LATTICE ENVELOPES

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Abstract

We introduce a class of countable groups by some abstract group-theoretic conditions. This class includes linear groups with finite amenable radical and finitely generated residually finite groups with some nonvanishing ℓ^2-Betti numbers that are not virtually a product of two infinite groups. Further, it includes acylindrically hyperbolic groups. For any group Γ in this class, we determine the general structure of the possible lattice embeddings of Γ, that is, of all compactly generated, locally compact groups that contain Γ as a lattice. This leads to a precise description of possible nonuniform lattice embeddings of groups in this class. Further applications include the determination of possible lattice embeddings of fundamental groups of closed manifolds with pinched negative curvature.

Contents

1. Introduction

1.1. Motivation and background

Let G be a *locally compact second countable* (*lcsc*)[1] group. Such a group carries a nonzero left-invariant Radon measure unique up to scalar multiple, known as the *Haar measure*. A subgroup $\Gamma < G$ is called a *lattice* if it is discrete and G/Γ carries a finite G-invariant measure or, equivalently, if the Γ-action on G admits a Borel fundamental

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[1]If G is, in addition, totally disconnected, then we call G a *tdlc* group.

domain of finite Haar measure. If G/Γ is compact, we say that Γ is a *uniform lattice*; otherwise, we say that Γ is a *nonuniform* lattice. The inclusion $\Gamma \hookrightarrow G$ is called a *lattice embedding*. We will also call G a *lattice envelope* of Γ.

The classical examples of lattices come from geometry and arithmetic. Starting from a locally symmetric manifold M of finite volume, we obtain a lattice embedding $\pi_1(M) \hookrightarrow \mathrm{Isom}(\tilde{M})$ of its fundamental group into the isometry group of its universal covering via the action by deck transformations. The real Lie group $\mathrm{Isom}(\tilde{M})$ is semisimple if $\tilde{M}$ has no Euclidean direct factors, and the lattice $\Gamma < \mathrm{Isom}(\tilde{M})$ is uniform if and only if M is compact.

Real semisimple Lie groups viewed as algebraic groups admit *arithmetic lattices* such as $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ (see [50, Theorem 3.2.4, p. 63]). Analogous constructions exist for products of real and p-adic algebraic semisimple groups, such as

$$\mathrm{SL}_d\left(\mathbb{Z}\left[\frac{1}{p_1},\dots,\frac{1}{p_k}\right]\right) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{SL}_d(\mathbb{Q}_{p_k}).$$

Our notation and conventions for arithmetic lattices are defined in Section 4.1 below.

A central theme in the study of lattices are the connections between the lattices and the ambient group. Mostow's strong rigidity (see [55] for the uniform case and [60] or [48] for the nonuniform case) asserts that an irreducible lattice in a semisimple real Lie group G that is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ determines the ambient Lie group among all Lie groups and determines its embedding in the ambient Lie group uniquely up to an automorphism of G.

It is natural to ask to what extent lattices determine their lattice envelopes among all lcsc groups and which countable groups have only trivial lattice envelopes. To make the question precise, we introduce the notion of *virtual isomorphism* between lattice embeddings in Definition 3.1. A lattice embedding is called *trivial* if it is virtually isomorphic to the identity homomorphism of a countable discrete group, which can be regarded as a lattice embedding. Virtual isomorphism is an equivalence relation. We refer to Section 3.1 for more information.

Problem 1.1
Given a countable group Γ, describe all of its possible lattice envelopes up to virtual isomorphism.

In the study of lattices, the nonuniform lattices are often harder to work with when compared to uniform ones. For example, the condition of *integrability* of lattices (e.g., required for different purposes in [5], [39], [49], [53], [67]) is automatically satisfied by uniform lattices and can be proved for some examples of nonuniform lattices, often using elaborate arguments. This motivates the following.

Problem 1.2
How prevalent are nonuniform lattice embeddings? What groups admit both nontrivial uniform and nonuniform embeddings?

1.2. On the structure of lattice envelopes
To state Theorem A below, which is a step toward Problem 1.1, we need to put some conditions on the group Γ.

