{m})$ is a $(\pi(G), \rho(H))$-coupling which is a quotient of $(\mathrm{Ker}(\pi) \times \mathrm{Ker}(\rho))\backslash(\Omega, m)$.

Proof We shall first construct a homomorphism $\rho : H \to \mathcal{G}$ and the $G \times H$-equivariant map $\Psi : \Omega \to \mathcal{G}$. Consider the space $\Omega^3 = \Omega \times \Omega \times \Omega$ and the three maps $p_{1,2}$, $p_{2,3}$, $p_{1,3}$, where

$$p_{i,j} : \Omega^3 \longrightarrow \Omega^2 \longrightarrow \Omega \times_H \Omega^*$$

is the projection to the i-th and j-th factor followed by the natural projection. Consider the $G^3 \times H$-action on Ω^3:

$$(g_1, g_2, g_3, h) : (\omega_1, \omega_2, \omega_3) \mapsto \big((g_1, h)\omega_1, (g_2, h)\omega_2, (g_3, h)\omega_3\big).$$

For $i \in \{1,2,3\}$ denote by G_i the corresponding G-factor in G^3. For $i, j \in \{1,2,3\}$ with $i \neq j$ the group $G_i \times G_j < G_1 \times G_2 \times G_3$ acts on $\Omega \times_H \Omega^*$ and on $\mathcal{G}$ by

$$(g_i, g_j) : [\omega_1, \omega_2] \mapsto [g_i \omega_1, g_j \omega_2],$$
$$(g_i, g_j) : x \mapsto \pi(g_i)\, x\, \pi(g_j)^{-1} \quad (x \in \mathcal{G})$$

respectively. Let $\{i, j, k\} = \{1,2,3\}$. The map $p_{i,j} : \Omega^3 \to \Omega \times_H \Omega^*$ is $G_k \times H$-invariant and $G_i \times G_j$-equivariant. This is also true of the maps

$$F_{i,j} = \Phi \circ p_{i,j} : \Omega^3 \xrightarrow{p_{i,j}} \Omega \times_H \Omega^* \xrightarrow{\Phi} \mathcal{G},$$

where $\Phi : \Omega \times_H \Omega^* \to \mathcal{G}$ is the tautening map. For $\{i, j, k\} = \{1,2,3\}$, the three maps $F_{i,j}$, $F_{j,i}^{-1}$ and $F_{i,k} \cdot F_{k,j}$ are all $G_k \times H$-invariant, hence factor through the natural map

$$\Omega^3 \to \Sigma_k = (G_k \times H)\backslash \Omega^3.$$

By an obvious variation on the argument in Appendix A.1.3 one verifies that Σ_k is a (G_i, G_j)-coupling. The three maps $F_{i,j}$, $F_{j,i}^{-1}$ and $F_{i,k} \cdot F_{k,j}$ are $G_i \times G_j$-equivariant. Since $\mathcal{G}$ is strongly ICC relative to $\pi(G)$, there is at most one $G_i \times G_j$-equivariant measurable map $\Sigma_k \to \mathcal{G}$ according to Lemma A.8. Therefore, we get m^3-a.e. identities

$$F_{i,j} = F_{j,i}^{-1} = F_{i,k} \cdot F_{k,j}. \tag{2.2}$$

Denote by $\bar{\Phi} : \Omega^2 \to \mathcal{G}$ the composition $\Omega^2 \longrightarrow \Omega \times_H \Omega^* \xrightarrow{\Phi} \mathcal{G}$. By Fubini's theorem, (2.2) implies that for m-a.e. $\omega_2 \in \Omega$, for $m \times m$-a.e. (ω_1, ω_3)

$$\bar{\Phi}(\omega_1, \omega_3) = \bar{\Phi}(\omega_1, \omega_2) \cdot \bar{\Phi}(\omega_2, \omega_3) = \bar{\Phi}(\omega_1, \omega_2) \cdot \bar{\Phi}(\omega_3, \omega_2)^{-1}.$$

Fix such a generic $\omega_2 \in \Omega$ and define $\Psi : \Omega \to \mathcal{G}$ by $\Psi(\omega) = \bar{\Phi}(\omega, \omega_2)$. Then for a.e. $[\omega, \omega'] \in \Omega \times_H \Omega$

$$\Phi\left([\omega, \omega']\right) = \bar{\Phi}\left(\omega, \omega'\right) = \Psi(\omega) \cdot \Psi\left(\omega'\right)^{-1}. \tag{2.3}$$

We proceed to construct a representation $\rho : H \to \mathcal{G}$. Equation (2.3) implies that for every $h \in H$ and for a.e. $\omega, \omega' \in \Omega$:

$$\Psi(h\omega)\Psi\left(h\omega'\right)^{-1} = \bar{\Phi}\left(h\omega, h\omega'\right) = \bar{\Phi}\left(\omega, \omega'\right) = \Psi(\omega)\Psi\left(\omega'\right)^{-1},$$

and in particular, we get

$$\Psi(h\omega)^{-1}\Psi(\omega) = \Psi\left(h\omega'\right)^{-1}\Psi\left(\omega'\right).$$

Observe that the left hand side is independent of $\omega' \in \Omega$, while the right hand side is independent of $\omega \in \Omega$. Hence both are m-a.e. constant, and we denote by $\rho(h) \in \mathcal{G}$ the constant value. Being coboundaries the above expressions are cocycles; being independent of the space variable they give a homomorphism $\rho : H \to \mathcal{G}$. To see this explicitly, for $h, h' \in H$ we compute using m-a.e. $\omega \in \Omega$:

$$\begin{aligned}\rho(hh') &= \Psi(hh'\omega)^{-1}\Psi(\omega) \\ &= \Psi(hh'\omega)^{-1}\Psi(h'\omega)\Psi(h'\omega)^{-1}\Psi(\omega) \\ &= \rho(h)\rho(h').\end{aligned}$$

Since the homomorphism ρ is measurable, it is also continuous [63, Theorem B.3 on p. 198]. By definition of ρ we have for $h \in H$ and m-a.e. $\omega \in \Omega$:

$$\Psi(h\omega) = \Psi(\omega)\rho(h)^{-1}. \tag{2.4}$$

Since $\Psi(\omega) = \bar{\Phi}(\omega, \omega_2)$, it also follows that for $g \in G$ and m-a.e. $\omega \in \Omega$

$$\Psi(g\omega) = \pi(g)\Psi(\omega). \tag{2.5}$$

Consider the collection of all pairs (Ψ, ρ) satisfying (2.4) and (2.5). Clearly, $\mathcal{G}$ acts on this set by $x : (\Psi, \rho) \mapsto (\Psi^x, \rho^x) = (\Psi \cdot x, x^{-1}\rho x)$; and we claim that this action is transitive. Let (Ψ_i, ρ_i), $i = 1, 2$, be two such pairs in the above set. Then

$$\tilde{\Phi}_i(\omega, \omega') = \Psi_i(\omega)\Psi_i(\omega')^{-1} \quad (i = 1, 2)$$

are $G \times G$-equivariant measurable maps $\Omega \times \Omega \to \mathcal{G}$, which are invariant under H. Hence they descend to $G \times G$-equivariant maps $\Phi_i : \Omega \times_H \Omega^* \to \mathcal{G}$. The assumption that $\mathcal{G}$ is strongly ICC relative to $\pi(G)$, implies a.e. identities $\Phi_1 = \Phi_2$, $\tilde{\Phi}_1 = \tilde{\Phi}_2$. Hence for a.e. ω, ω'

$$\Psi_1(\omega)^{-1}\Psi_2(\omega) = \Psi_1(\omega')^{-1}\Psi_2(\omega').$$

Since the left hand side depends only on ω, while the right hand side only on ω', it follows that both sides are a.e. constant $x \in \mathcal{G}$. This gives $\Psi_1 = \Psi_2^x$. The a.e. identity

$$\Psi_1(\omega)\rho_1(h) = \Psi_1(h^{-1}\omega) = \Psi_2(h^{-1}\omega)x = \Psi_2(\omega)\rho_2(h)x = \Psi_1(\omega)x^{-1}\rho_2(h)x$$

implies $\rho_1 = \rho_2^x$. This completes the proof of the first part of the theorem.

Next, we assume that $\mathrm{Ker}(\pi)$ is compact and $\pi(G)$ is closed in $\mathcal{G}$, and will show that the kernel $K = \mathrm{Ker}(\rho)$ is compact, the image $\bar{H} = \rho(H)$ is closed in $\mathcal{G}$, and that $\mathcal{G}/\bar{H}$, $\pi(G)\backslash\mathcal{G}$ carry finite measures. These properties will be deduced from the assumption on π and the existence of the measurable map $\Psi:\Omega\to\mathcal{G}$ satisfying (2.4) and (2.5). We need the next lemma, which says that Ω has measure space isomorphisms as in (1.1) with special properties. □

Lemma 2.7 *Let $\rho:H\to\mathcal{G}$ and $\Psi:\Omega\to\mathcal{G}$ be as above. Then there exist measure space isomorphisms $i:G\times Y\cong\Omega$ and $j:H\times X\cong\Omega$ as in* (1.1) *that satisfy in addition*

$$\Psi\big(i(g,y)\big)=\pi(g),\qquad \Psi\big(j(h,x)\big)=\rho(h).$$

Proof We start from some measure space isomorphisms $i_0:G\times Y\cong\Omega$ and $j_0:H\times X\cong\Omega$ as in (1.1) and will replace them by

$$i(g,y)=i_0(gg_y,y),\qquad j(h,x)=j_0(hh_x,x)$$

for some appropriately chosen measurable maps $Y\to G$, $y\mapsto g_y$ and $X\to H$, $x\mapsto h_x$. The conditions (1.1) remain valid after any such alteration.

Let us construct $y\mapsto g_y$ with the required property; the map $x\mapsto h_x$ can be constructed in a similar manner. By (2.5) for $m_G\times\nu$-a.e. $(g_1,y)\in G\times Y$ the value

$$\pi(g)^{-1}\Psi\circ i_0(gg_1,y)$$

is m_G-a.e. independent of g; denote it by $f(g_1,y)\in\mathcal{G}$. Fix $g_1\in G$ for which

$$\Psi\circ i_0(gg_1,y)=\pi(g)f(g_1,y)$$

holds for m_G-a.e. $g\in G$ and ν-a.e. $y\in Y$. There exists a Borel cross section $\mathcal{G}\to G$ to $\pi:G\to\mathcal{G}$. Using such, we get a measurable choice for g_y so that

$$\pi(g_y)=f(g_1,y)^{-1}\pi(g_1).$$

Setting $i(g,y)=i_0(gg_y,y)$, we get $m_G\times\nu$-a.e. that $\Psi\circ i(g,y)=\pi(g)$. □

Lemma 2.8 *Given a neighborhood of the identity $V\subset H$ and a compact subset $Q\subset\mathcal{G}$, the set $\rho^{-1}(Q)$ can be covered by finitely many translates of V:*

$$\rho^{-1}(Q)\subset h_1V\cup\cdots\cup h_NV.$$

Proof Since $\pi:G\to\mathcal{G}$ is assumed to be continuous, having closed image and compact kernel, for any compact $Q\subset\mathcal{G}$ the preimage $\pi^{-1}(Q)\subset G$ is also compact. Let $W\subset H$ be an open neighborhood of identity so that $W\cdot W^{-1}\subset V$; we may assume W has compact closure in H. Then $\pi^{-1}(Q)\cdot W$ is precompact. Hence there is an open precompact set $U\subset G$ with $\pi^{-1}(Q)\cdot$

$W \subset U$. Consider subsets $A = j(W \times X)$, and $B = i(U \times Y)$ of Ω, where i and j are as in the previous lemma. Then

$$m(A) = m_H(W) \cdot \nu(Y) > 0, \qquad m(B) = m_G(U) \cdot \mu(X) < \infty.$$

Let $\{h_1, \dots, h_n\} \subset \rho^{-1}(Q)$ be such that $h_k W \cap h_l W = \emptyset$ for $k \neq l \in \{1, \dots, n\}$. Then the sets $h_k A = j(h_k W \times X)$ are also pairwise disjoint and have $m(h_k A) = m(A) > 0$ for $1 \le k \le n$. Since

$$\Psi(h_k A) = \rho(h_k W) = \rho(h_k)\rho(W) \subset Q \cdot \rho(W) \subset \rho(U),$$

it follows that $h_k A \subset B$ for every $1 \le k \le n$. Hence $n \le m(B)/m(A)$. Choosing a *maximal* such set $\{h_1, \dots, h_N\}$, we obtain the desired cover. □

Continuation of the proof of Theorem 2.6 Lemma 2.8 implies that the closed subgroup $K = \mathrm{Ker}(\rho)$ is compact. More generally, it implies that the continuous homomorphism $\rho : H \to \mathcal{G}$ is proper, that is, preimages of compact sets are compact. Therefore $\bar{H} = \rho(H)$ is closed in $\mathcal{G}$.

We push forward the measure m to a measure $\bar{m}$ on $\mathcal{G}$ via the map $\Psi : \Omega \to \mathcal{G}$. The measure $\bar{m}$ is invariant under the action $x \mapsto \pi(g)\, x\, \rho(h)$. Since $\bar{H} = \rho(H) \cong H/\ker(\rho)$ is closed in $\mathcal{G}$, the space $\mathcal{G}/\bar{H}$ is Hausdorff. As $\mathrm{Ker}(\rho)$ and $\mathrm{Ker}(\pi)$ are compact normal subgroups in H and G, respectively, the map $\Psi : \Omega \to \mathcal{G}$ factors through

$$(\Omega, m) \longrightarrow \big(\Omega', m'\big) = \big(\mathrm{Ker}(\pi) \times \mathrm{Ker}(\rho)\big)\backslash(\Omega, m) \xrightarrow{\Psi'} \mathcal{G}.$$

Let $\bar{G} = G/\mathrm{Ker}(\pi)$. Starting from measure isomorphisms as in Lemma 2.7, we obtain equivariant measure isomorphisms $(\Omega', m') \cong (\bar{H} \times X, m_{\bar{H}} \times \mu)$ and $(\Omega', m') \cong (\bar{G} \times Y, m_{\bar{G}} \times \nu)$. In particular, (Ω', m') is a $(\bar{G}, \bar{H})$-coupling. The measure $\bar{m}$ on $\mathcal{G}$ descends to the $\bar{G}$-invariant finite measure on $\mathcal{G}/\bar{H}$ obtained by pushing forward μ. Similarly, $\bar{m}$ descends to the $\bar{H}$-invariant finite measure on $\bar{G}\backslash\mathcal{G}$ obtained by pushing forward ν. This completes the proof of Theorem 2.6. □

Proof of Theorem 2.1 Theorem 2.1 immediately follows from Theorem 2.6. In case of L^p-conditions, one observes that if (Ω, m) is an L^p-integrable (G, H)-coupling, then $\Omega \times_H \Omega^*$ is an L^p-integrable (G, G)-coupling (Lemma A.2); so it is taut under the assumption that G is p-taut. □

2.3 Lattices in taut groups

Proposition 2.9 (Taut groups and lattices) *Let G be a unimodular lcsc group, $\mathcal{G}$ a Polish group, $\pi : G \to \mathcal{G}$ a continuous homomorphism. Assume that $\mathcal{G}$ is* *strongly ICC relative to* $\pi(G)$ *and let* $\Gamma < G$ *be a lattice* (*resp. a* p*-integrable lattice*).

Then G *is taut* (*resp.* p*-taut*) *relative to* $\pi : G \to \mathcal{G}$ *iff* Γ *is taut* (*resp.* p*-taut*) *relative to* $\pi|_\Gamma : \Gamma \to \mathcal{G}$. *In particular,* G *is taut iff any/all lattices in* G *are taut relative to the inclusion* $\Gamma < G$.

For the proof of this proposition we shall need the following.

Lemma 2.10 (Induction) *Let* G *be a unimodular lcsc group,* $\mathcal{G}$ *a Polish group,* $\pi : G \to \mathcal{G}$ *a continuous homomorphism, and* $\Gamma_1, \Gamma_2 < G$ *lattices. Let* (Ω, m) *be a* (Γ_1, Γ_2)*-coupling, and assume that the* (G, G)*-coupling* $\bar{\Omega} = G \times_{\Gamma_1} \Omega \times_{\Gamma_2} G$ *is taut relative to* $\pi : G \to \mathcal{G}$. *Then there exists a* $\Gamma_1 \times \Gamma_2$*-equivariant map* $\Omega \to \mathcal{G}$.

Proof It is convenient to have a concrete model for $\bar{\Omega}$. Choose Borel cross-sections σ_i from $X_i = G/\Gamma_i$ to G, and form the cocycles $c_i : G \times X_i \to \Gamma_i$ by

$$c_i(g, x) = \sigma_i(g \,.\, x)^{-1} g \sigma_i(x) \quad (i = 1, 2).$$

Then, suppressing the obvious measure from the notations, $\bar{\Omega}$ identifies with $X_1 \times X_2 \times \Omega$, while the $G \times G$-action is given by

$$(g_1, g_2) : (x_1, x_2, \omega) \mapsto \big(g_1 \,.\, x_1, g_2 \,.\, x_2, (\gamma_1, \gamma_2)\omega\big) \quad \text{where } \gamma_i = c_i(g_i, x_i).$$

By the assumption there exists a measurable map $\bar{\Phi} : \bar{\Omega} \to \mathcal{G}$ so that

$$\bar{\Phi}\big((g_1, g_2)(x_1, x_2, \omega)\big) = \pi(g_1) \cdot \bar{\Phi}(x_1, x_2, \omega) \cdot \pi(g_2)^{-1} \quad (g_1, g_2 \in G)$$

for a.e. $(x_1, x_2, \omega) \in \bar{\Omega}$. Fix a generic pair $(x_1, x_2) \in X_1 \times X_2$, denote $h_i = \sigma_i(x_i)$ and consider $g_i = \gamma_i^{h_i}$ $(= h_i \gamma_i h_i^{-1})$, where $\gamma_i \in \Gamma_i$ for $i \in \{1, 2\}$. Then $g_i \,.\, x_i = x_i$, $c_i(g_i, x_i) = \gamma_i$ and the map $\Phi' : \Omega \to \mathcal{G}$ defined by $\Phi'(\omega) = \bar{\Phi}(x_1, x_2, \omega)$ satisfies m-a.e.

$$\begin{aligned} \Phi'\big((\gamma_1, \gamma_2)\omega\big) &= \bar{\Phi}\big((g_1, g_2)(x_1, x_2, \omega)\big) = \pi(g_1) \cdot \Phi'(\omega) \cdot \pi(g_2)^{-1} \\ &= \pi\left(\gamma_1^{h_1}\right) \cdot \Phi'(\omega) \cdot \pi\left(\gamma_2^{h_2}\right)^{-1}. \end{aligned}$$

Thus $\Phi(\omega) = \pi(h_1)^{-1} \Phi'(\omega) \pi(h_2)$ is a $\Gamma_1 \times \Gamma_2$-equivariant measurable map $\Omega \to \mathcal{G}$, as required. □

Proof of Proposition 2.9 Assuming that G is taut relative to $\pi : G \to \mathcal{G}$ and $\Gamma < G$ is a lattice, we shall show that Γ is taut relative to $\pi|_\Gamma : \Gamma \to \mathcal{G}$.

Let (Ω, m) be a (Γ, Γ)-coupling. Then the (G, G)-coupling $\bar{\Omega} = G \times_\Gamma \Omega \times_\Gamma G$ is taut relative to $\mathcal{G}$, and by Lemma 2.10, Ω admits a $\Gamma \times \Gamma$-tautening map $\Phi : \Omega \to \mathcal{G}$. Since $\mathcal{G}$ is strongly ICC relative to $\pi(G) < \mathcal{G}$, the map $\Phi : \Omega \to \mathcal{G}$ is unique as a $\Gamma \times \Gamma$-equivariant map (Lemma A.8.(3)). This shows that Γ is taut relative to $\mathcal{G}$.