Definition 1.3
We say that a countable group Γ has property

- (BT) if there is an upper bound on the order of finite subgroups of Γ,
- (Irr) if Γ is not virtually isomorphic to a product of two infinite groups,
- (CAF) if every amenable commensurated subgroup of Γ is finite,
- (wNbC) if for every normal subgroup $N \lhd \Gamma$ and every commensurated subgroup $M < \Gamma$ such that $N \cap M = \{1\}$ there is a finite-index subgroup of M that commutes with N,
- (NbC) if every quotient of Γ by a finite normal subgroup has (wNbC).

The abbreviations stand for *bound on torsion (BT)*, *irreducibility (Irr)*, *commensurated amenable is finite (CAF)*, *weak normal by commensurated (wNbC)*, and *normal by commensurated (NbC)*, respectively (for the definition of commensurated subgroup, see Section 2.1).

In Section 2 we prove the following result, which shows that large classes of groups satisfy the above conditions, and we also describe these classes in more detail.

THEOREM 1.4
The following countable groups have property (CAF)*:*
(1) *groups with a nonvanishing ℓ^2-Betti number in some degree,*
(2) *linear groups with finite amenable radical,*
(3) *groups in the class $\mathcal{D}_{\mathrm{reg}}$ introduced in [71] (this class contains all acylindrically hyperbolic groups and thus nonelementary convergence groups).*
Further, the following countable groups have property (NbC)*:*
(1′) *groups with nonvanishing first ℓ^2-Betti number,*
(2′) *linear groups,*
(3′) *finitely generated residually finite groups with finite amenable radical,*
(4′) *groups in the class $\mathcal{D}_{\mathrm{reg}}$.*
The groups in (1′) and (4′) also have the property (Irr).

The statement about (NbC) for the groups of class $(3')$ is due to [20, Corollary 18] and the fact that the class $(3')$ is closed under passing to quotients by finite normal subgroups.

The next theorem was announced in [7, Theorem 1.3] with a slightly stronger formulation of the (NbC) and (CAF) condition. It gives a partial answer to Problem 1.1 that concerns possible lattice envelopes of a given group Γ. We impose no restriction on the lattice envelope, except for it being compactly generated. This condition can also be removed if Γ is assumed to be finitely generated (see Lemma 3.7).

THEOREM A (Structure of possible lattice envelopes)
Let Γ be a countable group with properties (Irr), (CAF), *and* (NbC), *and let $\Gamma < G$ be a nontrivial lattice embedding into a compactly generated lcsc group G. Then the lattice embedding $\Gamma \hookrightarrow G$ is virtually isomorphic to one of the following lattice embeddings:*

(1) *an irreducible lattice in a connected, center-free, semisimple real Lie group without compact factors;*
(2) *an S-arithmetic lattice or an S-arithmetic lattice up to tree extension as in Definition 4.4;*
(3) *a lattice in a nondiscrete totally disconnected locally compact group with trivial amenable radical.*

If in addition Γ has (BT), *then the lattice in (3) is uniform.*

(For the notion of tree extension, see Definition 4.3. The three cases are distinct; see Remark 4.2. We show in Example 2.14 the necessity of the condition (NbC) for Theorem A.)

The classical case of lattices in semisimple real Lie groups includes both arithmetic and nonarithmetic examples. (Nonarithmetic lattices can appear only in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$.) The case of S-arithmetic lattices in the above statement refers to irreducible lattices in a product of finitely many real semisimple Lie groups and p-adic ones, where factors of both types are present. Case (3) contains a large class of examples that includes lattices in p-adic groups and fundamental groups Γ of some finite cube complexes whose universal cover has a nondiscrete group of automorphisms. This last case remains quite mysterious although a structure theory of nondiscrete, simple, totally disconnected groups emerged in the last decade. (In this regard, Caprace's survey [19] is recommended.)

Remark 1.5
The structure of lattices and their envelopes has been closely connected to (super) rigidity questions ever since the celebrated works of Mostow and of Margulis in the

1970's. Since the 1990's there has a been a surge of research activity toward extending rigidity theory from Lie and algebraic groups to locally compact groups (e.g., [17], [18], [21], [35], [39], [53], [67]). Remarkable results about the structure of lattices in locally compact groups went hand in hand with it (e.g., [18], [22], [23], [67]). Theorem A builds heavily on these previous works, and its statement should be compared with the results in [22, Section 5] by Caprace and Monod. The novelty in the statement of Theorem A is that we neither assume that G is a product nor impose any linearity assumption on Γ; we only impose conditions on Γ, and they are of an abstract group-theoretical nature.

By assuming property (BT), the last case of Theorem A does not allow nonuniform lattices. Hence, we obtain the following partial solution to Problem 1.2.

COROLLARY 1.6 (Classification of nonuniform lattices)
Let Γ be a countable group with properties (BT), (Irr), (CAF), *and* (NbC). *Then every nonuniform lattice embedding $\Gamma \hookrightarrow G$ of Γ is virtually isomorphic to either*

(1) *an irreducible lattice in a connected, center-free, semisimple real Lie group without compact factors, or*
(2) *an S-arithmetic lattice, possibly up to tree extension, with both real and non-Archimedean factors present.*

In particular, every lattice embedding of such Γ is square-integrable, with the exception of lattice embeddings into Lie groups that are locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ *or* $\mathrm{SL}_2(\mathbb{C})$.

This is a direct corollary except for the last statement, which is a consequence of results about the square-integrability of classical lattices from Shalom's work (see [67, Section 2] for the S-arithmetic and higher-rank case and [68, Theorem 3.7] for the rank-1 case).

1.3. Applications
As applications of our main result, we now present more precise classification results of lattice envelopes for specific groups. Their proofs can be found in Section 6.

THEOREM B (Mostow rigidity with locally compact targets)
Let $\Gamma < H$ be an irreducible lattice embedding into a center-free, real semisimple Lie group H without compact factors that is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. *Then every nontrivial lattice embedding of Γ into an lcsc group G is virtually isomorphic to $\Gamma < H$. If $\Gamma < H$ is an S-arithmetic lattice embedding as in Definition 4.4, then*

every nontrivial lattice embedding of Γ *into an lcsc group* G *is virtually isomorphic to* $\Gamma < H$ *or to a tree extension* $\Gamma < H^*$ *if* H *includes rank-*1 *non-Archimedean factors.*

This theorem may be viewed as a generalization of Mostow's strong rigidity. Mostow's celebrated result corresponds to the special case where G and H are semisimple real Lie groups. (In most accounts, one even assumes $G = H$, and the focus is on aligning two given lattice embeddings of Γ in G by an automorphism of G.)

In fact, Mostow considered the case of uniform lattice embeddings (see [55]); the nonuniform were later obtained by Prasad in [60] and Margulis in [48]. In [35], the second author proved Theorem B for the case of a simple real Lie group H with $\mathrm{rk}_{\mathbb{R}}(H) \geq 2$ and a general lattice envelope $\Gamma < G$ and for the case of a rank-1 real Lie group H and a uniform lattice envelope $\Gamma < G$.

The case of $H = \mathrm{PSL}_2(\mathbb{R})$ is excluded from Theorem B for two reasons. First, strong rigidity does not hold in this setting—the moduli space of embeddings of a surface group in the same group $G = H = \mathrm{PSL}_2(\mathbb{R})$ is multidimensional. Second, nonuniform lattices in $\mathrm{PSL}_2(\mathbb{R})$, which are virtually free groups of finite rank $2 \leq n < \infty$, can be embedded as lattices into completely different lcsc groups such as $\mathrm{PGL}_2(\mathbb{Q}_p)$ or the automorphism group $\mathrm{Aut}(T)$ of a locally finite tree T. Our next results show that all lattice envelopes of free groups are related to the ones mentioned above.