Observe, that if G is assumed to be only p-taut, while $\Gamma < G$ to be L^p-integrable, then the preceding argument for the existence of $\Gamma \times \Gamma$-tautening map for a L^p-integrable (Γ, Γ)-coupling Ω still applies. Indeed, the composed coupling $\bar{\Omega} = G \times_\Gamma \Omega \times_\Gamma G$ is then L^p-integrable and therefore admits a $G \times G$-tautening map $\bar{\Phi} : \bar{\Omega} \to \mathcal{G}$, leading to a $\Gamma \times \Gamma$-tautening map $\Phi : \Omega \to \mathcal{G}$. Finally, $\mathcal{G}$ is strongly ICC relative to $\pi(\Gamma)$ by Lemma A.6, and the uniqueness of the Γ tautening map follows from Lemma A.8(1).

Next assume that $\Gamma < G$ is a lattice and Γ is taut (resp. p-taut) relative to $\pi|_\Gamma : \Gamma \to \mathcal{G}$. Let (Ω, m) be a (G, G)-coupling (resp. a L^p-integrable one). Then (Ω, m) is also a (Γ, Γ)-coupling (resp. a L^p-integrable one). Since Γ is assumed to be taut (resp. p-taut) there is a $\Gamma \times \Gamma$-equivariant map $\Phi : \Omega \to \mathcal{G}$. As $\mathcal{G}$ is strongly ICC relative to $\pi(G)$ it follows from (4) in Lemma A.8 that $\Phi : \Omega \to \mathcal{G}$ is automatically $G \times G$-equivariant. The uniqueness of tautening maps follows from the strong ICC assumption by Lemma A.8(1). $\square$

Remark 2.11 The explicit assumption that $\mathcal{G}$ is strongly ICC relative to $\pi(G)$ is superfluous. If no integrability assumptions are imposed, the strong ICC follows from the tautness assumption by Lemma A.7. However, if one assumes merely p-tautness, the above lemma yields strong ICC property for a restricted class of measures; and the argument that this is sufficient becomes unjustifiably technical in this case.

3 Some boundary theory

Throughout this section we refer to the following

Setup 3.1 We fix $n \geq 2$ and introduce the following setting:

- $G = \mathrm{Isom}(\mathbf{H}^n)$.
- $\Gamma < G^0$ be a torsion-free uniform lattice in the connected component of the identity.
- $M = \Gamma \backslash \mathbf{H}^n$ is a compact hyperbolic manifold with $\Gamma \simeq \pi_1(M)$.
- The boundary $\partial \mathbf{H}^n$ is denoted by B; we identify it with the $(n-1)$-sphere and equip it with the Lebesgue measure ν.[7]
- (Ω, m) is an ergodic (Γ, Γ)-coupling.

[7] Any probability measure in the Lebesgue measure class would do it.

- For better readability we denote the left copy of Γ by Γ_l and the right copy by Γ_r. We fix a Γ_l-fundamental domain $X \subset \Omega$, and denote by μ the restriction of m to X.
- Identifying X with $\Gamma_l \backslash \Omega$, we get an ergodic action of Γ_r on (X, μ). We denote by $\alpha : \Gamma_r \times X \to \Gamma_l < G$ the associated cocycle.

Boundary theory, in the sense of Furstenberg [18] (see [7, Corollary 3.2], or [42, Proposition 3.3] for a detailed argument applying to our situation), yields the existence of an essentially unique measurable map, called *boundary map* or *Furstenberg map*,

$$\phi : X \times B \to B \quad \text{satisfying} \quad \phi(\gamma x, \gamma b) = \alpha(\gamma, x)\phi(x, b) \tag{3.1}$$

for every $\gamma \in \Gamma$ and a.e. $(x, b) \in X \times B$.

Proposition 3.2 *Let $\mathcal{G} < \mathrm{Homeo}(B)$ be a closed subgroup containing Γ and denote the inclusion $\Gamma < \mathcal{G}$ by π. If for a.e $x \in X$ the function $\phi(x, \cdot) : B \to B$ coincides a.e with an element of $\mathcal{G}$ then Ω is taut with respect to the inclusion $\pi : \Gamma \to \mathcal{G}$.*

Proof Consider the set $\mathrm{F}(B, B)$ of measurable functions $B \to B$, where two functions are identified if they agree on ν-conull set. We endow $\mathrm{F}(B, B)$ with the topology of convergence in measure. The Borel σ-algebra of this topology turns $\mathrm{F}(B, B)$ into a standard Borel space [13, Sect. 2A]. By [32, Corollary 15.2 on p. 89] the measurable injective map $j : \mathcal{G} \to \mathrm{F}(B, B)$ is a Borel isomorphism of $\mathcal{G}$ onto its measurable image.

The map ϕ gives rise to a measurable map $f : X \to \mathrm{F}(B, B)$ defined for almost every $x \in X$ by $f(x) = \phi(x, \cdot)$ [13, Corollary 2.9], which can be regarded as a measurable map $f : X \to \mathcal{G}$. Equation (3.1) gives

$$\pi \circ \alpha(\gamma, x) = f(\gamma \,.\, x)\pi(\gamma) f(x)^{-1}, \tag{3.2}$$

thus by Lemma A.4, there is a tautening map $\Omega \to \mathcal{G}$.

Note that by Proposition 2.3(2), $\mathrm{Homeo}(B)$ is strongly ICC relative to G. It follows by Lemma A.6 that it is also strongly ICC relative to Γ. Therefore $\mathcal{G}$ is strongly ICC relative to Γ, and by Lemma A.8(1) the tautening is unique. □

3.1 Preserving maximal simplices of the boundary

Recall that a geodesic simplex in $\bar{\mathbf{H}}^n = \mathbf{H}^n \cup \partial \mathbf{H}^n$ is called *regular* if any permutation of its vertices can be realized by an element in $\mathrm{Isom}(\mathbf{H}^n)$. The set of ordered $(n+1)$-tuples on the boundary B that form the vertex set of an ideal regular simplex is denoted by Σ^{reg}. The set Σ^{reg} is a disjoint union $\Sigma^{\mathrm{reg}} = \Sigma_+^{\mathrm{reg}} \cup \Sigma_-^{\mathrm{reg}}$ of two subsets that correspond to the positively and negatively oriented ideal regular n-simplices, respectively. We denote by $v_{\max}$ the maximum possible volume of an ideal simplex.

Lemma 3.3 (Key facts from Thurston's proof of Mostow rigidity)

(1) *The diagonal G-action on Σ^{reg} is simply transitive. The diagonal G^0-action on Σ_-^{reg} and Σ_+^{reg} are simply transitive, respectively.*
(2) *An ideal simplex has non-oriented volume $v_{\max}$ if and only if it is regular.*
(3) *Let $n \geq 3$. Let σ, σ' be two regular ideal simplices having a common face of codimension one. Let ρ be the reflection along the hyperspace spanned by this face. Then $\sigma = \rho(\sigma')$.*

Proof (1) See the proof of [53, Theorem 11.6.4 on p. 568].

(2) The statement is trivial for $n = 2$, as all non-degenerate ideal triangles in $\bar{\mathbf{H}}^2$ are regular, and G acts simply transitively on them. The case $n = 3$ is due to Milnor, and Haagerup and Munkholm [27] proved the general case $n \geq 3$.

(3) This is a key feature distinguishing the $n \geq 3$ case from the $n = 2$ case where Mostow rigidity fails. See [53, Lemma 13 on p. 567]. □

We shall need the following lemma, which is due to Thurston [59, p. 133/134] in dimension $n = 3$. Recall that $B = \partial \mathbf{H}^n$ is equipped with the Lebesgue measure class. We consider the natural measure $m_{\Sigma_+^{\mathrm{reg}}}$ on Σ_+^{reg} corresponding to the Haar measure on G^0 under the simply transitive action of G^0 on Σ_+^{reg}.

Lemma 3.4 *Let $n \geq 3$ and $\phi : B \to B$ be a Borel map such that $\phi^{n+1} = \phi \times \cdots \times \phi$ maps a.e. point in Σ_+^{reg} into Σ_+^{reg}. Then there exists a unique $g_0 \in G^0 = \mathrm{Isom}_+(\mathbf{H}^n)$ with $\phi(b) = g_0 b$ for a.e. $b \in B$.*

Proof Fix a regular ideal simplex $\sigma = (b_0, \ldots, b_n) \in \Sigma_+^{\mathrm{reg}}$, and identify G^0 with Σ_+^{reg} via $g \mapsto g\sigma$. Then there is a Borel map $f : G^0 \to G^0$ such that for a.e. $g \in G^0$

$$\big(\phi(gb_0), \ldots, \phi(gb_n)\big) = \big(f(g)b_0, \ldots, f(g)b_n\big). \tag{3.3}$$

Interchanging b_0, b_1 identifies Σ_+^{reg} with Σ_-^{reg}, and allows to extend f to a measurable map $G \to G$ satisfying (3.3) for a.e. $g \in G$. Let $\rho_0, \ldots, \rho_n \in G$ denote the reflections in the codimension one faces of σ. Then Lemma 3.3 (3) implies that

$$f(g\rho) = f(g)\rho \quad \text{for a.e. } g \in G$$

for ρ in $\{\rho_0, \ldots, \rho_n\}$. It follows that the same applies to each ρ in the countable group $R < G$ generated by $\rho_0, \ldots, \rho_n$. We claim that there exists $g_0 \in G$ so that $f(g) = g_0 g$ for a.e. $g \in G$, which implies that $\phi(b) = g_0 b$ also holds a.e. on B.

The case $n = 3$ is due to Thurston [59, p. 133/134]. So hereafter we focus on $n > 3$, and will show that in this case the group R is dense in G

(for $n = 2, 3$ it forms a lattice in G). Consequently the R-action on G is ergodic with respect to the Haar measure. Since $g \mapsto f(g)g^{-1}$ is a measurable R-invariant map on G, it follows that it is a.e. a constant $g_0 \in G^0$, i.e., $f(g) = g_0 g$ a.e. proving the lemma.

It remains to show that for $n > 3$, R is dense in G. Not being able to find a convenient reference for this fact, we include the proof here.

For $i \in \{0, \dots, n\}$ denote by $P_i < G$ the stabilizer of $b_i \in \partial\mathbf{H}^n$, and let $U_i < P_i$ denote its unipotent radical. We shall show that U_i is contained in the closure $\overline{R \cap P_i} < P_i$ (in fact, $\overline{R \cap P_i} = P_i$ but we shall not need this). Since unipotent radicals of any two opposite parabolics, say U_0 and U_1, generate the whole connected simple Lie group G^0, this would show $G^0 < \bar{R} < G$. Since R is not contained in G^0, it follows that $\bar{R} = G$ as claimed.

Let $f_i : \partial\mathbf{H}^n \to \mathbf{E}^{n-1} \cup \{\infty\}$ denote the stereographic projection taking b_i to the point at infinity. Then $f_i P_i f_i^{-1}$ is the group of similarities $\mathrm{Isom}(\mathbf{E}^{n-1}) \rtimes \mathbb{R}_+^\times$ of the Euclidean space $\mathbf{E}^{n-1}$. We claim that the subgroup of translations $\mathbb{R}^{n-1} \cong U_i < P_i$ is contained in the closure of $R_i = R \cap P_i$. To simplify notations we assume $i = 0$. The set of all n-tuples $(z_1, \dots, z_n)$ in $\mathbf{E}^{n-1}$ for which $(b_0, f_0^{-1}(z_1), \dots, \dots, f_0^{-1}(z_n))$ is a regular ideal simplex in $\bar{\mathbf{H}}^n$ is precisely the set of all regular Euclidean simplices in $\mathbf{E}^{n-1}$ [53, Lemma 3 on p. 519]. So conjugation by f_0 maps the group $R_0 = R \cap P_0$ to the subgroup of $\mathrm{Isom}(\mathbf{E}^{n-1})$ generated by the reflections in the faces of the Euclidean simplex $\Delta = (z_1, \dots, z_n)$, where $z_i = f_0(b_i)$. For $1 \le j < k \le n$ denote by r_{jk} the composition of the reflections in the jth and kth faces of Δ; it is a rotation leaving fixed the co-dimension two affine hyperplane L_{jk} containing $\{z_i \mid i \neq j, k\}$. The angle of this rotation is $2\theta_n$, where θ_n is the dihedral angle of the simplex Δ. One can easily check that $\cos(\theta_n) = -1/(n-1)$, using the fact that the unit normals v_i to the faces of Δ satisfy $v_1 + \dots + v_n = 0$ and $\langle v_i, v_j \rangle = \cos(\theta_n)$ for all $1 \le i < j \le n$. Thus $w = \exp(\theta_n\sqrt{-1})$ satisfies $w + 1/w = -2/(n-1)$. Equivalently, w is a root of

$$p_n(z) = (n-1)z^2 + 2z + (n-1).$$

This condition on w implies that θ_n is not a rational multiple of π. Indeed, otherwise, w is a root of unit, and therefore is a root of some cyclotomic polynomial

$$c_m(z) = \prod_{k \in \{1..m-1 \mid \gcd(k,m)=1\}} \left(z - e^{\frac{2\pi k i}{m}}\right)$$

whose degree is Euler's totient function $\deg(c_m) = \phi(m)$. The cyclotomic polynomials are irreducible over $\mathbb{Q}$. So $p_n(z)$ and $c_m(z)$ share a root only if they are proportional, which in particular implies $\phi(m) = 2$. The latter happens only for $m = 3$, $m = 4$ and $m = 6$; corresponding to $c_3(z) = z^2 + z + 1$, $c_4(z) = z^2 + 1$, and $c_6(z) = z^2 - z + 1$. The only proportionality between these polynomials is $p_3(z) = 2c_2(z)$; and it is ruled out by the assumption $n > 3$.

Thus the image of R_0 in $\mathrm{Isom}(\mathbf{E}^{n-1})$ is not discrete. Let

$$\pi : \overline{R \cap P_0} \to \mathrm{Isom}\big(\mathbf{E}^{n-1}\big) \to \mathrm{O}\big(\mathbb{R}^{n-1}\big)$$

denote the homomorphism defined by taking the linear part. Then $\pi(r_{jk})$ is an irrational rotation in $\mathrm{O}(\mathbb{R}^{n-1})$ leaving invariant the linear subspace parallel to L_{jk}. The closure of the subgroup generated by this rotation is a subgroup $C_{jk} < \mathrm{O}(\mathbb{R}^{n-1})$, isomorphic to $\mathrm{SO}(2)$. The group $K < \mathrm{O}(\mathbb{R}^{n-1})$ generated by all such C_{jk} acts irreducibly on $\mathbb{R}^{n-1}$, because there is no subspace orthogonal to all L_{jk}. Since $\overline{R \cap P_0}$ is not compact (otherwise there would be a point in $\mathbf{E}^{n-1}$ fixed by all reflections in faces of Δ), the epimorphism $\pi : \overline{R \cap P_0} \to K$ has a non-trivial kernel $V < \mathbb{R}^{n-1}$, which is invariant under K. As the latter group acts irreducibly, $V = \mathbb{R}^{n-1}$ or, equivalently, $U_0 < \overline{R \cap P_0}$. This completes the proof of the lemma. $\square$

3.2 Boundary simplices in general position

Definition 3.5 For $0 \le k \le n$, a $(k+1)$-tuple of points in B, $(z_0, \dots, z_k) \in B^{k+1}$ is said to be in *general position* if the following equivalent conditions hold:

(1) The k-volume of the ideal k-simplex with vertices $\{z_0, \dots, z_k\}$ is positive,
(2) The points $\{z_0, \dots, z_k\}$ lie on the boundary of a unique isometrically embedded copy of $\mathbf{H}^k$ in $\mathbf{H}^n$,
(3) The points $\{z_0, \dots, z_k\}$ do not lie on the boundary of some isometrically embedded copy of $\mathbf{H}^{k-1}$ in $\mathbf{H}^n$.

The set of $(k+1)$-tuples in a general position in B^{k+1} is denoted $B^{(k+1)}$.

We shall use the term $(k-1)$-*sphere* to denote the boundary of an isometrically embedded copy of $\mathbf{H}^k$ in $\mathbf{H}^n$; with 0-spheres meaning pairs of distinct points.

Lemma 3.6 *Consider the boundary map $\phi(x, \cdot) = \phi_x$ from (3.1). For $\mu \times \nu^{n+1}$-a.e. point $(x, b_0, \dots, b_n)$, the $(n+1)$-tuple $(\phi_x(b_0), \dots, \phi_x(b_n))$ is in general position.*

Remark 3.7 In fact we prove a more general statement. The only important properties of our setting are the fact that α is Zariski dense (in particular, is not measurably cohomologous to a cocycle taking values in a stabilizer of $\mathbf{H}^k \subset \mathbf{H}^n$ with $k < n$) and that the diagonal measure class-preserving action

$$\Gamma \curvearrowright (X \times B \times B, \mu \times \nu \times \nu)$$

is ergodic, which, in our setting, follows from the Howe-Moore theorem.

Proof of Lemma 3.6 Denote by $\eta_x \in \mathrm{Prob}(B)$ the push-forward of ν under the map $\phi_x : B \to B$. For $k \in \{2, \dots, n\}$ and $x \in X$ let

$$E_k = \left\{x \in X \mid \eta_x^{k+1}\left(B^{k+1} \setminus B^{(k+1)}\right) > 0\right\}.$$

This is a measurable subset of X, which is Γ-invariant since $\eta_{\gamma.x} = \alpha(\gamma, x)_* \eta_x$ while $B^{(k+1)}$ is a Borel, in fact open, G-invariant subset of B^{k+1}. Ergodicity of $\Gamma \curvearrowright (X, \mu)$ implies that $\mu(E_k) = 0$ or $\mu(E_k) = 1$. The sets E_k are also nested: $E_{k-1} \subset E_k$, because any subset of a $(k+1)$-tuple in general position, is itself in general position.

We claim that $\mu(E_n) = 0$. By contradiction, let k be the smallest integer in $\{2, \dots, n\}$ with $\mu(E_k) > 0$. Then, in fact, $\mu(E_k) = 1$ by the ergodicity argument above. Since $\mu(E_{k-1}) = 0$ for μ-a.e. $x \in X$ and ν^k-a.e. $(b_1, \dots, b_k) \in B^k$ the points $(\phi_x(b_1), \dots, \phi_x(b_k))$ are in general position, and therefore define a unique $(k-2)$-sphere

$$S_x(b_1, \dots, b_k) \subset B.$$

On the other hand, $\mu(E_k) = 1$ means that for μ-a.e. $x \in X$

$$\nu^{k+1}\left\{(b_0, \dots, b_k) \mid \phi_x(b_0) \in S_x(b_1, \dots, b_k)\right\} > 0.$$

By Fubini's theorem, there is a measurable family of measurable subsets $A_x \subset B^k$ with $\nu^k(A_x) > 0$, so that for $(b_1, \dots, b_k) \in A_x$

$$\eta_x\left(S_x(b_1, \dots, b_k)\right) > 0.$$

Denote by $\mathcal{S}$ the space of all $(k-2)$-spheres $S \subset B$, and let

$$\mathcal{S}_x = \left\{S \in \mathcal{S} \mid \eta_x(S) > 0\right\}.$$

Using $\eta_{g.x} = \alpha(g, x)_* \eta_x$ we deduce that

$$\mathcal{S}_{\gamma.x} = \alpha(\gamma, x)\, \mathcal{S}_x.$$

Hence the set $\{x \in X \times B \mid \mathcal{S}_x \neq \emptyset\}$ is measurable and Γ-invariant. We just argued above that this set has positive measure, hence by ergodicity of $\Gamma \curvearrowright (X, \mu)$, it has full measure.

Our main claim is that $\mathcal{S}_x$ consists of a single $(k-2)$-sphere:

$$\mathcal{S}_x = \{S_x\}. \tag{3.4}$$

This claim leads to a desired contradiction as follows: equivariance of $\mathcal{S}_x$ becomes the μ-a.e. identity $\alpha(\gamma, x)S_x = S_{\gamma.x}$. Fix a $(k-2)$-sphere S_0 and a measurable map $f : X \to G$ with $S_x = f(x)S_0$. Then the f-conjugate of α

$$\alpha^f(\gamma, x) = f(\gamma\,.\,x)^{-1}\alpha(\gamma, x)f(x)$$

takes values in the stabilizer of S_0 in G, which is a proper algebraic subgroup $\mathrm{Isom}(\mathbf{H}^k) < \mathrm{Isom}(\mathbf{H}^n) = G$. But this is impossible for an ME-cocycle.

It remains to show (3.4). Consider any two measurable families $S_x, S'_x \in \mathcal{S}_x$ indexed by $x \in X$, and let

$$F = \left\{x \in X \mid S_x \neq S'_x \quad \text{and} \quad \eta_x\left(S_x \cap S'_x\right) > 0\right\}.$$

We claim that $\mu(F) = 0$. Indeed, for $x \in F$ the intersection $R_x = S_x \cap S'_x$ is a sphere of dimension $\leq (k-3)$, and therefore k-tuples of points in R_x are not in general position. This implies

$$\eta_x^k\left(B^k \setminus B^{(k)}\right) \geq \eta_x^k\left(R_x^k\right) = \left(\eta_x(R_x)\right)^k > 0$$

meaning that $x \in E_{k-1}$. As $\mu(E_{k-1}) = 0$, it follows that $\mu(F) = 0$.

We now claim that a.e. $\mathcal{S}_x$ has at most countably many elements (spheres). It suffices to show that for every $\epsilon > 0$ for μ-a.e. x the set

$$\mathcal{S}_x^{>\epsilon} = \left\{S \in \mathcal{S}_x \mid \eta_x(S) > \epsilon\right\}.$$

is finite. We will show that its cardinality is bounded by $1/\epsilon$. Otherwise it is possible to find a positive measure set $Y \subset X$ and $m > 1/\epsilon$ maps $S_{i,y} \in \mathcal{S}_y^{>\epsilon}$, $y \in Y$, $1 \leq i \leq m$, so that for $i \neq j$ one has $S_{i,y} \neq S_{j,y}$. But this is impossible, because for a.e. $y \in Y$ one has $\eta_y(S_{i,y} \cap S_{j,y}) = 0$ for every pair $i \neq j$, and therefore

$$1 \geq \eta_y\left(\bigcup_{i=1}^m S_{i,y}\right) = \sum_{i=1}^m \eta_y(S_{i,y}) > m\epsilon > 1.$$

Therefore, a.e. $\mathcal{S}_x$ is countable, and one can enumerate these collections by a fixed sequence $\mathcal{S}_x = \{S_{i,x}\}_{i=1}^\infty$ of $(k-2)$-spheres with $S_{i,x}$ varying measurably in $x \in X$. For $x \in X$ let

$$P_{i,x} = \left\{\left(b, b'\right) \in B \times B \mid \phi_x(b), \phi_x\left(b'\right) \in S_{i,x}\right\}.$$

We have $\nu^2(P_{i,x}) = \eta_x(S_{i,x})^2 > 0$. The union

$$P_x = \bigcup_{i=1}^\infty P_{i,x} = \left\{\left(b, b'\right) \mid \exists_{S \in \mathcal{S}_x} \phi_x(b), \phi_x\left(b'\right) \in S\right\}$$

satisfies $\alpha(\gamma, x)P_x = P_{\gamma.x}$. Therefore $\{(x, b, b') \mid (b, b') \in P_x\}$ is a measurable, Γ-invariant set of positive $\mu \times \nu \times \nu$-measure. Hence from the ergodicity of the measure-class preserving action $\Gamma \curvearrowright (X \times B \times B, \mu \times \nu \times \nu)$, this set has full measure. In particular, for μ-a.e. $x \in X$, one has

$$\sum_i \eta_x(S_{i,x})^2 = \sum_i \nu^2(P_{i,x}) = \nu^2(P_x) = 1$$

while

$$\sum_i \eta_x(S_{i,x}) = \eta_x\left(\bigcup_{S \in \mathcal{S}_x} S\right) \le 1.$$

This is possible, only if exactly one $S_{i,x}$ has full η_x-measure, i.e., if $\mathcal{S}_x$ consists of a single sphere $\mathcal{S}_x = \{S_x\}$, as claimed. This completes the proof of the lemma. $\square$

3.3 A Lebesgue differentiation lemma

Lemma 3.8 *Fix points $o \in \mathbf{H}^n$ and $b_0 \in \partial\mathbf{H}^n$. Denote by $d = d_o$ the visual metric on $\partial\mathbf{H}^n$ associated with o. Let $\{z^{(k)}\}_{k=1}^{\infty}$ be a sequence in $\mathbf{H}^n$ converging radially to b_0. Let $\phi\colon B \to B$ be a measurable map. For every $\epsilon > 0$ and for a.e. $g \in G$ we have*

$$\lim_{k\to\infty} \nu_{z^{(k)}}\big\{b \in B \mid d\big(\phi(gb), \phi(gb_0)\big) > \epsilon\big\} = 0.$$

Proof For the domain of ϕ, it is convenient to represent $\partial\mathbf{H}^n$ as the boundary $\hat{\mathbb{R}}^n = \{(x_1, \dots, x_n, 0) \mid x_i \in \mathbb{R}\} \cup \{\infty\}$ of the upper half space model

$$\mathbf{H}^n = \big\{(x_1, \dots, x_{n+1}) \mid x_{n+1} > 0\big\} \subset \mathbb{R}^{n+1}.$$

We may assume that $o = (0, \dots, 0, 1)$ and $b_0 = 0 \in \mathbb{R}^n \subset \hat{\mathbb{R}}^n$. The points $z^{(k)}$ lie on the line l between o and b_0. The subgroup of G consisting of reflections along hyperplanes containing l and perpendicular to $\{x_{n+1} = 0\}$ leaves the measures $\nu_{z^{(k)}}$ invariant, i.e. each $\nu_{z^{(k)}}$ is $\mathrm{O}(n)$-invariant. Since the probability measure $\nu_{z^{(k)}}$ is in the Lebesgue measure class, the Radon-Nikodym theorem, combined with the $\mathrm{O}(n)$-invariance, yields the existence of a measurable functions $h_k : [0, \infty) \to [0, \infty)$ such that for any bounded measurable function l

$$\int l \, d\nu_{z^{(k)}} = \int_0^\infty \left(\frac{1}{\mathrm{vol}(B(0, r))} \int_{B(0,r)} l(y)\, dy\right) h_k(r)\, dr$$

holds[8] and

$$\int_0^\infty h_k(r)\,dr = 1.$$

Since the $\nu_{z(k)}$ weakly converge to the Dirac measure at $0 \in \mathbb{R}^n$, we have for every $r_0 > 0$

$$\lim_{k\to\infty} \int_{r_0}^\infty h_k(r)\,dr = 0. \tag{3.5}$$

For the target of ϕ, we represent $B = \partial \mathbf{H}^n$ as the boundary $S^{n-1} \subset \mathbb{R}^n$ of the Poincare disk model. The visual metric is then just the standard metric of the unit sphere. Considering coordinates in the target, it suffices to prove that every measurable function $f : \hat{\mathbb{R}}^n \to [-1, 1]$ satisfies

$$\lim_{k\to\infty} \int_{\hat{\mathbb{R}}^n} \left| f(gx) - f(g0) \right| d\nu_{z(k)}(x) = 0.$$

for a.e. $g \in G$. By the Lebesgue differentiation theorem the set L_f of points $x \in \mathbb{R}^n$ with the property

$$\lim_{r\to 0} \frac{1}{\mathrm{vol}(B(0,r))} \int_{B(x,r)} \left| f(y) - f(x) \right| dy = 0 \tag{3.6}$$

is conull in $\mathbb{R}^n$. The subset of elements $g \in G$ such that $g0 \in L_f$ and $g0 \neq \infty$ is conull with respect to the Haar measure. From now on we fix such an element $g \in G$. By compactness there is $L > 0$ such that the diffeomorphism of $\hat{\mathbb{R}}^n$ given by g has Lipschitz constant at most L and its Jacobian satisfies $|\mathrm{Jac}(g)| > 1/L$ everywhere on $\mathbb{R}^n \subset \hat{\mathbb{R}}^n$. Let $\epsilon > 0$. According to (3.6) choose $r_0 > 0$ such that for all $r < r_0$

$$\frac{L}{\mathrm{vol}(B(0,r))} \int_{B(g0,Lr)} \left| f(y) - f(g0) \right| dy < \frac{\epsilon}{2}. \tag{3.7}$$

According to (3.5) choose $k_0 \in \mathbb{N}$ such that

$$\int_{r_0}^\infty h_k(r)\,dr < \frac{\epsilon}{4}$$

[8] $\mathrm{vol}(B(0,r))$ is here the Lebesgue measure of the Euclidean ball of radius r around $0 \in \mathbb{R}^n$.

for every $k > k_0$. So we obtain that

$$\begin{aligned}
&\int_{\hat{\mathbb{R}}^n} \left| f(gx) - f(g0) \right| d\nu_{z^{(k)}} \\
&\quad < \int_0^{r_0} \frac{1}{\mathrm{vol}(B(0,r))} \int_{B(0,r)} \left| f(gx) - f(g0) \right| dx\, h_k(r)\, dr + \frac{\epsilon}{2} \\
&\quad \le \int_0^{r_0} \frac{L}{\mathrm{vol}(B(0,r))} \int_{gB(0,r)} \left| f(y) - f(g0) \right| dy\, h_k(r)\, dr + \frac{\epsilon}{2}
\end{aligned}$$

for $k > k_0$. Because of $gB(0,r) \subset B(g0, Lr)$ and (3.7) we obtain that for $k > k_0$

$$\int_{\hat{\mathbb{R}}^n} \left| f(gx) - f(g0) \right| d\nu_{z^{(k)}} < \epsilon. \qquad \square$$

4 Cohomological tools

The aim of this section is to prove that the boundary map $\phi_x = \phi(x, \cdot) : B \to B$, which is associated to a (Γ, Γ)-coupling with $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ and introduced in Setup 3.1, satisfies the assumption of Lemma 3.4. This will be achieved in Corollary 4.13. The conclusion of Lemma 3.4 is a crucial ingredient in the proof of Theorem B. To prove Corollary 4.13 we have to develop and rely on a fair amount of cohomological machinery. For the reader's convenience a brief introduction to the subject of bounded cohomology is given in Appendix B.

4.1 The cohomological induction map

The *cohomological induction map* associated to an arbitrary ME-coupling was introduced by Monod and Shalom [43].

Proposition 4.1 (Monod-Shalom) *Let (Ω, m) be a (Γ, Λ)-coupling. Let $Y \subset \Omega$ be a measurable fundamental domain for the Γ-action. Let $\chi : \Omega \to \Gamma$ be the measurable Γ-equivariant map uniquely defined by $\chi(\omega)^{-1}\omega \in Y$ for $\omega \in \Omega$. The maps*

$$\begin{aligned}
&\mathrm{C}_\mathrm{b}^\bullet(\chi) : \mathrm{C}_\mathrm{b}^\bullet\big(\Gamma, \mathrm{L}^\infty(\Omega)\big) \to \mathrm{C}_\mathrm{b}^\bullet\big(\Lambda, \mathrm{L}^\infty(\Omega)\big) \\
&\mathrm{C}_\mathrm{b}^k(\chi)(f)(\lambda_0, \dots, \lambda_k)(y) = f\big(\chi\big(\lambda_0^{-1} y\big)\big), \dots, \chi\big(\lambda_k^{-1} y\big)(y)
\end{aligned}$$

defines a $\Gamma \times \Lambda$-equivariant chain morphism with regard to the following actions: The $\Gamma \times \Lambda$-action on $\mathrm{C}_\mathrm{b}^\bullet(\Gamma, \mathrm{L}^\infty(\Omega)) \cong \mathrm{L}^\infty(\Gamma^{\bullet+1} \times \Omega)$ is induced *by Γ acting diagonally on $\Gamma^{\bullet+1}\times\Omega$ and by Λ acting only on Ω. The $\Gamma\times\Lambda$-action on $\mathrm{C}^\bullet_\mathrm{b}(\Lambda,\mathrm{L}^\infty(\Omega))\cong\mathrm{L}^\infty(\Lambda^{\bullet+1}\times\Omega)$ is induced by Λ acting diagonally on $\Lambda^{\bullet+1}\times\Omega$ and by Γ acting only on Ω.*

The chain map $\mathrm{C}^\bullet_\mathrm{b}(\chi)$ induces, after taking $\Gamma\times\Lambda$-invariants and identifying $\mathrm{L}^\infty(\Gamma\backslash\Omega)$ with $\mathrm{L}^\infty(\Omega)^\Gamma$ and similarly for Λ, an isometric isomorphism

$$\mathrm{H}^\bullet_\mathrm{b}(\chi)\colon \mathrm{H}^\bullet_\mathrm{b}\big(\Gamma,\mathrm{L}^\infty(\Lambda\backslash\Omega)\big)\xrightarrow{\cong}\mathrm{H}^\bullet_\mathrm{b}\big(\Lambda,\mathrm{L}^\infty(\Gamma\backslash\Omega)\big).$$

in cohomology. This map does not depend on the choice of Y, or equivalently χ, and will be denoted by $\mathrm{H}^\bullet_\mathrm{b}(\Omega)$. We call $\mathrm{H}^\bullet_\mathrm{b}(\Omega)$ the cohomological induction map associated to Ω.

Proof Apart from the fact that the isomorphism is isometric, this is exactly Proposition 4.6 in [43] (with $S=\Omega$ and $E=\mathbb{R}$). The proof therein relies on [41, Theorem 7.5.3 in §7], which also yields the isometry statement. $\square$

Proposition 4.2 *Retain the setting of the previous proposition. Let $\alpha\colon\Lambda\times Y\to\Gamma$ be the corresponding ME-cocycle. Let B_Γ and B_Λ be standard Borel spaces endowed with probability Borel measures and measure-class preserving Borel actions of Γ and Λ, respectively. Let $\phi\colon B_\Lambda\times\Gamma\backslash\Omega\to B_\Gamma$ be a measurable α-equivariant map (upon identifying Y with $\Gamma\backslash\Omega$). Then the chain morphism (see Appendix B.1 for notation)*

$$\begin{aligned}&\mathrm{C}^\bullet_\mathrm{b}(\phi)\colon \mathcal{B}^\infty\big(B_\Gamma^{\bullet+1},\mathbb{R}\big)\to\mathrm{L}^\infty_{\mathrm{w}*}\big(B_\Lambda^{\bullet+1},\mathrm{L}^\infty(\Omega)\big)\\&\mathrm{C}^k_\mathrm{b}(\phi)(f)(\dots,b_i,\dots)(\omega)=f\big(\dots,\chi(\omega)\phi\big(b_i,[\omega]\big),\dots\big)\end{aligned}$$

is $\Gamma\times\Lambda$-equivariant with regard to the following actions: The action on $\mathcal{B}^\infty(B_\Gamma^{\bullet+1},\mathbb{R})$ is induced from Γ acting diagonally $B^{\bullet+1}$ and Λ acting trivially. The action on $\mathrm{L}^\infty_{\mathrm{w}*}(B_\Lambda^{\bullet+1},\mathrm{L}^\infty(\Omega))\cong\mathrm{L}^\infty(B_\Lambda^{\bullet+1}\times\Omega)$ is induced from Λ acting diagonally on $B_\Lambda^{\bullet+1}\times\Omega$ and from Γ acting only on Ω.

Proof Firstly, we show equivariance of $\mathrm{C}^\bullet_\mathrm{b}(\phi)$. By definition we have

$$\mathrm{C}^\bullet_\mathrm{b}(\phi)\big((\gamma,\lambda)f\big)(\dots,b_i,\dots)(\omega)=f\big(\dots,\gamma^{-1}\chi(\omega)\phi\big(b_i,[\omega]\big),\dots\big).$$

By definition, Γ-equivariance of χ, and α-equivariance of ϕ we have

$$\begin{aligned}&\mathrm{C}^\bullet_\mathrm{b}(\phi)(f)\big(\dots,\lambda^{-1}b_i,\dots\big)\big(\gamma^{-1}\lambda^{-1}\omega\big)\\&\quad=f\big(\dots,\gamma^{-1}\chi\big(\lambda^{-1}\omega\big)\alpha\big(\lambda^{-1},[\omega]\big)\phi\big(b_i,[\omega]\big),\dots\big).\end{aligned}$$

It remains to check that

$$\chi\big(\lambda^{-1}\omega\big)\alpha\big(\lambda^{-1},[\omega]\big)=\chi(\omega).$$

Since both sides are Γ-equivariant, we may assume that $\omega \in Y$, i.e., $\chi(\omega) = 1$. In this case it follows from the defining properties of χ and α. $\square$

Remark 4.3 The map $\mathrm{C}_{\mathrm{b}}^{\bullet}(\phi)$ cannot be defined on $\mathrm{L}^{\infty}(B_{\Gamma}^{\bullet+1}, \mathbb{R})$ since we do not assume that ϕ preserves the measure class. The idea to work with the complex $\mathcal{B}^{\infty}(B_{\Gamma}^{\bullet+1}, \mathbb{R})$ to circumvent this problem in the context of boundary maps is due to Burger and Iozzi [4].

4.2 The Euler number in terms of boundary maps

In this subsection we retain the notation in Setup 3.1. In Burger-Monod's functorial theory of bounded cohomology [6, 41] the measurable map

$$\mathrm{dvol}_b \colon B^{n+1} \to \mathbb{R} \tag{4.1}$$

that assigns to $(b_0, \ldots, b_n)$ the oriented volume of the geodesic, ideal simplex with vertices $b_0, \ldots, b_n$ is a Γ-invariant (even G^0-invariant) cocycle and defines an element $\mathrm{dvol}_b \in \mathrm{H}_{\mathrm{b}}^{n}(\Gamma, \mathbb{R})$ (Theorem B.4). The forgetful map (comparison map) from bounded cohomology to ordinary cohomology is denoted by

$$\mathrm{comp}^{\bullet} \colon \mathrm{H}_{\mathrm{b}}^{\bullet}(\Gamma, \mathbb{R}) \to \mathrm{H}^{\bullet}(\Gamma, \mathbb{R}).$$

We consider the induction homomorphism

$$\mathrm{H}_{\mathrm{b}}^{\bullet}(\Omega) \colon \mathrm{H}_{\mathrm{b}}^{\bullet}\big(\Gamma_l, \mathrm{L}^{\infty}(\Gamma_r \backslash \Omega)\big) \to \mathrm{H}_{\mathrm{b}}^{\bullet}\big(\Gamma_r, \mathrm{L}^{\infty}(\Gamma_l \backslash \Omega)\big)$$

in bounded cohomology associated to Ω (see Sect. 4.1). Let

$$\mathrm{H}_{\mathrm{b}}^{\bullet}\big(j^{\bullet}\big) \colon \mathrm{H}_{\mathrm{b}}^{\bullet}(\Gamma_l, \mathbb{R}) \to \mathrm{H}_{\mathrm{b}}^{\bullet}\big(\Gamma_l, \mathrm{L}^{\infty}(\Gamma_r \backslash \Omega)\big)$$
$$\mathrm{H}_{\mathrm{b}}^{\bullet}\big(\mathrm{I}^{\bullet}\big) \colon \mathrm{H}_{\mathrm{b}}^{\bullet}\big(\Gamma_r, \mathrm{L}^{\infty}(\Gamma_l \backslash \Omega)\big) \to \mathrm{H}_{\mathrm{b}}^{\bullet}(\Gamma_r, \mathbb{R})$$

be the homomorphisms induced by inclusion of constant functions in the coefficients and by integration in the coefficients, respectively. Inspired by the classical Euler number of a surface representation we define:

Definition 4.4 (Higher-dimensional Euler number) Denote by $[\Gamma] \in \mathrm{H}_n(\Gamma, \mathbb{R}) \cong \mathrm{H}_n(\Gamma \backslash \mathbf{H}^n, \mathbb{R})$ the homological fundamental class of the manifold $\Gamma \backslash \mathbf{H}^n$. The *Euler number* $\mathrm{eu}(\Omega)$ of Ω is the evaluation of the cohomology class $\mathrm{comp}^n \circ \mathrm{H}_{\mathrm{b}}^{n}(\mathrm{I}^{\bullet}) \circ \mathrm{H}_{\mathrm{b}}^{n}(\Omega) \circ \mathrm{H}_{\mathrm{b}}^{n}(j^{\bullet})(\mathrm{dvol}_b)$ against the fundamental class $[\Gamma]$

$$\mathrm{eu}(\Omega) = \big\langle \mathrm{comp}^n \circ \mathrm{H}_{\mathrm{b}}^{n}\big(\mathrm{I}^{\bullet}\big) \circ \mathrm{H}_{\mathrm{b}}^{n}(\Omega) \circ \mathrm{H}_{\mathrm{b}}^{n}\big(j^{\bullet}\big)(\mathrm{dvol}_b), [\Gamma]\big\rangle. \tag{4.2}$$

In a recent paper [3] Bucher-Burger-Iozzi use a related notion to study maximal (in a similar sense as in Corollary 4.12) representations of $\mathrm{SO}_{n,1}$. In the Burger-Monod approach to bounded cohomology one can realize bounded cocycles in the bounded cohomology of Γ as cocycles on the boundary B. However, it is not immediately clear how the evaluation of a bounded n-cocycle realized on B at the fundamental class of $\Gamma\backslash\mathbf{H}^n$ can be explicitly computed since the fundamental class is not defined in terms of the boundary. Lemma 4.6 below achieves just that. Let us now describe two important ingredients that enter the proof of Lemma 4.6.