THEOREM C (Lattice embeddings of free groups)
Let Γ *be a finite extension of a finitely generated non-Abelian free group* F_n*, and let* $\Gamma < G$ *be a nontrivial lattice embedding.*

(1) *If* $\Gamma < G$ *is nonuniform, then it is virtually isomorphic to a nonuniform lattice embedding into* $\mathrm{PSL}_2(\mathbb{R})$.

(2) *If* $\Gamma < G$ *is uniform, then it is virtually isomorphic to a lattice embedding into a closed cocompact subgroup of the automorphism group of a tree.*

The second case includes such examples as $\mathrm{PGL}_2(\mathbb{Q}_p)$ or other rank-1 non-Archimedean groups. For uniform lattices in $\mathrm{PSL}_2(\mathbb{R})$, such as surface groups, the possibilities for lattice embeddings are even more restricted.

THEOREM D (Lattice embeddings of surface groups)
Let Γ *be a uniform lattice in* $\mathrm{PSL}_2(\mathbb{R})$. *Any other nontrivial lattice embedding of* Γ *is virtually isomorphic to a uniform lattice embedding into* $\mathrm{PSL}_2(\mathbb{R})$.

Let M be a manifold that admits a Riemannian metric with strictly negative sectional curvature, but is not homeomorphic to a locally symmetric one. We conjecture

that fundamental groups of such manifolds $\Gamma = \pi_1(M)$ have no nontrivial lattice envelopes. Under an additional pinching assumption, we are able to prove this.

THEOREM E (Lack of lattice embeddings in pinched negative curvature)
The fundamental group of a closed Riemannian manifold M of dimension $n \geq 5$ whose sectional curvatures range in

$$\left[-\left(1+\frac{1}{n-1}\right)^2, -1\right]$$

does not admit a nontrivial lattice embedding unless M is homeomorphic to a closed hyperbolic manifold.

Some remarks about this result are in order. Gromov and Thurston in [40] construct infinitely many examples of negatively curved manifolds with the above pinching in dimensions $n \geq 4$ that are not homeomorphic to hyperbolic manifolds. By taking connected sums of a hyperbolic manifold with an exotic sphere, Farrell and Jones construct closed smooth manifolds that are homeomorphic to a hyperbolic manifold but whose smooth structure does not support a hyperbolic Riemannian metric (see [32]). We recommend the survey [34] by Farrell, Jones, and Ontaneda on these issues.

Finally, we obtain the following surprising characterization of free groups.

THEOREM F (Nonuniform lattice embeddings of groups with $\beta_1^{(2)} > 0$)
Let Γ be a group with nonvanishing first ℓ^2-Betti number that has an upper bound on the order of its finite subgroups. If Γ possesses a nonuniform compactly generated lattice envelope, then Γ has a non-Abelian free subgroup of finite index.

Remark 1.7
Classification of lattice envelopes can also be obtained from *measure equivalence rigidity (ME-rigidity)* properties as in [35, Theorem A]. ME-rigidity results are expected to be rare and are currently known only for very special classes of groups. In [43] Kida proved ME-rigidity for mapping class groups, and as a corollary he deduced that mapping class groups do not admit nontrivial lattice envelopes. In [44] Kida proved ME-rigidity for certain amalgams of ME-rigid groups, and in [26] Chifan and Kida discuss (among other things) ME-rigidity for certain surface braid groups. These are additional examples of groups with only trivial lattice envelopes.

Let us note that Theorem A can be applied to many of these groups to deduce that their lattice envelopes must be uniform and must be tdlc. However, showing that these envelopes actually must be discrete would require further work, most likely involving quasi-isometric rigidity.

Let us also mention the work [29] of Dymarz, who studied uniform lattice envelopes for a class of solvable groups. These groups do not satisfy the (CAF) property and, therefore, are out of the scope of the current paper.

1.4. Structure of the paper

We devote most of Section 2 to the proof of Theorem 1.4. We discuss an example by Burger and Mozes in Section 2.4 that shows the necessity of the condition (NbC); if one drops (NbC) while keeping (CAF), there are exotic lattice embeddings that are not covered by the three cases of Theorem A.