The first ingredient is the *cohomological Poisson transform* which is expressed by the visual measures on $B=\partial\mathbf{H}^n$.

Definition 4.5 For $z\in\mathbf{H}^n$ let ν_z be the *visual measure* at z on the boundary $B=\partial\mathbf{H}^n$ at infinity, that is, ν_z is the push-forward of the Lebesgue measure on the unit tangent sphere $\mathrm{T}^1_z\mathbf{H}^n$ under the homeomorphism $\mathrm{T}^1_z\mathbf{H}^n\to\partial\mathbf{H}^n$ given by the exponential map. For a $(k+1)$-tuple $\sigma=(z_0,\dots,z_k)$ of points in $\mathbf{H}^n$ we denote the product of the ν_{z_i} on B^{k+1} by ν_σ.

The measure ν_z is the unique Borel probability measure on B that is invariant with respect to the stabilizer of z. All visual measures are in the same measure class. Moreover, we have

$$\nu_{gz}=g_*\nu_z=\nu_z\big(g^{-1}\_\big)\quad\text{for every } g\in G.$$

The cohomological Poisson transform (see Definition B.5 for its general formulation) is the Γ-morphism of chain complexes $\mathrm{PT}^\bullet:\mathrm{L}^\infty(B^{\bullet+1},\mathbb{R})\to\mathrm{C}^\bullet_{\mathrm{b}}(\Gamma,\mathbb{R})$ with

$$\begin{aligned}\mathrm{PT}^n(f)(\gamma_0,\dots,\gamma_n)&=\int_{B^{n+1}}f(\gamma_0b_0,\dots,\gamma_nb_n)\,d\nu_{x_0}\dots d\nu_{x_0}\\&=\int_{B^{n+1}}f(b_0,\dots,b_n)\,d\nu_{(\gamma_0x_0,\dots,\gamma_nx_0)}\end{aligned}\tag{4.3}$$

where $x_0\in\mathbf{H}^n$ is a base point. The map $\mathrm{PT}^\bullet$ is independent of the choice of x_0 (see the remark after Definition B.5).

The second ingredient is Thurston's description of singular homology by *measure cycles* [59]: Let M be a topological space. We equip the space $\mathcal{S}_k(M)=\mathrm{Map}(\Delta^k,M)$ of continuous maps from the standard k-simplex to M with the compact-open topology. The group $\mathrm{C}^{\mathrm{m}}_k(M)$ is the vector space of all signed, compactly supported Borel measures on $\mathcal{S}_k(M)$ with finite total variation. The usual face maps $\partial_i:\mathcal{S}_k(M)\to\mathcal{S}_{k-1}(M)$ are measurable, and the maps $\mathrm{C}^{\mathrm{m}}_k(M)\to\mathrm{C}^{\mathrm{m}}_{k-1}(M)$ that send μ to $\sum_{i=0}^k(-1)^i(\partial_i)_*\mu$ turn $\mathrm{C}^{\mathrm{m}}_\bullet(M)$

into a chain complex. The map

$$\mathrm{D}_\bullet\colon \mathrm{C}_\bullet(M) \to \mathrm{C}^{\mathrm{m}}_\bullet(M),\ \sigma \mapsto \delta_\sigma$$

that maps a singular simplex σ to the point measure concentrated at σ is a chain map that induces an (isometric) homology isomorphism provided M is homeomorphic to a CW-complex [37, 60].

Next we recall Thurston's *smearing construction*, which describes an explicit representative of the fundamental class of a closed hyperbolic manifold $M = \Gamma\backslash\mathbf{H}^n$.

For any positively oriented geodesic n-simplex σ in $\mathbf{H}^n$, let $\mathrm{sm}(\sigma)$ denote the push-forward of the normalized Haar measure on $G^0 = \mathrm{Isom}(\mathbf{H}^n)^0$ under the measurable map

$$\Gamma\backslash G^0 \to \mathrm{Map}\bigl(\Delta^n, \Gamma\backslash\mathbf{H}^n\bigr), \qquad g \mapsto \mathrm{pr}(g\sigma).$$

Let $\rho \in G$ be the orientation reversing isometry that maps $(z_0, z_1, \dots, z_n)$ to $(z_1, z_0, \dots, z_n)$. By [53, Theorem 11.5.4 on p. 551] the image of the fundamental class in $\mathrm{H}_n(\Gamma\backslash\mathbf{H}^n, \mathbb{R})$ under the map $\mathrm{H}_n(\mathrm{D}_\bullet)$ is represented by the signed measure[9]

$$\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^n)}{2\,\mathrm{vol}(\sigma_0)}\bigl(\mathrm{sm}(\sigma_0) - \mathrm{sm}(\rho\circ\sigma_0)\bigr) \tag{4.4}$$

for any positively oriented geodesic n-simplex σ_0 in $\mathbf{H}^n$.

Lemma 4.6 *Let $\Gamma \subset G^0$ be a torsion-free and uniform lattice. Let $\sigma_0 = (z_0, \dots, z_n)$ be a positively oriented geodesic simplex in $\mathbf{H}^n$. Let $[\Gamma] \in \mathrm{H}_n(\Gamma, \mathbb{R}) \cong \mathrm{H}_n(\Gamma\backslash\mathbf{H}^n, \mathbb{R})$ be the fundamental class of $\Gamma\backslash\mathbf{H}^n$. Let $f \in \mathrm{L}^\infty(B^{n+1}, \mathbb{R})^\Gamma$ be an alternating cocycle. Then*

$$\begin{aligned}&\bigl\langle \mathrm{comp}^n \circ \mathrm{H}^n_{\mathrm{b}}\bigl(\mathrm{PT}^\bullet\bigr)\bigl([f]\bigr), [\Gamma]\bigr\rangle\\ &\quad= \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^n)}{\mathrm{vol}(\sigma_0)}\int_{B^{n+1}}\int_{\Gamma\backslash G^0} f(gb_0, \dots, gb_n)\,d\nu_{\sigma_0}\,dg.\end{aligned}$$

Proof Fix a basepoint $x_0 \in \mathbf{H}^n$. Consider the Γ-equivariant chain homomorphism $j_k\colon \mathrm{C}_k(\Gamma) \to \mathrm{C}_k(\mathbf{H}^n)$ that maps $(\gamma_0, \dots, \gamma_k)$ to the geodesic simplex with vertices $(\gamma_0 x_0, \dots, \gamma_k x_0)$. Let $\mathcal{B}^\infty(\mathcal{S}_\bullet(\mathbf{H}^n), \mathbb{R}) \subset \mathrm{C}^\bullet(\mathbf{H}^n, \mathbb{R})$ be the subcomplex of bounded measurable singular cochains on $\mathbf{H}^n$. From (4.3) we see that the Poisson transform $\mathrm{PT}^\bullet$ factorizes as

$$\mathrm{L}^\infty\bigl(B^{\bullet+1}, \mathbb{R}\bigr) \xrightarrow{\mathrm{P}^\bullet} \mathcal{B}^\infty\bigl(\mathcal{S}_\bullet\bigl(\mathbf{H}^n\bigr), \mathbb{R}\bigr) \xrightarrow{\mathrm{R}^\bullet} \mathrm{C}^\bullet_{\mathrm{b}}(\Gamma, \mathbb{R})$$

[9] The reader should note that in *loc. cit.* the Haar measure is normalized by $\mathrm{vol}(\Gamma\backslash\mathbf{H}^n)$ whereas we normalize it by 1.

where

$$\mathrm{P}^k(l)(\sigma) = \int_{B^{k+1}} l(b_0, \ldots, b_k)\, d\nu_\sigma \quad \text{for } \sigma \in \mathcal{S}_k\big(\mathbf{H}^n\big), \quad \text{and}$$

$$\mathrm{R}^k(f) = f \circ j_k.$$

For every $k \geq 0$ there is a Borel section $s_k : \mathcal{S}_k(\Gamma\backslash\mathbf{H}^n) \to \mathcal{S}_k(\mathbf{H}^n)$ of the projection [37, Theorem 4.1]. The following pairing is independent of the choice of s_k and descends to cohomology:

$$\langle \_, \_\rangle_m : \mathcal{B}^\infty\big(\mathcal{S}_\bullet\big(\mathbf{H}^n\big), \mathbb{R}\big)^\Gamma \otimes \mathrm{C}^{\mathrm{m}}_\bullet\big(\Gamma\backslash\mathbf{H}^n\big) \to \mathbb{R}$$

$$\langle l, \mu\rangle_m = \int_{\mathcal{S}_\bullet(\Gamma\backslash\mathbf{H}^n)} l\big(s_\bullet(\sigma)\big)\, d\mu(\sigma)$$

One sees directly from the definitions that for every $x \in \mathrm{H}_n(\Gamma, \mathbb{R})$

$$\begin{aligned}\big\langle \mathrm{comp}^n \circ \mathrm{H}^n\big(\mathrm{PT}^\bullet\big)\big([f]\big), x\big\rangle &= \big\langle \mathrm{comp}^n \circ \mathrm{H}^n\big(\mathrm{R}^\bullet\big) \circ \mathrm{H}^n\big(\mathrm{P}^\bullet\big)(f), x\big\rangle \\ &= \big\langle \mathrm{H}^n\big(\mathrm{P}^\bullet\big)\big([f]\big), \mathrm{H}_n(\mathrm{D}_\bullet \circ j_\bullet)(x)\big\rangle_m. \end{aligned} \tag{4.5}$$

Now we plug in $x = [\Gamma]$. Since the homology class $\mathrm{H}_n(\mathrm{D}_\bullet \circ j_\bullet)([\Gamma])$ is represented by the measure cycle (4.4) and f is alternating, the assertion is implied. □

Theorem 4.7 is known to experts; we prove it for the lack of a good reference. Although it can be seen as a special case of Theorem 4.8 we separate the proofs. The proofs of Theorems 4.7 and 4.8 are given at the end of the subsection.

Theorem 4.7 *Let $\Gamma \subset G^0$ be a torsion-free and uniform lattice. Then*

$$\big\langle \mathrm{comp}^n(\mathrm{dvol}_b), [\Gamma]\big\rangle = \mathrm{vol}\big(\Gamma\backslash\mathbf{H}^n\big).$$

Equivalently, this means that $\mathrm{comp}^n(\mathrm{dvol}_b) = \mathrm{dvol}$.

Theorem 4.8 *Let (Ω, m) be an ergodic (Γ, Γ)-coupling of a torsion-free and uniform lattice $\Gamma \subset G^0$. Let*

$$\phi : X \times B \to B$$

be the α-equivariant boundary map from (3.1), where $\alpha : \Gamma \times X \to \Gamma$ is a ME-cocycle for Ω. If $\sigma = (z_0, \ldots, z_n)$ with $z_i \in B$ is a positively oriented ideal regular simplex, then the Euler number of Ω satisfies

$$\mathrm{eu}(\Omega) = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^n)}{v_{\max}} \int_{\Gamma\backslash G^0} \int_X \mathrm{vol}\big(\phi_x(gz_0), \ldots, \phi_x(gz_n)\big)\, d\mu(x)\, dg,$$

where $v_{\max}$ is the volume of a positively oriented ideal maximal simplex in B^{n+1} and the quotient $\Gamma\backslash G^0$ carries the normalized Haar measure.

Note that the function

$$g \mapsto \int_X \operatorname{vol}\bigl(\phi_x(gz_0), \ldots, \phi_x(gz_n)\bigr)\, d\mu(x)$$

in the previous statement is Γ-invariant by α-equivariance of ϕ, G^0-invariance of the volume, and Γ-invariance of μ. So the integral in Theorem 4.8 makes sense.

The following immediate corollary, which we will not use in this paper, can be viewed as a higher-dimensional cocycle analog of the Milnor-Wood inequality for homomorphisms of a surface group into $\mathrm{Homeo}_+(S^1)$. We will present an independent stronger result, valid under an integrability assumption, in Corollary 4.12.

Corollary 4.9 (Higher-dimensional Milnor-Wood inequality) *In the setting of Theorem 4.8 we have $|\mathrm{eu}(\Omega)| \le \operatorname{vol}(\Gamma\backslash\mathbf{H}^n)$.*

We shall need the auxiliary Lemma 4.10 before we conclude the proof of Theorem 4.8 at the end of this subsection. We retain the setting of Theorem 4.8 for the rest of this subsection.

Lemma 4.10 *If $\sigma = (z_0, \ldots, z_n)$ with $z_i \in \mathbf{H}^n$ is a positively oriented geodesic simplex, then the Euler number of Ω satisfies*

$$\begin{aligned}\mathrm{eu}(\Omega) = {} & \frac{\operatorname{vol}(\Gamma\backslash\mathbf{H}^n)}{\operatorname{vol}(\sigma)} \\ & \times \int_{B^{n+1}} \int_{G^0/\Gamma} \int_X \operatorname{vol}\bigl(\phi_x(gb_0), \ldots, \phi_x(gb_n)\bigr)\, d\mu(x)\, dg\, d\nu_\sigma .\end{aligned}$$

Proof For better readability, we keep the notational distinction between Γ_l and Γ_r from Setup 3.1 and denote the copy of B on which Γ_l acts by B_l; similarly for B_r.

Consider the diagram below. The unlabeled maps are the obvious ones, sending a function to its equivalence class up to null sets and inclusion of constant functions.

All the maps are $\Gamma_l \times \Gamma_r$-equivariant chain morphisms as explained now. On $\mathrm{L}^\infty_{\mathrm{w}*}(B_l^{\bullet+1}, \mathbb{R})$ and $\mathrm{C}_\mathrm{b}^\bullet(\Gamma_l, \mathbb{R})$ we have the usual Γ_l-actions and the trivial Γ_r-actions. The Poisson transform in the lower row is then clearly $\Gamma_l \times \Gamma_r$-equivariant. The actions on the domain and target of the maps $\mathrm{C}_\mathrm{b}^\bullet(\chi)$ and $\mathrm{C}_\mathrm{b}^\bullet(\phi)$ are defined in Propositions 4.1 and 4.2, and is proved there that these maps are $\Gamma_l \times \Gamma_r$-equivariant. The Poisson transform in the upper row, which is Γ_r-equivariant, is also Γ_l-equivariant, since Γ_l acts only by its natural action on Ω.

$$\begin{tikzcd}
\mathcal{B}^\infty(B_l^{\bullet+1}, \mathbb{R}) \arrow[r, "\mathrm{C}_\mathrm{b}^\bullet(\phi)"] \arrow[d] & \mathrm{L}^\infty_{\mathrm{w}*}(B_r^{\bullet+1}, \mathrm{L}^\infty(\Omega)) \arrow[r, "\mathrm{PT}^\bullet"] & \mathrm{C}_\mathrm{b}^\bullet(\Gamma_r, \mathrm{L}^\infty(\Omega)) \\
\mathrm{L}^\infty_{\mathrm{w}*}(B_l^{\bullet+1}, \mathbb{R}) \arrow[r, "\mathrm{PT}^\bullet"] & \mathrm{C}_\mathrm{b}^\bullet(\Gamma_l, \mathbb{R}) \arrow[r] & \mathrm{C}_\mathrm{b}^\bullet(\Gamma_l, \mathrm{L}^\infty(\Omega)) \arrow[u, "\mathrm{C}_\mathrm{b}^\bullet(\chi)"']
\end{tikzcd}$$

The diagram describes two $\Gamma_l \times \Gamma_r$-equivariant chain morphisms

$$\phi, \psi : \mathcal{B}^\infty\big(B_l^{\bullet+1}, \mathbb{R}\big) \to \mathrm{C}_\mathrm{b}^\bullet\big(\Gamma_r, \mathrm{L}^\infty(\Omega)\big)$$

for which we want to prove, using Theorem B.1, that they are $\Gamma_l \times \Gamma_r$-chain homotopic. By Proposition B.3 the source $\mathcal{B}^\infty(B_l^{\bullet+1}, \mathbb{R})$ is a strong $\Gamma_l \times \Gamma_r$-resolution of $\mathbb{R}$. It is shown in [43, Proof of Proposition 4.6.] that the target $\mathrm{C}_\mathrm{b}^\bullet(\Gamma_r, \mathrm{L}^\infty(\Omega))$ is a relatively injective and strong $\Gamma_l \times \Gamma_r$-resolution of $L^\infty(\Omega)$. Both ϕ and ψ as the lower map make the diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d] & \mathrm{L}^\infty(\Omega) \arrow[d] \\
\mathrm{C}_\mathrm{b}^\bullet(\Gamma_r, \mathrm{L}^\infty(\Omega)) \arrow[r] & \mathrm{C}_\mathrm{b}^\bullet(\Gamma_r, \mathrm{L}^\infty(\Omega))
\end{tikzcd}$$

where the upper map is the inclusion of constant functions, commutative, that is, ϕ and ψ are morphisms between the augmented resolutions. By Theorem B.1, ϕ and ψ are equivariantly chain homotopic. Taking invariants and cohomology, this means that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{H}^n(\mathcal{B}^\infty(B_l^{\bullet+1}, \mathbb{R})^{\Gamma_l}) \arrow[r, "\mathrm{H}^n(\phi)"] \arrow[d] & \mathrm{H}^n(\mathrm{L}^\infty_{\mathrm{w}*}(B_r^{\bullet+1}, \mathrm{L}^\infty(\Gamma_l\backslash\Omega))) \arrow[r, "\mathrm{H}^n(\mathrm{PT}^\bullet)"] & \mathrm{H}_\mathrm{b}^n(\Gamma_r, \mathrm{L}^\infty(\Gamma_l\backslash\Omega)) \\
\mathrm{H}^n(\mathrm{L}^\infty_{\mathrm{w}*}(B_l^{\bullet+1}, \mathbb{R})^{\Gamma_l}) \arrow[r, "\mathrm{H}^n(\mathrm{PT}^\bullet)"] & \mathrm{H}_\mathrm{b}^n(\Gamma_l, \mathbb{R}) \arrow[r, "\mathrm{H}_\mathrm{b}^n(j^\bullet)"] & \mathrm{H}_\mathrm{b}^n(\Gamma_l, \mathrm{L}^\infty(\Gamma_r\backslash\Omega)) \arrow[u, "\mathrm{H}_\mathrm{b}^n(\Omega)"']
\end{tikzcd}$$

The volume cocycle dvol_b, which we defined as a cocycle in $\mathrm{L}^\infty_{\mathrm{w}*}(B^{n+1}, \mathbb{R})$, is everywhere defined and everywhere Γ-invariant and strictly satisfies the cocycle condition; hence it lifts to a cocycle in $\mathcal{B}^\infty(B^{n+1}, \mathbb{R})$ which we denote by $\mathrm{dvol}_{\mathrm{strict}}$. Now we have

$$\mathrm{eu}(\Omega) = \big\langle \mathrm{comp}^n \circ \mathrm{H}_\mathrm{b}^n\big(\mathrm{I}^\bullet\big) \circ \mathrm{H}_\mathrm{b}^n(\Omega) \circ \mathrm{H}_\mathrm{b}^n\big(j^\bullet\big) \circ \mathrm{H}_\mathrm{b}^n\big(\mathrm{PT}^\bullet\big)(\mathrm{dvol}_b), [\Gamma]\big\rangle$$

$$= \left\langle \mathrm{comp}^n \circ \mathrm{H}^n_{\mathrm{b}}\big(\mathrm{I}^\bullet\big) \circ \mathrm{H}^n_{\mathrm{b}}\big(\mathrm{PT}^\bullet\big) \circ \mathrm{H}^n_{\mathrm{b}}(\phi)(\mathrm{dvol}_{\mathrm{strict}}), [\Gamma] \right\rangle$$
$$= \left\langle \mathrm{comp}^n \circ \mathrm{H}^n_{\mathrm{b}}\big(\mathrm{PT}^\bullet\big) \circ \mathrm{H}^n_{\mathrm{b}}\big(\mathrm{I}^\bullet\big) \circ \mathrm{H}^n_{\mathrm{b}}(\phi)(\mathrm{dvol}_{\mathrm{strict}}), [\Gamma] \right\rangle.$$

Here the first equality is just the definition of the Euler class as in Definition 4.4; just be aware that there we denoted $\mathrm{H}^n_{\mathrm{b}}(\mathrm{PT}^\bullet)(\mathrm{dvol}_b)$ by the same symbol dvol_b since $\mathrm{H}^n_{\mathrm{b}}(\mathrm{PT}^\bullet)$ is a canonical isomorphism between two resolutions computing bounded cohomology in the functorial approach. The second equality follows by the commutativity of the above diagram. The third equality is true since the cohomological Poisson transform is natural in the coefficients, hence the integration $\mathrm{H}^n_{\mathrm{b}}(\mathrm{I}^\bullet)$ in the coefficients and $\mathrm{H}^n_{\mathrm{b}}(\mathrm{PT}^\bullet)$ interchange. We invoke Lemma 4.6 with $[f] = \mathrm{H}^n_{\mathrm{b}}(\mathrm{I}^\bullet) \circ \mathrm{H}^n_{\mathrm{b}}(\phi)(\mathrm{dvol}_{\mathrm{strict}})$ to conclude the proof. □

Proofs of Theorems 4.7 and 4.8 We start with the proof of Theorem 4.8. For every $i \in \{0, \dots, n\}$ we pick a sequence $(z_i^{(k)})_{k\in\mathbb{N}}$ on the geodesic ray from a basepoint $o \in \mathbf{H}^n$ to z_i converging to z_i. Let σ_k be the geodesic simplex spanned by the vertices $z_0^{(k)}, \dots, z_n^{(k)}$. By Lemma 4.10,

$$\mathrm{eu}(\Omega) = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}^n)}{\mathrm{vol}(\sigma)} \times \int_{B^{n+1}} \int_{\Gamma\backslash G^0} \int_X \mathrm{vol}\big(\phi_x(gb_0), \dots, \phi_x(gb_n)\big)\, d\mu(x)\, dg\, d\nu_\sigma.$$

We now let k go to ∞. Note that the left hand side does not depend on k. First of all, the volumes $\mathrm{vol}(\sigma_k)$ converge to $\mathrm{vol}(\sigma) = v_{\max}$. By Lemma 3.8,

$$\lim_{k\to\infty} \nu_{\sigma_k}\big\{(b_0, \dots, b_n) \mid d\big(\phi_x(gz_i), \phi_x(gb_i)\big) < \epsilon\big\} = 1$$

for every $\epsilon > 0$ and a.e. $(x, g) \in X \times G$.

It is shown in [53, Theorem 11.4.2 on p. 541] that the volume, vol, is a continuous function on the open set $B^{(n+1)}$ of all $(n+1)$-tuples in general position (see Definition 3.5). Thus, by Lemma 3.6, vol is continuous at a.e. $(\phi_x(gz_0), \dots, \phi_x(gz_n))$, and therefore

$$\lim_{k\to\infty} \int_{B^{n+1}} \mathrm{vol}\big(\phi_x(gb_0), \dots, \phi_x(gb_n)\big)\, d\nu_{\sigma_k} = \mathrm{vol}\big(\phi_x(gz_0), \dots, \phi_x(gz_n)\big),$$

for a.e. $(x, g) \in X \times G$, which finally yields Theorem 4.8 by the dominated convergence theorem. The proof of Theorem 4.7 is even easier since it does not require Lemma 3.8. One obtains from Lemma 4.6 that

$$\begin{aligned}&\left\langle \mathrm{comp}^n(\mathrm{dvol}_b), [\Gamma]\right\rangle\\ &\quad= \frac{\mathrm{vol}(\Gamma\backslash \mathbf{H}^n)}{\mathrm{vol}(\sigma)} \int_{B^{n+1}} \int_{\Gamma\backslash G^0} \int_X \mathrm{vol}(gb_0, \ldots, gb_n)\, d\mu(x)\, dg\, d\nu_\sigma,\end{aligned}$$

which converges for $k \to \infty$ to $\mathrm{vol}(\Gamma\backslash \mathbf{H}^n)$ by continuity of the function vol at a.e. point $(\phi_x(gb_0), \ldots, \phi_x(gb_n)) \in B^{n+1}$ (Lemma 3.6) and the weak convergence of $\nu_{z_i^{(k)}}$ to the point measure at z_i for every $i \in \{0, \ldots, n\}$. □

4.3 Adding integrability assumption

In this subsection we appeal to a general result from our companion paper [1], which relies on the integrability of the coupling. We get that, in the presence of such an integrability assumption, the Milnor-Wood inequality given in Corollary 4.9 becomes an equality, see Corollary 4.12 below.

Theorem 4.11 ([1, Theorem 5.12] and [1, Corollary 1.11]) *Let M and N be closed, oriented, negatively curved manifolds of dimension n. Let (Ω, μ) be an ergodic, integrable ME-coupling (Ω, μ) of the fundamental groups $\Gamma = \pi_1(M)$ and $\Lambda = \pi_1(N)$, and set $c = \frac{\mu(\Lambda\backslash\Omega)}{\mu(\Gamma\backslash\Omega)}$. Suppose that $x^b_\Gamma \in \mathrm{H}^n_\mathrm{b}(\Gamma, \mathbb{R})$ is an element that maps to the cohomological fundamental class $x_\Gamma \in \mathrm{H}^n(\Gamma, \mathbb{R}) \cong \mathrm{H}^n(M, \mathbb{R})$ of M under the comparison map. Define $x_\Lambda \in \mathrm{H}^n(\Lambda, \mathbb{R})$ analogously. Then the composition*

$$\begin{aligned}&\mathrm{H}^n_\mathrm{b}(\Gamma, \mathbb{R}) \xrightarrow{\mathrm{H}^n_\mathrm{b}\left(j^\bullet\right)} \mathrm{H}^n_\mathrm{b}\big(\Gamma, \mathrm{L}^\infty(\Lambda\backslash\Omega)\big) \xrightarrow{\mathrm{H}^n_\mathrm{b}(\Omega)} \mathrm{H}^n_\mathrm{b}\big(\Lambda, \mathrm{L}^\infty(\Gamma\backslash\Omega)\big)\\ &\qquad \xrightarrow{\mathrm{H}^n_\mathrm{b}\left(\mathrm{I}^\bullet\right)} \mathrm{H}^n_\mathrm{b}(\Lambda, \mathbb{R}) \xrightarrow{\mathrm{comp}^n} \mathrm{H}^n(\Lambda, \mathbb{R}) \end{aligned}\tag{4.6}$$

sends x^b_Γ to $\pm c \cdot x_\Lambda$. Furthermore, if $\Gamma \cong \Lambda$, then $c = 1$.

Corollary 4.12 (Maximality of the Euler class) *Retain the setting of Theorem 4.8. If, in addition, the coupling Ω is integrable, then*

$$\mathrm{eu}(\Omega) = \pm\, \mathrm{vol}\big(\Gamma\backslash \mathbf{H}^n\big). \tag{4.7}$$

Proof We apply Theorem 4.11 to $M = N = \Gamma\backslash \mathbf{H}^n$ and $\Lambda = \Gamma$. One has

$$\mathrm{dvol} = \mathrm{vol}(M) \cdot x_\Gamma$$

because the top degree cohomology is one-dimensional, and the evaluation against the homological fundamental class gives the equality. By Theorem 4.7

$$\mathrm{dvol} = \mathrm{comp}^n(\mathrm{dvol}_b).$$

Thus, $x_\Gamma^b = x_\Lambda^b = \mathrm{dvol}_b \,/\, \mathrm{vol}(M)$ satisfy the conditions of the theorem. Since in this case $c = 1$, we conclude that dvol_b is mapped to $\pm\,\mathrm{dvol}$ under (4.6). Equation (4.2) of Definition 4.4 gives

$$\begin{aligned}\mathrm{eu}(\Omega) &= \big\langle \mathrm{comp}^n \circ \mathrm{H}_\mathrm{b}^n\big(\mathrm{I}^\bullet\big) \circ \mathrm{H}_\mathrm{b}^n(\Omega) \circ \mathrm{H}_\mathrm{b}^n\big(j^\bullet\big)(\mathrm{dvol}_b), [\Gamma]\big\rangle \\ &= \big\langle \pm\,\mathrm{dvol}, [\Gamma]\big\rangle = \pm\,\mathrm{vol}\big(\Gamma\backslash\mathbf{H}^n\big). \qquad \square\end{aligned}$$

Recall that Σ_+^{reg} (resp. Σ_-^{reg}) denotes the set of positively (resp. negatively) oriented regular ideal simplices (see Sect. 3.1). We think of Σ_+^{reg} and Σ_-^{reg} as subsets of B^{n+1}—the $(n+1)$-tuples of points on the boundary $B = \partial\mathbf{H}^n$. Since Σ_+^{reg} (resp. Σ_-^{reg}) is a single G^0-orbit, terms like *a.e. point on* Σ_+^{reg} refer to the Haar measure on G^0.

Corollary 4.13 *Retain the setting of Theorem 4.8. If, in addition, the coupling Ω is integrable, then the Borel map*

$$\phi_x^{n+1} = \phi_x \times \cdots \times \phi_x : B^{n+1} \to B^{n+1}$$

either maps a.e. point in Σ_+^{reg} into Σ_+^{reg} for μ-a.e $x \in X$, or a.e. point of Σ_+^{reg} is mapped into Σ_-^{reg} for μ-a.e $x \in X$.

Proof Fix $(z_0, \dots, z_n) \in \Sigma_+^{\mathrm{reg}}$. Combining Corollary 4.12 with Theorem 4.8 we get

$$\int_{\Gamma\backslash G^0} \int_X \mathrm{vol}\big(\phi_x(gz_0), \dots, \phi_x(gz_n)\big)\, d\mu(x)\, dg = \pm v_{\max}.$$

By Lemma 3.3 (2), an ideal simplex has an oriented volume $v_{\max}$ iff it is in Σ_+^{reg} and $-v_{\max}$ iff it is in Σ_-^{reg}. Combining this with the fact that the absolute value of the integrand on the left hand side in the above formula is a priori at most $v_{\max}$ implies that either for a.e. $(g, x) \in G^0 \times X$, $(\phi_x(gz_0), \dots, \phi_x(gz_n)) \in \Sigma_+^{\mathrm{reg}}$ or for a.e. $(g, x) \in G^0 \times X$, $(\phi_x(gz_0), \dots, \phi_x(gz_n)) \in \Sigma_-^{\mathrm{reg}}$. But by Lemma 3.3 (1), G^0 acts simply transitive on Σ_+^{reg}, thus for the set of $g \in G^0$ satisfying the above generic condition, the set of ideal simplices of the form $(gz_0, \dots, gz_n)$ is of full measure in Σ_+^{reg}. The proof now follows by an application of Fubini's theorem. $\square$

5 Proofs of the main results

5.1 Proof of Theorems B and C

Proof of Theorem B We aim to show that the group $G = \mathrm{Isom}(\mathbf{H}^n)$ is 1-taut for any $n \geq 3$. Fix a cocompact torsion-free lattice $\Gamma < G^0$ in the connected component of $e \in G$. By Proposition 2.3 G is strongly ICC, thus Proposition 2.9 applies and it is enough to show that Γ is 1-taut relative to G (note that a cocompact lattice is integrable). By Lemma A.5 (applied to Γ) it is enough to show that every integrable ergodic (Γ, Γ)-coupling is taut relative to G.

Let (Ω, m) be an integrable ergodic (Γ, Γ)-coupling. We adopt the Setup 3.1 and consider the boundary map (3.1) $\phi : X \times B \to B$. Let $\Sigma_+^{\mathrm{reg}}, \Sigma_-^{\mathrm{reg}} \in B^{n+1}$ denote the sets of positively and negatively oriented regular ideal simplices, as defined in Sect. 3.1. Then Corollary 4.13 implies that either for μ-a.e $x \in X$, $\phi_x^{n+1} = \phi_x \times \cdots \times \phi_x$ maps a.e. point in Σ_+^{reg} into Σ_+^{reg}, or for μ-a.e $x \in X$, ϕ_x^{n+1} maps a.e. point in Σ_+^{reg} into Σ_-^{reg}.

We now use the assumption that $n \geq 3$, and apply Lemma 3.4 to deduce that for a.e $x \in X$ there exists a unique $g_x \in G$ with $\phi_x(b) = g_x b$ for a.e. $b \in B$. Proposition 3.2 applied to $\mathcal{G}$ being the image of G in $\mathrm{Homeo}(B)$ yields that Ω is taut with respect to G. □

We now set the stage for the proof of Theorem C which deals with the case $n = 2$. If we normalize the volume cocycle (4.1) by the volume $v_{\max}$ of a non-degenerate positively oriented ideal 2-simplex in $\mathbf{H}^2 \cup \bar{\mathbf{H}}^2$—they all have the same volume—we obtain the *orientation cocycle* c defined on triples of points on the circle $S^1 = B = \partial\mathbf{H}^2$ by

$$c(b_0, b_1, b_3) = v_{\max}^{-1} \cdot \mathrm{vol}(b_0, b_1, b_2).$$

It takes values in $\{-1, 0, 1\}$ with $c(b_0, b_1, b_2) = 1$ if the triple (b_0, b_1, b_2) consists of distinct points in the positive orientation/cyclic order, $c = -1$ if the cyclic order is reversed, and $c = 0$ if the triple is degenerate. Let ν denote a probability measure in the Lebesgue class, and suppose that $\phi : (S^1, \nu) \to S^1$ is a measurable map so that for ν^3-a.e. (b_0, b_1, b_2):

$$c\big(\phi(b_0), \phi(b_1), \phi(b_2)\big) = c(b_0, b_1, b_2).$$

It follows from [30, Proposition 5.5] that the following conditions on such measurable orientation preserving $\phi : (S^1, \nu) \to S^1$ are equivalent:

(1) The push-forward measure $\phi_* \nu$ has full support;
(2) ϕ agrees a.e. with a homeomorphism $f \in \mathrm{Homeo}(S^1)$.

Let $\Gamma < G = \mathrm{Isom}(\mathbf{H}^2)$ be a lattice. Let $\alpha : \Gamma \times X \to \Gamma$ be the ME-cocycle associated with an ergodic (Γ, Γ)-coupling (Ω, m) and an identification $i : \Gamma \times X \to \Omega$. Let $\phi_x : (S^1, \nu) \to S^1$, $x \in X$, be the boundary map 3.1 associated to α as in Sect. 3.

Proposition 5.1 *If the orientation cocycle is preserved by ϕ_x a.e., that is,*

$$c\big(\phi_x(b_0), \phi_x(b_1), \phi_x(b_2)\big) = c(b_0, b_1, b_2) \quad \nu^3\text{-}a.e.$$

for a.e. $x \in X$, then the map ϕ_x agree a.e. with a homeomorphism $f_x \in$ $\mathrm{Homeo}(S^1)$ *for a.e. $x \in X$.*

Proof We prove that the measurable family of open sets $U_x = S^1 \setminus \mathrm{supp}(\phi_x \nu)$ satisfies a.e. $U_x = \emptyset$. The fact that ν is Γ-quasi-invariant and the identity

$$\phi_{\gamma.x}(\gamma b) = \alpha(\gamma, x)\phi_x(b)$$

imply the following *a priori* equivariance of $\{U_x \mid x \in X\}$

$$U_{\gamma.x} = \alpha(\gamma, x)\, U_x. \tag{5.1}$$

Since $U_x \neq S^1$ for every $x \in X$, the proposition is implied by Lemma 2.5 and the fact that the action of $G = \mathrm{PSL}_2(\mathbb{R})$ and of its lattices on the circle S^1 is minimal and strongly proximal [19, Propositions 4.2 and 4.4]. □

Proof of Theorem C We fix a cocompact torsion-free lattice $\Gamma < G^0$ (a surface group) in the connected component of $G = \mathrm{Isom}(\mathbf{H}^2)$. We identify S^1 as $B = \partial\mathbf{H}^2$ and embed G in the Polish group $\mathcal{G} := \mathrm{Homeo}(S^1)$ accordingly. Exactly as at the start of the proof of Theorem B ($n \geq 3$ was not needed for that) one sees that it suffices to show that Γ is 1-taut relative to $\mathcal{G}$.

By Lemma A.5 it is enough to show that every integrable ergodic (Γ, Γ)-coupling is taut relative to $\mathcal{G}$. Let (Ω, m) be such a coupling. We adopt the Setup 3.1 and consider the boundary map (3.1) $\phi : X \times B \to B$.

Corollary 4.13 implies that there is $\sigma \in \{1, -1\}$ such that for μ-a.e. $x \in X$ and a.e. triple $(b_1, b_2, b_3) \in (S^1)^3$:

$$c\big(\phi_x(b_1), \phi_x(b_2), \phi_x(b_3)\big) = \sigma \cdot c(b_1, b_2, b_3).$$

That is, either a.e. ϕ_x preserves the cyclic order of a.e. triple, or a.e. ϕ_x reverses the cyclic order of a.e. triple. In either case, by Proposition 5.1 we conclude that for a.e. $x \in X$, ϕ_x agree a.e. with a homeomorphism $f_x \in \mathrm{Homeo}(S^1)$. It follows with Proposition 3.2 that the ergodic integrable (Γ, Γ)-coupling Ω is taut relative to $\mathcal{G} = \mathrm{Homeo}(S^1)$. □

5.2 Measure equivalence rigidity: Theorem D

Let $G = \mathrm{Isom}(\mathbf{H}^n)$, $n \geq 3$. Let $\Gamma < G$ be a lattice, and Λ a finitely generated group which admits an integrable (Γ, Λ)-coupling (Ω, m). By Lemma A.2 the (Γ, Γ)-coupling $\Omega \times_\Lambda \Omega^*$ is integrable.

By Theorem B and Proposition 2.9 the lattice Γ is 1-taut relative to the inclusion $\Gamma < G$. Hence the coupling $\Omega \times_\Lambda \Omega^*$ is taut. By Proposition 2.3 the group G is strongly ICC relative to $\Gamma < G$. Applying Theorem 2.6 we obtain a continuous homomorphism $\rho : \Lambda \to G$ with finite kernel F, image

$\bar{\Lambda} = \rho(\Lambda)$ being discrete in G, and a measurable $\mathrm{id}_\Gamma \times \rho$-equivariant map $\Psi : \Omega \to G$.

To complete the proof of statement (1) of Theorem D and case $n \geq 3$ of Theorem A it remains to show that $\bar{\Lambda}$ is not merely discrete, but is actually a lattice in G. This can be deduced from the application of Ratner's theorem below which is needed for the precise description of the push-forward measure $\Psi_* m$ on G as stated in part (2) of Theorem D. Let us also give the following direct argument which relies only on the strong ICC property of G.

Consider the composition (G, Λ)-coupling $\widetilde{\Omega} = G \times_\Gamma \Omega$, and the (G, G)-coupling $\widetilde{\Omega} \times_\Lambda \widetilde{\Omega}^*$. Since Γ is an integrable lattice in G (Theorem 1.9) by Lemma A.2 both $\widetilde{\Omega}$ and $\widetilde{\Omega} \times_\Lambda \widetilde{\Omega}^*$ are integrable couplings. Theorem B provides a unique tautening map

$$\widetilde{\Phi} : \widetilde{\Omega} \times_\Lambda \widetilde{\Omega}^* \to G.$$

Applying Theorem 2.1 (a special case of Theorem 2.6 with $\mathcal{G} = G$), we obtain a homomorphism $\widetilde{\rho} : \Lambda \to G$ with finite kernel and image being a lattice in G. There is also a $\mathrm{Id}_G \times \widetilde{\rho}$-equivariant measurable map

$$\widetilde{\Psi} : \widetilde{\Omega} = G \times_\Gamma \Omega \to G.$$

We claim that $\rho, \widetilde{\rho} : \Lambda \to G$ are conjugate representations. To see this observe that since G is strongly ICC, there is only one tautening map $\widetilde{\Omega} \times_\Lambda \widetilde{\Omega}^* \to G$. This implies the a.e. identity

$$\widetilde{\Psi}\big([g_1, \omega_1]\big)\widetilde{\Psi}\big([g_2, \omega_2]\big)^{-1} = g_1 \Psi(\omega_1)\Psi(\omega_2)^{-1} g_2^{-1}.$$

Equivalently, we have a.e. identity

$$\Psi(\omega_1)^{-1} g_1^{-1} \widetilde{\Psi}\big([g_1, \omega_1]\big) = \Psi(\omega_2)^{-1} g_2^{-1} \widetilde{\Psi}\big([g_2, \omega_2]\big).$$

Hence the value of both sides are a.e. equal to a constant $g_0 \in G$. It follows that for a.e. $g \in G$ and $\omega \in \Omega$

$$g^{-1} \widetilde{\Psi}\big([g, \omega]\big) = \Psi(\omega) g_0.$$

Finally, the fact that $\Psi, \widetilde{\Psi}$ are ρ-, $\widetilde{\rho}$- equivariant respectively, implies:

$$\widetilde{\rho}(\lambda) = g_0 \rho(\lambda) g_0^{-1} \quad (\lambda \in \Lambda).$$

In particular, $\bar{\Lambda} = g_0^{-1} \widetilde{\rho}(\Lambda) g_0$ is a lattice in G.

We proceed with the proof of statement (2): given the $\mathrm{Id}_\Gamma \times \rho$-equivariant measurable map $\Psi : \Omega \to G$ we shall describe the pushforward $\Psi_* m$ on G. (We shall use the discreteness of $\bar{\Lambda} = \rho(\Lambda)$, but the fact that it is a lattice will not be needed; in fact, it will follow from the application of Ratner's theorem.) Recall that the measure Ψ_*m is invariant under the action $x \mapsto \gamma x \rho(\lambda)^{-1}$, and descends to a finite Γ-invariant measure μ on $G/\bar{\Lambda}$ and to a finite $\bar{\Lambda}$-invariant measure ν on $\Gamma\backslash G$. Assuming m was $\Gamma \times \Lambda$-ergodic, μ and ν are ergodic under the Γ- and $\bar{\Lambda}$-action, respectively. One can now apply Ratner's theorem [54] to describe μ, and thereby Ψ_*m, as in [15, Lemma 4.6]. For the reader's convenience we sketch the arguments.

Let $\Lambda^0 = \bar{\Lambda} \cap G^0$; so either $\Lambda^0 = \bar{\Lambda}$ or $[\bar{\Lambda} : \Lambda^0] = 2$. In the first case we set $\mu' = \mu$, in the latter case let μ' denote the 2-to-1 lift of μ to G/Λ^0. Let $\Gamma^0 = \Gamma \cap G^0$, and let μ^0 be an ergodic component of μ' supported on G^0/Λ^0. We consider the homogeneous space $Z = G^0/\Gamma^0 \times G^0/\Lambda^0$ which is endowed with the following probability measure

$$\tilde{\mu}^0 = \int_{G^0/\Gamma^0} \delta_{g\Gamma^0} \times g_*\mu^0 \, dm_{G^0/\Gamma^0}.$$

Observe that $\tilde{\mu}^0$ well defined because μ^0 is Γ^0-invariant. Moreover, $\tilde{\mu}^0$ is invariant and ergodic for the action of the diagonal $\Delta(G^0) \subset G^0 \times G^0$ on Z. Since G^0 is a connected group generated by unipotent elements, Ratner's theorem shows that $\tilde{\mu}^0$ is *homogeneous*. This means that there is a connected Lie subgroup $L < G^0 \times G^0$ containing $\Delta(G^0)$ and a point $z \in Z$ such that the stabilizer L_z of z is a lattice in L and $\tilde{\mu}^0$ is the push-forward of the normalized Haar measure m_{L/L_z} to the L-orbit $Lz \subset Z$. Since G^0 is a simple group, there are only two possibilities for L: either (i) $L = G^0 \times G^0$ or (ii) $L = \Delta(G^0)$.

In case (i), $\tilde{\mu}^0$ is the Haar measure on $G^0/\Gamma^0 \times G^0/\Lambda^0$, and μ^0 is the Haar measure on G^0/Λ^0. (In particular, Λ^0 is a lattice in G^0, and Λ is a lattice in G.) The original measure μ may be either the G-invariant measure $m_{G/\bar{\Lambda}}$, or a G^0-invariant measure on $G/\bar{\Lambda}$. In the latter case, by possibly multiplying Φ and conjugating ρ with some $x \in G \setminus G^0$, we may assume that μ is the G^0-invariant probability measure on $G^0/\bar{\Lambda}$.

In case (ii), the fact that L_z is lattice in $L = \Delta(G^0)$, implies that μ^0 and the original measure μ are atomic. Since Γ acts ergodically on $(G/\bar{\Lambda}, \mu)$, this atomic measure is necessarily supported and equidistributed on a finite Γ-orbit of some $g_0\bar{\Lambda} \in G/\bar{\Lambda}$. It follows that $\Gamma \cap g_0^{-1}\bar{\Lambda}g_0$ has finite index in Γ. (This also implies that $\bar{\Lambda}$ is a lattice in G.) Upon multiplying Ψ and conjugating ρ by $g_0 \in G$, we may assume that Φ_*m is equidistributed on the double coset $\Gamma e \bar{\Lambda}$ and that Γ, $\bar{\Lambda}$ are commensurable lattices. This completes the proof of Theorem D.

5.3 Convergence actions on the circle: case $n = 2$ of Theorem A

Let Γ be a uniform lattice in $G = \mathrm{Isom}(\mathbf{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$. The group G is a subgroup of $\mathrm{Homeo}(S^1)$ by the natural action of $\mathrm{PGL}_2(\mathbb{R})$ on $S^1 \cong \mathbb{R}\mathrm{P}^1$.

Consider a compactly generated unimodular group H that is L^1-measure equivalent to Γ. We will prove a more general statement than in Theorem A, which is formulated for discrete $H = \Lambda$. Since Γ is uniform, hence integrable in G, we can induce any integrable (Γ, H)-coupling to an integrable (G, H)-coupling (Lemma A.2). Let (Ω, m) be an integrable (G, H)-coupling (Ω, m).

From Theorem 2.6 we obtain a continuous homomorphism $\rho : H \to \mathrm{Homeo}(S^1)$ with compact kernel and closed image $\bar{H} < \mathrm{Homeo}(S^1)$ and, by pushing forward m, a measure $\bar{m}$ on $\mathrm{Homeo}(S^1)$ that is invariant under all bilateral translations on $f \mapsto g f \rho(h)^{-1}$ with $g \in G$ and $h \in H$ and descends to a finite $\bar{H}$-invariant measure μ on $G\backslash \mathrm{Homeo}(S^1)$ and a finite G-invariant measure ν on $\mathrm{Homeo}(S^1)/\bar{H}$.

The next step is to show that $\bar{H}$ can be conjugated into G. To this end, we shall use the existence of the finite $\bar{H}$-invariant measure μ on $G\backslash \mathrm{Homeo}(S^1)$, which may be normalized to a probability measure. We need the following theorem which we prove relying on the deep work by Gabai [20] and Casson-Jungreis [8] on the determination of convergence groups as Fuchsian groups.

Theorem 5.2 *Let μ be a Borel probability measure on $G\backslash \mathrm{Homeo}(S^1)$. Then the stabilizer $H_\mu = \{f \in \mathrm{Homeo}(S^1) \mid f_*\mu = \mu\}$ for the action by the right translations is conjugate to a closed subgroup of G.*

Proof We fix a metric d on the circle, say $d(x, y) = \measuredangle(x, y)$. Let $\mathrm{Trp} \subset S^1 \times S^1 \times S^1$ be the space of distinct triples on the circle. The group $\mathrm{Homeo}(S^1)$ acts diagonally on Trp. We denote elements in Trp by bold letters $\mathbf{x} \in \mathrm{Trp}$; the coordinates of $\mathbf{x} \in \mathrm{Trp}$ or $\mathbf{y} \in \mathrm{Trp}$ will be denoted by x_i or y_i where $i \in \{1, 2, 3\}$, respectively. For $f \in \mathrm{Homeo}(S^1)$ we write $f(\mathbf{x})$ for $(f(x_1), f(x_2), f(x_3))$. We equip Trp with the metric, also denoted by d, given by

$$d(\mathbf{x}, \mathbf{y}) = \max_{i \in \{1,2,3\}} d(x_i, y_i).$$

The following lemma will eventually allow us to apply the work of Gabai-Casson-Jungreis. □

Lemma 5.3 *For every compact subset $K \subset \mathrm{Trp}$ and every $\epsilon > 0$ there is $\delta > 0$ so that for all $h, h' \in H_\mu$ and $\mathbf{y} \in K \cap h^{-1}K$ and $\mathbf{y}' \in K \cap h'^{-1}K$ one has the implication*:

$$d(\mathbf{y}, \mathbf{y}') < \delta \quad \text{and} \quad d\big(h(\mathbf{y}), h'(\mathbf{y}')\big) < \delta \quad \Longrightarrow \quad \sup_{x \in S^1} d\big(h(x), h'(x)\big) < \epsilon.$$

Proof For an arbitrary triple $\mathbf{z} \in \mathrm{Trp}$ and $x \in S^1 \setminus \{z_3\}$ consider the real valued *cross-ratio*

$$[x, z_1; z_2, z_3] = \frac{(x - z_1)(z_2 - z_3)}{(x - z_3)(z_2 - z_1)}.$$

In this formula we view the circle as the one-point compactification of the real line. Denote by $[z_1, z_2]_{z_3}$ the circle arc from z_1 to z_2 not including z_3. As a function in the first variable, $[\_, z_1; z_2, z_3]$ is a monotone homeomorphism between the closed arc $[z_1, z_2]_{z_3}$ and the interval $[0, 1]$. For $f \in \mathrm{Homeo}(S^1)$ and $\mathbf{z} \in \mathrm{Trp}$ we define the function

$$F_{\mathbf{z},f} : [z_1, z_2]_{z_3} \to [0, 1], \quad F_{\mathbf{z},f}(x) = \big[f(x), f(z_1); f(z_2), f(z_3)\big].$$

Since the cross-ratio is invariant under G [53, Theorem 4.3.1 on p. 116], we have $F_{\mathbf{z},gf}(x) = F_{\mathbf{z},f}(x)$ for any $g \in G$. Hence we may and will use the notation $F_{\mathbf{z},Gf}(x)$. We now average $F_{\mathbf{z},Gf}(x)$ with regard to the measure μ and obtain the function $\bar{F}_{\mathbf{z}} : [z_1, z_2]_{z_3} \to [0, 1]$ with

$$\bar{F}_{\mathbf{z}}(x) = \int\limits_{G\backslash \mathrm{Homeo}(S^1)} F_{\mathbf{z},Gf}(x)\, d\mu(Gf).$$

The H_μ-invariance of μ implies that

$$\bar{F}_{h(\mathbf{z})}\big(h(x)\big) = \bar{F}_{\mathbf{z}}(x) \tag{5.2}$$

for every $h \in H_\mu$ and every $x \in [z_1, z_2]_{z_3}$. Let us introduce the following notation: Whenever $K \subset \mathrm{Trp}$ is a subset, we denote by $\widetilde{K}$ the subset

$$\widetilde{K} = \big\{(x, \mathbf{z}) \mid \mathbf{z} \in K, x \in [z_1, z_2]_{z_3}\big\} \subset S^1 \times S^1 \times S^1 \times S^1.$$

Next let us establish the following continuity properties:

(1) For every compact $K \subset \mathrm{Trp}$ and every $\epsilon > 0$ there is $\eta > 0$ such that:

$$\forall_{(s,\mathbf{z}),(t,\mathbf{z}) \in \widetilde{K}} \quad \left(\big|\bar{F}_{\mathbf{z}}(t) - \bar{F}_{\mathbf{z}}(s)\big| < \eta \Rightarrow d(t, s) < \frac{\epsilon}{5}\right).$$

(2) For every compact $K \subset \mathrm{Trp}$ and every $\eta > 0$ there is $\delta > 0$ such that:

$$\forall_{(t,\mathbf{y}),(t,\mathbf{z}) \in \widetilde{K}} \quad \left(d(\mathbf{y}, \mathbf{z}) < \delta \Rightarrow \big|\bar{F}_{\mathbf{y}}(t) - \bar{F}_{\mathbf{z}}(t)\big| < \frac{\eta}{2}\right).$$

Proof of (1): Let $K \subset \mathrm{Trp}$ be compact and $\epsilon > 0$. Let $f \in \mathrm{Homeo}(S^1)$. The family of homeomorphisms $\bar{F}_{\mathbf{z},Gf} : [z_1, z_2]_{z_3} \to [0, 1]$ depends continuously on $\mathbf{z} \in \mathrm{Trp}$. The inverses of these functions are equicontinuous when $\mathbf{z}$ ranges in a compact subset. Hence there exists $\theta(Gf) > 0$ such that for every $\mathbf{z} \in K$ and all $t, s \in [z_1, z_2]_{z_3}$ we have the implication

$$\left|F_{\mathbf{z},Gf}(t) - F_{\mathbf{z},Gf}(s)\right| < \theta(Gf) \quad \Rightarrow \quad d(t,s) < \frac{\epsilon}{5}.$$

The set $G\backslash \mathrm{Homeo}(S^1)$ is the union of an increasing sequence of measurable sets

$$A_n = \left\{ Gf \in G\backslash H \mid \theta(Gf) > \frac{1}{n} \right\}.$$

Fix n large enough so that $\mu(A_n) > 1/2$. We claim that $\eta = (2n)^{-1}$ satisfies (1). Suppose that $\mathbf{z} \in K$ and $t, s \in [z_1, z_2]_{z_3}$ satisfy $d(t,s) > \epsilon/5$. Up to exchanging t and s, we may assume that $[s, z_1; z_2, z_3] \geq [t, z_1; z_2, z_3]$. Then $F_{\mathbf{z},Gf}(s) \geq F_{\mathbf{z},Gf}(t)$ for all $f \in \mathrm{Homeo}(S^1)$, and

$$\bar{F}_{\mathbf{z}}(s) - \bar{F}_{\mathbf{z}}(t) \geq \int_{A_n} \left(F_{\mathbf{z},Gf}(s) - F_{\mathbf{z},Gf}(t)\right) d\mu > \mu(A_n) \cdot \frac{1}{n} > \eta.$$

Proof of (2): Let $K \subset \mathrm{Trp}$ be compact, and let $\eta > 0$. Let $f \in \mathrm{Homeo}(S^1)$. Since $\widetilde{K}$ is compact, $\bar{F}_{\mathbf{z}}(x)$ as a function on $\widetilde{K}$ is equicontinuous. Hence there is $\delta(Gf) > 0$ such that for all $(x, \mathbf{y}) \in \widetilde{K}$ and $(x, \mathbf{z}) \in \widetilde{K}$ with $d(\mathbf{y}, \mathbf{z}) < \delta(Gf)$ we have

$$\left|F_{\mathbf{y},Gf}(x) - F_{\mathbf{z},Gf}(x)\right| < \frac{\eta}{2}.$$

The set $G\backslash \mathrm{Homeo}(S^1)$ is the union of an increasing sequence of measurable sets

$$B_n = \left\{ Gf \in G\backslash H \mid \delta(Gf) > \frac{1}{n} \right\}.$$

We choose $n \in \mathbb{N}$ with $\mu(B_n) > 1 - \eta/2$ and set $\delta = n^{-1}$. Then for $(x, \mathbf{y}) \in \widetilde{K}$ and $(x, \mathbf{z}) \in \widetilde{K}$ with $d(\mathbf{y}, \mathbf{z}) < \delta$ we have

$$\left|\bar{F}_{\mathbf{y}}(x) - \bar{F}_{\mathbf{z}}(x)\right| \leq \int_{B_n} \left|F_{\mathbf{y},Gf}(x) - F_{\mathbf{z},Gf}(x)\right| d\mu(Gf) + \frac{\eta}{2} < \eta,$$

proving (2).

We can now complete the proof of the lemma. Let $K \subset \mathrm{Trp}$ be a compact subset. Let $\epsilon > 0$. We can choose $r > 0$ such that

$$K \subset \left\{ \mathbf{x} \in \mathrm{Trp} \mid d(x_1, x_2), d(x_2, x_3), d(x_3, x_1) \geq r \right\}.$$

For the given ϵ and K let $\eta > 0$ be as in (1). For the given ϵ and K and this η let $\delta > 0$ be as in (2). We may also assume that

$$\delta < \frac{\epsilon}{5} < \frac{r}{3}.$$

Consider $h, h' \in H_\mu$ and $\mathbf{y}, \mathbf{y}' \in K$ where $\mathbf{z} = h(\mathbf{y})$, $\mathbf{z}' = h'(\mathbf{y}')$ are also in $K\mathring{A}$, and assume that $d(\mathbf{y}, \mathbf{y}') < \delta$ and $d(\mathbf{z}, \mathbf{z}') < \delta$. There are several possibilities for the cyclic order of the points $\{y_1, y_1', y_2, y_2', y_3, y_3'\}$, but since the pairs $\{y_i, y_i'\}$ of corresponding points in the triples $\mathbf{y}, \mathbf{y}'$ are closer ($d(y_i, y_i') < \delta < r/3$) than the separation between the points in the triples ($d(y_i, y_j), d(y_i', y_j') \ge r$), these points define a partition of the circle into three long arcs L_{ij} separated by three short arcs S_k (possibly degenerating into points) in the following cyclic order

$$S^1 = L_{12} \cup S_2 \cup L_{23} \cup S_3 \cup L_{31} \cup S_1.$$

The end points of the arc S_i are $\{y_i, y_i'\}$; and if $(i, j, k) = (1, 2, 3)$ up to a cyclic permutation, then

$$L_{ij} = [y_i, y_j]_{y_k} \cap \left[y_i', y_j'\right]_{y_k'}.$$

Note that for any $x \in L_{ij}$ we have

$$h(x), h'(x) \in [z_i, z_j]_{z_k} \cap \left[z_i', z_j'\right]_{z_k'}.$$

Using (2) and (5.2) we obtain

$$\begin{aligned}
\left|\bar{F}_{\mathbf{z}}\big(h(x)\big) - \bar{F}_{\mathbf{z}}\big(h'(x)\big)\right| &\le \left|\bar{F}_{\mathbf{z}}\big(h(x)\big) - \bar{F}_{\mathbf{z}'}\big(h'(x)\big)\right| \\
&\quad + \left|\bar{F}_{\mathbf{z}'}\big(h'(x)\big) - \bar{F}_{\mathbf{z}}\big(h'(x)\big)\right| \\
&\le \left|\bar{F}_{\mathbf{z}}\big(h(x)\big) - \bar{F}_{\mathbf{z}'}\big(h'(x)\big)\right| + \frac{\eta}{2} \\
&= \left|\bar{F}_{\mathbf{y}}(x) - \bar{F}_{\mathbf{y}'}(x)\right| + \frac{\eta}{2} < \eta.
\end{aligned}$$

By (1) it follows that $d(h(x), h'(x)) < \epsilon/5$ for every $x \in L_{12} \cup L_{23} \cup L_{31}$. It remains to consider points $x \in S_i$, $i = 1, 2, 3$, which can be controlled via the behavior of the endpoints y_i, y_i' of the short arc S_i.

First observe that the image $h(S_i)$ of S_i is the short arc defined by $h(y_i), h(y_i')$. Indeed, on one hand the two points are close:

$$d\big(h(y_i), h\big(y_i'\big)\big) \le d\big(h(y_i), h'\big(y_i'\big)\big) + d\big(h'\big(y_i'\big), h\big(y_i'\big)\big) < \delta + \frac{\epsilon}{5} < \frac{2}{5}\epsilon.$$

On the other hand, $S^1 \setminus S_i$ of S_i contains a point y_j with $j \in \{1,2,3\} \setminus \{i\}$; therefore $h(y_j) \notin h(S_i)$. Since $h(\mathbf{y}) \in K$ we have

$$d\big(h(y_i), h(y_j)\big) \geq r > 2\epsilon/5.$$

Hence $h(S_i)$ is the short arc defined by $2\epsilon/3$-close points $h(y_i), h(y_i')$, implying

$$d\big(h(x), h(y_i)\big) < \frac{2}{5}\epsilon \quad (x \in S_i).$$

Similarly, $h'(S_i)$ is the short arc defined by $2\epsilon/5$-close points $h'(y_i), h'(y_i')$, and

$$d\big(h'(x), h'(y_i)\big) < \frac{2}{5}\epsilon \quad (x \in S_i).$$

Since $y_i \in L_{ij}$, $d(h(y_i), h'(y_i)) < \epsilon/5$. Therefore for any $x \in S_i$

$$\begin{aligned} d\big(h(x), h'(x)\big) &\leq d\big(h(x), h(y_i)\big) + d\big(h(y_i), h'(y_i)\big) \\ &\quad + d\big(h'(x), h'(y_i)\big) < \epsilon. \end{aligned}$$

□

Continuation of the proof of Theorem 5.2 We claim that $H_\mu < \mathrm{Homeo}(S^1)$ is a *convergence group*, i.e., for any compact subset $K \subset \mathrm{Trp}$ the set

$$H(\mu, K) = \big\{h \in H_\mu \mid h^{-1}K \cap K \neq \emptyset\big\}$$

is compact. In particular, the Polish group H_μ is locally compact. Let us fix a compact subset $K \subset \mathrm{Trp}$. Since $H(\mu, K)$ is a closed subset in the Polish group $\mathrm{Homeo}(S^1)$, it suffices to show that any sequence $\{h_n\}_{n=1}^\infty$ in $H(\mu, K)$ contains a Cauchy subsequence. Choose triples $\mathbf{y}_n \in h_n^{-1}K \cap K$. Upon passing to a subsequence, we may assume that the points $\mathbf{y}_n$ converge to some $\mathbf{y} \in K$ and the points $\mathbf{z}_n = h_n(\mathbf{y}_n)$ converge to some $\mathbf{z} \in K$. Let $\epsilon > 0$. For the given ϵ and K let $\delta > 0$ be as in Lemma 5.3. Choose $N \in \mathbf{N}$ be large enough to ensure that $d(\mathbf{y}_n, \mathbf{y}_m) < \delta$ and $d(\mathbf{z}_n, \mathbf{z}_m) < \delta$ for all $n, m > N$. It follows from Lemma 5.3 that h_n and h_m are ϵ-close whenever $n, m > N$. This proves that H_μ is a convergence group on the circle.

Finally, it follows that H_μ is conjugate to a closed subgroup of G. For discrete groups this is a well known results of Gabai [20] and Casson–Jungreis [8]. The case of non-discrete convergence group $H_\mu < \mathrm{Homeo}(S^1)$ can be argued more directly. The closed convergence group H_μ is a locally compact subgroup of $\mathrm{Homeo}(S^1)$; the classification of all such groups is well known, and the only ones with convergence property are conjugate to $\mathrm{PGL}_2(\mathbb{R})$ [24, pp. 345–348, 16, pp. 51–54]. □

We return to the proof of Theorem A in case of $n = 2$. Starting from an integrable (G, H)-coupling (Ω, m) between $G = \mathrm{PGL}_2(\mathbb{R})$ and an unknown compactly generated unimodular group H a continuous representation $\rho : H \to \mathrm{Homeo}(S^1)$ with compact kernel and closed image was constructed. Theorem 5.2 implies that, up to conjugation, we may assume that

$$\bar{H} = \rho(H) < G = \mathrm{PGL}_2(\mathbb{R}).$$

Since $\bar{H}$ is measure equivalent to $G = \mathrm{PGL}_2(\mathbb{R})$, it is non-amenable.

Case (1): $\bar{H} < G = \mathrm{PGL}_2(\mathbb{R})$ is non-discrete. (This does not occur in the original formulation of Theorem A, but is included in the broader context of lcsc H adapted in this proof.) There are only two non-discrete non-amenable closed subgroups of G: the whole group G and its index two subgroup $G^0 = \mathrm{PSL}_2(\mathbb{R})$. Both of these groups may appear as $\bar{H}$; in fact, direct products of the form $H \cong G \times K$ or $H \cong G^0 \times K$ with compact K and certain almost direct products $G' \times K'/C$ as in [16, Theorem A] give rise to an integrable measure equivalence between H and G (cf. [16, Theorem C]).

Case (2): $\bar{H}$ is discrete. We claim that such $\bar{H}$ is a cocompact lattice in G. Indeed, every finitely generated discrete non-amenable subgroup of G is either cocompact or is virtually a free group $\mathbb{F}_2$. The latter possibility is ruled out by the following.

Lemma 5.4 *The free group $\mathbb{F}_2$ is not L^1-measure equivalent to G.*

Note that these groups are measure equivalent since $\mathbb{F}_2$ forms a lattice in G.

Proof Assuming $\mathbb{F}_2$ is L^1-measure equivalent to G, one can construct an integrable measure equivalence between G and the automorphism group $H = \mathrm{Aut}(\mathrm{Tree}_4)$ of the 4-regular tree, which contains $\mathbb{F}_2$ as a cocompact lattice. By Theorems C and 2.6 this would yield a continuous homomorphism $H \to \mathrm{Homeo}(S^1)$ with closed image. This leads to a contradiction, because H is totally disconnected and virtually simple [58, Théorème 4.5], while $\mathrm{Homeo}(S^1)$ has no non-discrete totally disconnected subgroups [24, Theorem 4.7 on p. 345]. □

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Appendix A: Measure equivalence

The appendix contains some general facts related to measure equivalence (Definition 1.1), the strong ICC property (Definition 2.2), and the notions of taut couplings and groups (Definition 1.3).

A.1 The category of couplings

Measure equivalence is an equivalence relation on unimodular lcsc groups. Let us describe explicitly the constructions which show reflexivity, symmetry and transitivity of measure equivalence.

A.1.1 Tautological coupling

The tautological coupling is the $(G \times G)$-coupling (G, m_G) given by $(g_1, g_2) : g \mapsto g_1 g g_2^{-1}$. It demonstrates reflexivity of measure equivalence.

A.1.2 Duality

Symmetry is implied by the following: Given a (G, H)-coupling (Ω, m) the dual (Ω^*, m^*) is the (H, G)-coupling Ω^* with the same underlying measure space (Ω, m) and the $H \times G$-action $(h, g) : \omega^* \mapsto (g, h)\omega^*$.

A.1.3 Composition of couplings

Compositions defined below shows that measure equivalence is a transitive relation. Let G_1, H, G_2 be unimodular lcsc groups, and (Ω_i, m_i) be a (G_i, H)-coupling for $i \in \{1, 2\}$. We describe the (G_1, G_2)-coupling $\Omega_1 \times_H \Omega_2^*$ modeled on the space of H-orbits on $(\Omega_1 \times \Omega_2, m_1 \times m_2)$ with respect to the diagonal H-action. Consider measure isomorphisms for (Ω_i, mi) as in (1.1): For $i \in \{1, 2\}$ there are finite measure spaces (X_i, μ_i) and (Y_i, ν_i), measure-preserving actions $G_i \curvearrowright (X_i, \mu_i)$ and $H \curvearrowright (Y_i, \nu_i)$, measurable cocycles $\alpha_i : G_i \times X_i \to H$ and $\beta_i : H \times Y_i \to G_i$, and measure space isomorphisms $G_i \times Y_i \cong \Omega_i \cong H \times X_i$ with respect to which the $G_i \times H$-actions are given by

$$(g_i, h) : \big(h', x\big) \mapsto \big(hh'\alpha_i(g_i, x)^{-1},\ g_i \,.\, x\big),$$
$$(g_i, h) : \big(g', y\big) \mapsto \big(g_i g' \beta(h, y)^{-1},\ h \,.\, y\big).$$

The space $\Omega_1 \times_H \Omega_2^*$ with its natural $G_1 \times G_2$-action is equivariantly isomorphic to $(X_1 \times X_2 \times H, \mu_1 \times \mu_2 \times m_H)$ endowed with the $G_1 \times G_2$-action

$$(g_1, g_2) : (x_1, x_2, h) \mapsto \big(g_1 \,.\, x_1,\ g_2 \,.\, x_2,\ \alpha_1(g_1, x_1) h \alpha_2(g_2, x_2)^{-1}\big).$$

To see that it is a (G_1, G_2)-coupling, we identify this space with $Z \times G_1$ equipped with the action

$$(g_1, g_2) : \bigl(g', z\bigr) \mapsto \bigl(g_1 g' c(g_2, z)^{-1}, g_2 . z\bigr) \quad \bigl(g' \in G_1,\ z \in Z\bigr)$$

where $Z = X_2 \times Y_1$, while the action $G_2 \curvearrowright Z$ and the cocycle $c : G_2 \times Z \to G_1$ are given by

$$\begin{aligned} g_2 &: (x, y) \mapsto \bigl(g_2 . x, \alpha_2(g_2, x) . y\bigr), \\ c\bigl(g_2, (x, y)\bigr) &= \beta_1\bigl(\alpha_2(g_2, x), y\bigr). \end{aligned} \tag{A.1}$$

Similarly, $\Omega_1 \times_H \Omega_2^* \cong W \times G_2$, for $W \cong X_1 \times Y_2$.

A.1.4 Morphisms

Let (Ω_i, m_i), $i \in \{1, 2\}$, be two (G, H)-couplings. Let $F : \Omega_1 \to \Omega_2$ be a measurable map such that for m_1-a.e. $\omega \in \Omega_1$ and every $g \in G$ and every $h \in H$

$$F\bigl((g, h)\omega\bigr) = (g, h)F(\omega).$$

Such maps are called *quotient maps* or *morphisms*.

A.1.5 Compact kernels

Let (Ω, m) be a (G, H)-coupling, and let

$$\{1\} \to K \to G \to \bar{G} \to \{1\}$$

be a short exact sequence where K is compact. Then the natural quotient space $(\bar{\Omega}, \bar{m}) = (\Omega, m)/K$ is a $(\bar{G}, H)$-coupling, and the natural map $F : \Omega \to \bar{\Omega}$, $F : \omega \mapsto K\omega$, is equivariant in the sense of $F((g, h)\omega) = (\bar{g}, h)F(\omega)$. This may be considered as an *isomorphism of couplings up to compact kernel.*

A.1.6 Passage to lattices

Let (Ω, m) be a (G, H)-coupling, and let $\Gamma < G$ be a lattice. By restricting the $G \times H$-action on (Ω, m) to $\Gamma \times H$ we obtain a (Γ, H)-coupling. Formally, this follows by considering (G, m_G) as a $\Gamma \times G$-coupling and considering the composition $G \times_G \Omega$ as Ω with the $\Gamma \times H$-action.

A.2 L^p-integrability conditions

Let G and H be compactly generated unimodular lcsc groups equipped with proper norms $|\cdot|_G$ and $|\cdot|_H$. Let $c : G \times X \to H$ be a measurable cocycle, and fix some $p \in [1, \infty)$. For $g \in G$ we define

$$\|g\|_{c,p} = \left(\int_X \big|c(g,x)\big|_H^p \, d\mu(x)\right)^{1/p}.$$

For $p = \infty$ we use the essential supremum. Assume that $\|g\|_{c,p} < \infty$ for a.e. $g \in G$. We claim that there are constants $a, A > 0$ so that for every $g \in G$

$$\|g\|_{c,p} \le A \cdot |g|_G + a. \tag{A.2}$$

Hence c is L^p-integrable in the sense of Definition 1.5. The key observation here is that $\| - \|_{c,p}$ is subadditive. Indeed, by the cocycle identity, subadditivity of the norm $| - |_H$, and the Minkowski inequality, for any $g_1, g_2 \in G$ we get

$$\begin{aligned}
\|g_2 g_1\|_{c,p} &\le \left(\int_X \left(\big|c(g_2, g_1 . x)\big|_H + \big|c(g_1,x)\big|_H\right)^p d\mu(x)\right)^{1/p} \\
&\le \left(\int_X \big|c(g_2, -)\big|_H^p \, d\mu\right)^{1/p} + \left(\int_X \big|c(g_1, -)\big|_H^p \, d\mu\right)^{1/p} \\
&= \|g_2\|_{c,p} + \|g_1\|_{c,p}.
\end{aligned}$$

For $t > 0$ denote $E_t = \{g \in G : \|g\|_{c,p} < t\}$. We have $E_t \cdot E_s \subseteq E_{s+t}$ for any $t, s > 0$. Fix t large enough so that $m_G(E_t) > 0$. By [11, Corollary 12.4 on p. 235], $E_{2t} \supseteq E_t \cdot E_t$ has a non-empty interior. Hence any compact subset of G can be covered by finitely many translates of E_{2t}. The subadditivity implies that $\|g\|_{c,p}$ is bounded on compact sets. This gives (A.2).

Lemma A.1 *Let G,H,L be compactly generated groups, $G \curvearrowright (X,\mu)$, $H \curvearrowright (Y,\nu)$ be finite measure-preserving actions, and $\alpha : G \times X \to H$ and $\beta : H \times Y \to L$ be L^p-integrable cocycles for some $1 \le p \le \infty$. Consider $Z = X \times Y$ and $G \curvearrowright Z$ by $g : (x, y) \mapsto (g . x, \alpha(g,x) . y)$. Then the cocycle $\gamma : G \times Z \to L$ given by*

$$\gamma\big(g, (x, y)\big) = \beta\big(\alpha(g, x), y\big).$$

is L^p-integrable.

Proof For $p = \infty$ the claim is obvious. Assume $p < \infty$. Let A, a, B, b be constants such that $\|h\|_{\beta,p} \leq B \cdot |h|_H + b$ and $\|g\|_{\alpha,p} \leq A \cdot |g|_G + a$. Then

$$\begin{aligned}
\|g\|_{\gamma,p}^p &= \int_{X\times Y} \left|\beta\big(\alpha(g,x), y\big)\right|_{\mathrm{L}}^p \, d\mu(x)\, d\nu(y) \\
&\leq \int_X \big(B \cdot \left|\alpha(g,x)\right|_H + b\big)^p \, d\mu(x) \\
&\leq \max(B,b)^p \cdot \|g\|_{\alpha,p}^p \leq (C \cdot |g|_G + c)^p
\end{aligned}$$

for appropriate constants $c > 0$ and $C > 0$. $\square$

Lemma A.2 *Let G_1, H, G_2 be compactly generated unimodular lcsc groups. For $i \in \{1,2\}$ let (Ω_i, m_i) be an L^p-integrable (G_i, H)-coupling. Then $\Omega_1 \times_H \Omega_2^*$ is an L^p-integrable (G_1, G_2)-coupling.*

Proof This follows from Lemma A.1 using the explicit description (A.1) of the cocycles for $\Omega_1 \times_H \Omega_2^*$. $\square$

We conclude that for each $1 \leq p \leq \infty$, L^p-measure equivalence is an equivalence relation between compactly generated unimodular lcsc groups.

A.3 Tautening maps

Lemma A.3 *Let G be a lcsc group, Γ a countable group and $j_1, j_2 : \Gamma \to G$ be homomorphisms with $\Gamma_i = j_i(\Gamma)$ being lattices in G. Assume that G is taut (resp. p-taut and Γ_i are L^p-integrable). Then there exists $g \in G$ so that*

$$j_2(\gamma) = g\, j_1(\gamma)\, g^{-1} \quad (\gamma \in \Gamma).$$

If $\pi : G \to \mathcal{G}$ is a continuous homomorphism into a Polish group and G is taut relative to $\pi : G \to \mathcal{G}$ (resp. G is p-taut relative to $\pi : G \to \mathcal{G}$ and Γ_i are L^p-integrable) then there exists $y \in \mathcal{G}$ with

$$\pi\big(j_2(\gamma)\big) = y\pi\big(j_1(\gamma)\big)y^{-1} \quad (\gamma \in \Gamma).$$

Proof We prove the more general second statement. The group

$$\Delta = \big\{\big(j_1(\gamma), j_2(\gamma)\big) \in G \times G \mid \gamma \in \Gamma\big\}$$

is a closed discrete subgroup in $G \times G$. The $G \times G$-space $\Omega = (G \times G)/\Delta$ equipped with the $G \times G$-invariant measure is easily seen to be a (G, G)-coupling. It will be L^p-integrable if Γ_1 and Γ_2 are L^p-integrable lattices. Let

$\Phi : \Omega \to \mathcal{G}$ be the tautening map. There are $a, b \in G$ and $x \in \mathcal{G}$ such that for all $g_1, g_2 \in G$

$$\Phi\big((g_1 a, g_2 b)\Delta_f\big) = \pi(g_1) x \pi(g_2)^{-1}.$$

Since (a, b) and $(j_1(\gamma)^a a,\ j_2(\gamma)^b b)$ are in the same Δ-coset, where $g^h = hgh^{-1}$, we get for all $g_1, g_2 \in G$ and every $\gamma \in \Gamma$

$$\pi(g_1) x \pi(g_2)^{-1} = \pi(g_1)\pi\big(j_1(\gamma)^a\big) x \pi\big(j_2(\gamma)^b\big)^{-1} \pi(g_2)^{-1}.$$

This implies that j_1 and j_2 are conjugate homomorphisms. □

The following lemma relates tautening maps $\Phi : \Omega \to G$ and cocycle rigidity for ME-cocycles.

Lemma A.4 *Let G be a unimodular lcsc group, $\mathcal{G}$ be a Polish group, $\pi : G \to \mathcal{G}$ a continuous homomorphism. Let (Ω, m) be a (G, G)-coupling and $\alpha : G \times X \to G$, $\beta : G \times Y \to G$ be the corresponding ME-cocycles. Then there is a tautening map $\Omega \to \mathcal{G}$ iff the $\mathcal{G}$-valued cocycle $\pi \circ \alpha$ is conjugate to π, that is,*

$$\pi \circ \alpha(g, x) = f(g \,.\, x)^{-1} \pi(g) f(x).$$

Moreover, Ω is taut relative to π if such a measurable map $f : X \to \mathcal{G}$ is unique. This is also equivalent to $\pi \circ \beta$ being uniquely conjugate to $\pi : G \to \mathcal{G}$.

Proof Let $\alpha : G \times X \to G$ be the ME-cocycle associated to a measure space isomorphism $i : (G, m_G) \times (X, \mu) \to (\Omega, m)$ as in (1.1). In particular,

$$(g_1, g_2) : i(g, x) \mapsto i\big(g_2 g \alpha(g_1, x)^{-1},\ g_1 \,.\, x\big).$$

We shall now establish a 1-to-1 correspondence between Borel maps $f : X \to \mathcal{G}$ with

$$\pi \circ \alpha(g, x) = f(g \,.\, x)^{-1} \pi(g) f(x)$$

and tautening maps $\Phi : \Omega \to \mathcal{G}$. Given f as above one verifies that

$$\Phi : \Omega \to \mathcal{G}, \qquad \Phi\big(i(g, x)\big) = f(x)\pi(g)^{-1}$$

is $G \times G$-equivariant.

For the converse direction, suppose $\Phi : \Omega \to G$ is a tautening map. Thus,

$$g_1 \Phi(g_0, x) g_2^{-1} = \Phi\big((g_1, g_2)(g_0, x)\big) = \Phi\big(g_2 g_0 \alpha(g_1, x)^{-1},\ g_1 \,.\, x\big).$$

For μ-a.e. $x \in X$ and a.e. $g \in G$ the value of $\Phi(g, x) g$ is constant $f(x)$. If we substitute $g_0 = g_1 = g$ and $g_2 = g\alpha(g, x) g^{-1}$ in the above identity, then we obtain $\alpha(g, x) = f(g \,.\, x)^{-1} g f(x)$. □

Lemma A.5 *Let G be a unimodular lcsc group, $\mathcal{G}$ be a Polish group, $\pi : G \to \mathcal{G}$ a continuous homomorphism. Then G is (p-)taut, that is every (p-integrable) (G, G)-coupling is taut relative to π, iff every ergodic (p-integrable) (G, G)-coupling is taut relative to π.*

Proof We give the proof in the p-integrable case, the case without the integrability condition being simpler. We assume that every ergodic p-integrable (G, G)-coupling is taut relative to π and let (Ω, m) be an arbitrary p-integrable coupling. We fix a fundamental domain (X, μ) for the second G action such that the associated cocycle $\alpha : G \times X \to G$ is p-integrable.

Let $\mu = \int \mu_t \, d\eta(t)$ be the G-ergodic decomposition of (X, μ). By [15, Lemma 2.2] it corresponds to the $(G \times G)$-ergodic decomposition of (Ω, m) into ergodic couplings (Ω, m_t), so that $m = \int m_t \, d\eta(t)$.

Let $|\_|\colon G \to \mathbb{N}$ be the length function associated to some word-metric on G. The p-integrability of α means that

$$\int \int_X \left|\alpha(g, x)\right|^p d\mu_t(x)\, d\eta(t) = \int_X \left|\alpha(g, x)\right|^p d\mu(x) < \infty$$

for every $g \in G$. By Fubini's theorem $\int_X |\alpha(g, x)|^p \, d\mu_t(x) < \infty$ for η-a.e. t. Hence (Ω, m_t) is p-integrable for η-a.e. m_t, and in particular it is taut relative to π, by our assumption. It follows by Lemma A.4 that the cocycle $\pi \circ \alpha$ is conjugate to the constant cocycle π over η-a.e. (X, μ_t). Then by [13, Corollary 3.6][10] $\pi \circ \alpha$ is conjugate to π over (X, μ). Again, by Lemma A.4 we conclude that (Ω, m) is taut relative to π. □

A.4 Strong ICC property

Lemma A.6 *Let G be a unimodular lcsc group, $\mathcal{G}$ a Polish group, $\pi : G \to \mathcal{G}$ a continuous homomorphism. Let $\Gamma < G$ be a lattice. Then $\mathcal{G}$ is strongly ICC relative to $\pi(G)$ if and only if it is strongly ICC relative to $\pi(\Gamma)$.*

Proof Clearly if $\mathcal{G}$ is strongly ICC relative to $\pi(\Gamma)$ then it is also strongly ICC relative to $\pi(G)$. The reverse implication follows by averaging a $\pi(\Gamma)$ invariant measure over G/Γ. □

Lemma A.7 *Let G be a unimodular lcsc group, $\mathcal{G}$ a Polish group, $\pi : G \to \mathcal{G}$ a continuous homomorphism. Suppose that $\mathcal{G}$ is not strongly ICC relative to $\pi(G)$. Then there is a (G, G)-coupling (Ω, m) with two distinct tautening maps to $\mathcal{G}$.*

[10] The target $\mathcal{G}$ is assumed to be locally compact in this reference but the proof therein works the same for a Polish group $\mathcal{G}$.

Proof Let μ be a Borel probability measure on $\mathcal{G}$ invariant under conjugations by $\pi(G)$. Consider $\Omega = G \times \mathcal{G}$ with the measure $m = m_G \times \mu$ where m_G denotes the Haar measure, and measure-preserving $G \times G$-action

$$(g_1, g_2) : (g, x) \mapsto \left(g_1 g g_2^{-1},\, \pi(g_2)\, x\, \pi(g_2)^{-1}\right).$$

This is clearly a (G, G)-coupling and the following measurable maps $\Phi_i : \Omega \to \mathcal{G}$, $i \in \{1, 2\}$, are $G \times G$-equivariant: $\Phi_1(g, x) = \pi(g)$ and $\Phi_2(g, x) = \pi(g) \cdot x$. Note that $\Phi_1 = \Phi_2$ on a conull set iff $\mu = \delta_e$. □

Lemma A.8 *Let G be a unimodular lcsc group and $\mathcal{G}$ a Polish group. Assume that $\mathcal{G}$ is strongly ICC relative to $\pi(G)$. Let (Ω, m) be a (G, G)-coupling. Then*:

(1) *There is at most one tautening map $\Phi\colon \Omega \to \mathcal{G}$.*
(2) *Let $F : (\Omega, m) \to (\Omega_0, m_0)$ be a morphism of (G, G)-couplings and suppose that there exists a tautening map $\Phi : \Omega \to \mathcal{G}$. Then it descends to Ω_0, i.e., $\Phi = \Phi_0 \circ F$ for a unique tautening map $\Phi_0 : \Omega_0 \to \mathcal{G}$.*
(3) *If $\Gamma_1, \Gamma_2 < G$ are lattices, then $\Phi : \Omega \to \mathcal{G}$ is unique as a $\Gamma_1 \times \Gamma_2$-equivariant map.*
(4) *If $\Gamma_1, \Gamma_2 < G$ are lattices, and (Ω, m) admits a $\Gamma_1 \times \Gamma_2$-equivariant map $\Phi : \Omega \to \mathcal{G}$, then Φ is $G \times G$-equivariant.*
(5) *If $\eta : \Omega \to \mathrm{Prob}(\mathcal{G})$, $\omega \mapsto \eta_\omega$, is a measurable $G \times G$-equivariant map to the space of Borel probability measures on $\mathcal{G}$ endowed with the weak topology, then it takes values in Dirac measures*: *We have $\eta_\omega = \delta_{\Phi(\omega)}$, where $\Phi : \Omega \to \mathcal{G}$ is the unique tautening map.*

Proof We start from the last claim and deduce the other ones from it.
(5). Given an equivariant map $\eta : \Omega \to \mathrm{Prob}(\mathcal{G})$ consider the convolution

$$\nu_\omega = \check{\eta}_\omega * \eta_\omega,$$

namely the image of $\eta_\omega \times \eta_\omega$ under the map $(a, b) \mapsto a^{-1} \cdot b$. Then

$$\nu_{(g,h)\omega} = \nu_\omega^{\pi(g)} \quad (g, h \in G),$$

where the latter denotes the push-forward of ν_ω under the conjugation

$$a \mapsto a^{\pi(g)} = \pi(g)^{-1} a \pi(g).$$

In particular, the map $\omega \mapsto \nu_\omega$ is invariant under the action of the second G-factor. Therefore ν_ω descends to a measurable map $\tilde{\nu} : \Omega/G \to \mathrm{Prob}(\mathcal{G})$, satisfying

$$\tilde{\nu}_{g.x} = \tilde{\nu}_x^{\pi(g)} \quad (x \in X = \Omega/G,\ g \in G).$$

Here we identify Ω/G with a finite measure space (X,μ) as in (1.1). Consider the center of mass

$$\bar{\nu} = \frac{1}{\mu(X)} \int_X \tilde{\nu}_x \, d\mu(x).$$

It is a probability measure on $\mathcal{G}$, which is invariant under conjugations. By the strong ICC property relative to $\pi(G)$ we get $\bar{\nu} = \delta_e$. Since δ_e is an extremal point of $\mathrm{Prob}(\mathcal{G})$, it follows that m-a.e. $\nu(\omega) = \delta_e$. This implies that $\eta_\omega = \delta_{\Phi(\omega)}$ for some measurable $\Phi : \Omega \to \mathcal{G}$. The latter is automatically $G \times G$-equivariant.

(1). If $\Phi_1, \Phi_2 : \Omega \to \mathcal{G}$ are tautening maps, then $\eta_\omega = \frac{1}{2}(\delta_{\Phi_1(\omega)} + \delta_{\Phi_2(\omega)})$ is an equivariant map $\Omega \to \mathrm{Prob}(\mathcal{G})$. By (5) it takes values in Dirac measures, which is equivalent to the m-a.e. equality $\Phi_1 = \Phi_2$.

(2). Disintegration of m with respect to m_0 gives a $G \times G$-equivariant measurable map $\Omega_0 \to \mathcal{M}(\Omega)$, $\omega \mapsto m_{\omega_0}$, to the space of finite measures on Ω. Then the map $\eta : \Omega_0 \to \mathrm{Prob}(\mathcal{G})$, given by

$$\eta_{\omega_0} = \|m_{\omega_0}\|^{-1} \cdot \Phi_*(m_{\omega_0})$$

is $G \times G$-equivariant. Hence by (5), $\eta_{\omega_0} = \delta_{\Phi_0(\omega_0)}$ for the unique tautening map $\Phi_0 : \Omega_0 \to \mathcal{G}$. The relation $\Phi = \Phi_0 \circ F$ follows from the fact that Dirac measures are extremal.

(3) follows from (4) and (1).

(4). The claim is equivalent to: For m-a.e. $\omega \in \Omega$ the map $F_\omega \colon G \times G \to \mathcal{G}$ with

$$F_\omega(g_1, g_2) = \pi(g_1)^{-1} \, \Phi\big((g_1, g_2)\omega\big) \, \pi(g_2)$$

is $m_G \times m_G$-a.e. constant $\Phi_0(\omega)$. Note that the family $\{F_\omega\}$ has the following equivariance property: For $g_1, g_2, h_1, h_2 \in G$ we have

$$\begin{aligned} F_{(h_1,h_2)\omega}(g_1, g_2) &= \pi(g_1)^{-1} \Phi\big((g_1 h_1, g_2 h_2)\omega\big) \pi(g_2) \\ &= \pi(h)_1^{-1} F_\omega(g_1 h_1, g_2 h_2) \pi(h_2). \end{aligned}$$

Since Φ is $\Gamma_1 \times \Gamma_2$-equivariant, for m-a.e. $\omega \in \Omega$ the map F_ω descends to $G/\Gamma_1 \times G/\Gamma_2$. Let $\eta_\omega \in \mathrm{Prob}(\mathcal{G})$ denote the distribution of $F_\omega(\cdot, \cdot)$ over the probability space $G/\Gamma_1 \times G/\Gamma_2$, that is, for a Borel subset $E \subset \mathcal{G}$

$$\eta_\omega(E) = m_{G/\Gamma_1} \times m_{G/\Gamma_2} \big\{(g_1, g_2) \mid F_\omega(g_1, g_2) \in E\big\}.$$

Since this measure is invariant under translations by $G \times G$, it follows that η_ω is a $G \times G$-equivariant maps $\Omega \to \mathrm{Prob}(\mathcal{G})$. By (5) one has $\eta_\omega = \delta_{f(\omega)}$ for some measurable $G \times G$-equivariant map $f : \Omega \to \mathcal{G}$. Hence $F_\omega(g_1, g_2) =$

$f(\omega)$ for a.e. $g_1, g_2 \in G$; it follows that

$$\Phi\big((g_1, g_2)\omega\big) = \pi(g_1)\Phi(\omega)\pi(g_2)^{-1} \tag{A.3}$$

holds for $m_G \times m_G \times m$-a.e. (g_1, g_2, ω). □

Corollary A.9 *Let $\pi : G \to \mathcal{G}$ be as above and assume that $\mathcal{G}$ is strongly ICC relative to $\pi(G)$. Then the collection of all (G, G)-couplings which are taut relative to $\pi : G \to \mathcal{G}$ is closed under the operations of taking the dual, compositions, quotients and extensions.*

Proof The uniqueness of tautening maps follow from the relative strong ICC property (Lemma A.8.(1)). Hence we focus on the existence of such maps.

Let $\Phi : \Omega \to \mathcal{G}$ be a tautening map. Then $\Psi(\omega^*) = \Phi(\omega)^{-1}$ is a tautening map $\Omega^* \to \mathcal{G}$.

Let $\Phi_i : \Omega_i \to \mathcal{G}$, $i = 1, 2$, be tautening maps. Then $\Psi([\omega_1, \omega_2]) = \Phi(\omega_1) \cdot \Phi(\omega_2)$ is a tautening map $\Omega_1 \times_G \Omega_2 \to \mathcal{G}$.

If $F : (\Omega_1, m_1) \to (\Omega_2, m_2)$ is a quotient map and $\Phi_1 : \Omega_1 \to \mathcal{G}$ is a tautening map, then, by Lemma A.8.(2), Φ_1 factors as $\Phi_1 = \Phi_2 \circ F$ for a tautening map $\Phi_2 : \Omega_2 \to \mathcal{G}$. On the other hand, given a tautening map $\Phi_2 : \Omega_2 \to \mathcal{G}$, the map $\Phi_1 = \Phi_2 \circ F$ is tautening for Ω_1. □

Appendix B: Bounded cohomology

Our background references for bounded cohomology, especially for the functorial approach to it, are [6, 41]. We summarize what we need from Burger-Monod's theory of bounded cohomology. Since we restrict to discrete groups, some results we quote from this theory already go back to Ivanov [31].

B.1 Banach modules

All Banach spaces are over the field $\mathbb{R}$ of real numbers. By the dual of a Banach space we understand the normed topological dual. The dual of a Banach space E is denoted by E^*. Let Γ be a discrete and countable group. A *Banach Γ-module* is a Banach space E endowed with a group homomorphism π from Γ into the group of isometric linear automorphisms of E. We use the module notation $\gamma \cdot e = \pi(\gamma)(e)$ or just $\gamma e = \pi(\gamma)(e)$ for $\gamma \in \Gamma$ and $e \in E$ whenever the action is clear from the context. The submodule of Γ-invariant elements is denoted by E^Γ. Note that $E^\Gamma \subset E$ is closed.

If E and F are Banach Γ-modules, a *Γ-morphism* $E \to F$ is a Γ-equivariant continuous linear map. The space $\mathcal{B}(E, F)$ of continuous, linear maps $E \to F$ is endowed with a natural Banach Γ-module structure via

$$(\gamma \cdot f)(e) = \gamma f\big(\gamma^{-1} e\big). \tag{B.1}$$

The *contragredient* Banach Γ-module structure $E^\sharp$ associated to E is by definition $\mathcal{B}(E,\mathbb{R})=E^*$ with the Γ-action (B.1). A *coefficient Γ-module* is a Banach Γ-module E contragredient to some separable continuous Banach Γ-module denoted by $E^\flat$. The choice of $E^\flat$ is part of the data. The specific choice of $E^\flat$ defines a weak-$*$ topology on E. The only examples that appear in this paper are $E=\mathrm{L}^\infty(X,\mu)$ with $E^\flat=L^1(X,\mu)$ and $E=E^\flat=\mathbb{R}$.

For a coefficient Γ-module E let $\mathrm{C}_\mathrm{b}^k(\Gamma,E)$ be the Banach Γ-module $\mathrm{L}^\infty(\Gamma^{k+1},E)$ consisting of bounded maps from Γ^{k+1} to E endowed with the supremum norm and the Γ-action:

$$(\gamma\cdot f)(\gamma_0,\dots,\gamma_k)=\gamma\cdot f\big(\gamma^{-1}\gamma_0,\dots,\gamma^{-1}\gamma_k\big). \tag{B.2}$$

For a coefficient Γ-module E and a standard Borel Γ-space S with quasi-invariant measure let $\mathrm{L}^\infty_{\mathrm{w}*}(S,E)$ be the space of weak-$*$-measurable essentially bounded maps from S to E, where maps are identified if they only differ on a null set. The space $\mathrm{L}^\infty_{\mathrm{w}*}(S,E)$ is endowed with the essential supremum norm and the Γ-action (B.2). For a measurable space X the Banach space $\mathcal{B}^\infty(X,E)$ is the space of weak-$*$-measurable bounded maps from X to E endowed with supremum norm [4, Sect. 2] and the Γ-action (B.2).

B.2 Injective resolutions

Let Γ be a discrete group and E be a Banach Γ-module. The sequence of Banach Γ-modules $\mathrm{C}_\mathrm{b}^k(\Gamma,E)$, $k\geq 0$, becomes a chain complex of Banach Γ-modules via the standard homogeneous coboundary operator

$$d(f)(\gamma_0,\dots,\gamma_k)=\sum_{i\geq 0}^{k}(-1)^i f(\gamma_0,\dots,\hat{\gamma}_i,\dots,\gamma_k). \tag{B.3}$$

The *bounded cohomology* $\mathrm{H}_\mathrm{b}^\bullet(\Gamma,E)$ of Γ with coefficients E is the cohomology of the complex of Γ-invariants $\mathrm{C}_\mathrm{b}^\bullet(\Gamma,E)^\Gamma$. The bounded cohomology $\mathrm{H}_\mathrm{b}^\bullet(\Gamma,E)$ inherits a semi-norm from $\mathrm{C}_\mathrm{b}^\bullet(\Gamma,E)$: The *(semi-)norm* of an element $x\in\mathrm{H}_\mathrm{b}^k(\Gamma,E)$ is the infimum of the norms of all cocycles in the cohomology class x.

Next we briefly recall the functorial approach to bounded cohomology as introduced by Ivanov [31] for discrete groups and further developed by Burger-Monod [6, 41]. We refer for the definition of *relative injectivity* of a Banach Γ-module to [41, Definition 4.1.2 on p. 32]. A *strong resolution* $E^\bullet$ of E is a resolution, i.e. an acyclic complex,

$$0\to E\to E^0\to E^1\to E^2\to\cdots$$

of Banach Γ-modules that has a chain contraction which is contracting with respect to the Banach norms. The key to the functorial definition of bounded cohomology are the following two theorems:

Theorem B.1 ([6, Proposition 1.5.2]) *Let E and F be Banach Γ-modules. Let $E^\bullet$ be a strong resolution of E. Let $F^\bullet$ be a resolution F by relatively injective Banach Γ-modules. Then any Γ-morphism $E \to F$ extends to a Γ-morphism of resolutions $E^\bullet \to F^\bullet$ which is unique up to Γ-homotopy. Hence $E \to F$ induces functorially continuous linear maps* $\mathrm{H}^\bullet(E^{\bullet\Gamma}) \to \mathrm{H}^\bullet(F^{\bullet\Gamma})$.

Theorem B.2 ([41, Corollary 7.4.7 on p. 80]) *Let E be a Banach Γ-module. The complex $E \to \mathrm{C}_\mathrm{b}^\bullet(\Gamma, E)$ with $E \to \mathrm{C}_\mathrm{b}^0(\Gamma, E)$ being the inclusion of constant functions is a strong, relatively injective resolution.*

For a coefficient Γ-module, a measurable space X with measurable Γ-action, and a standard Borel Γ-space S with quasi-invariant measure we obtain chain complexes $\mathcal{B}^\infty(X^{\bullet+1}, E)$ and $\mathrm{L}_{\mathrm{w}*}(S^{\bullet+1}, E)$ of Banach Γ-modules via the standard homogeneous coboundary operators (similar as in (B.3)).

The following result is important for expressing induced maps in bounded cohomology in terms of boundary maps [4].

Proposition B.3 ([4, Proposition 2.1]) *Let E be a coefficient Γ-module. Let X be a measurable space with measurable Γ-action. The complex $E \to \mathcal{B}^\infty(X^{\bullet+1}, E)$ with $E \to \mathcal{B}^\infty(X, E)$ being the inclusion of constant functions is a strong resolution of E.*

The next theorem is one of the main results of the functorial approach to bounded cohomology by Burger-Monod:

Theorem B.4 ([6, Corollary 2.3.2, 41, Theorem 7.5.3 on p. 83]) *Let S be a regular Γ-space and be E a coefficient Γ-module. Then $E \to \mathrm{L}_{\mathrm{w}*}(S^{\bullet+1}, E)$ with $E \to \mathrm{L}_{\mathrm{w}*}(S^{\bullet+1}, E)$ being the inclusion of constant functions is a strong resolution. If, in addition, S is amenable in the sense of Zimmer* [41, *Definition* 5.3.1], *then each $\mathrm{L}_{\mathrm{w}*}(S^{k+1}, E)$ is relatively injective, and according to Theorem B.1 the cohomology groups $\mathrm{H}^\bullet(\mathrm{L}_{\mathrm{w}*}(S^{\bullet+1}, E)^\Gamma)$ are canonically isomorphic to $\mathrm{H}_\mathrm{b}^\bullet(\Gamma, E)$.*

Definition B.5 Let S be a standard Borel Γ-space with a quasi-invariant probability measure μ. Let E be a coefficient Γ-module. The *Poisson transform* $\mathrm{PT}^\bullet : \mathrm{L}_{\mathrm{w}*}(S^{\bullet+1}, E) \to \mathrm{C}_\mathrm{b}^\bullet(\Gamma, E)$ is the Γ-morphism of chain complexes defined by

$$\mathrm{PT}^k(f)(\gamma_0, \dots, \gamma_k) = \int_{S^{k+1}} f(\gamma_0 s_0, \dots, \gamma_k s_k)\, d\mu(s_0) \cdots d\mu(s_k).$$

If S is amenable, then the Poisson transform induces a canonical isomorphism in cohomology (Theorem B.4). By the same theorem this isomorphism does not depend on the choice of μ within a given measure class.

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