The definition of virtual isomorphism of lattice embeddings and the tools of the trade for the proof of Theorem A are provided in Section 3. The most difficult proof in Section 3 is the result about outer automorphisms of S-arithmetic lattices (Theorem 3.14). Although this result is known or expected to hold by the experts, there has been no proof so far in the required generality. We hope its proof provides a useful reference.

The bulk of our work here is devoted to the proof of Theorem A in Section 5. The paper [7], in which we announced the results of this paper, provided proofs in special cases that might be helpful for the reader. In the first step of the proof, we rely on property (CAF) and the positive solution of Hilbert's fifth problem to show that the given lattice embedding $\Gamma < G$ is virtually isomorphic to one into a product of a semisimple Lie group L and a totally disconnected group D. So we may assume that $G = L \times D$. Let $U < D$ be a compact-open subgroup. In the second step of the proof we split L further as $L = L_0 \times L_1$ in such a way that the projection of Γ to L_1 is dense. Depending on the finiteness of $\Gamma \cap (L \times U)$ and $\Gamma \cap (\{1\} \times L_1 \times U)$, we distinguish three cases in the third step which correspond to the cases (1), (2), or (3) of the statement of Theorem A. The second case is the most sophisticated. Here we have to identify the lattice $\Gamma \cap (\{1\} \times L_1 \times D') < L_1 \times D'$, where $D' < D$ is a certain closed cocompact subgroup, as an S-arithmetic lattice, and also identify D' with the corresponding non-Archimedean factor. This identification is an arithmeticity theorem which is proved in our companion paper [8] and heavily relies on Margulis's commensurator rigidity. We prove in [8] a result of greater generality which does not assume compact generation of the ambient group. The version needed here is only slightly more general than the arithmeticity theorem by Caprace and Monod in [22, Theorem 5.20]. Our basic setup for S-arithmetic groups and a detailed explanation of the arithmeticity theorem are provided in Section 4. Further, we use Margulis' normal subgroup theorem and the aforementioned result on outer automorphism groups to conclude that $L = L_1$ and $L_0 = \{1\}$ in the second case. The condition on (NbC) appears just once in the whole proof, namely, in the third step of the proof, in Lemma 5.9. The final step of the proof

identifies the difference between $L \times D'$ and $L \times D$ as a tree extension in the sense of Setup 4.1.

In the proof of the applications, that is, Theorems B–F, a major step is always to identify the most mysterious case, namely, (3), in Theorem A. To this end, we appeal to ideas of geometric group theory and quasi-isometric rigidity. This is based on the fact that a lattice with property (BT) is quasi-isometric to any of its totally disconnected lattice envelopes.

2. The properties (BT), (Irr), (CAF), and (NbC)

We describe the classes in Theorem 1.4 in more detail. In Section 2.3 we prove Theorem 1.4. In Section 2.4 we describe an exotic lattice embedding of a group that does not have property (NbC).

2.1. *Commensurated and commensurable groups*

Definition 2.1

A subgroup A of a group G is *commensurated* by a subset $S \subset G$ if $s^{-1}As \cap A$ has finite index in $s^{-1}As$ and A for every $s \in S$. If $A < G$ is commensurated by all of G, we just say that $A < G$ is *commensurated*.

Definition 2.2

Two lcsc groups G and G' are *commensurable* if there are open subgroups $H < G$ and $H' < G'$ of finite index such that $H \cong H'$ as topological groups. Further, G and G' are *weakly commensurable* or *virtually isomorphic* if there are open subgroups $H < G$ and $H' < G'$ of finite-index and compact normal subgroups $K \lhd H$ and $K' \lhd H'$ such that $H/K \cong H'/K'$ as topological groups.

A topological isomorphism $f : H/K \to H'/K'$ is called a *virtual isomorphism* from G to G'. We say it *restricts* to subgroups $A < G$ and $A' < G'$ if f restricts to an isomorphism $(A \cap H)/(A \cap K) \to (A' \cap H')/(A' \cap K')$, which is then a virtual isomorphism between A and A'.

If in the above definition the groups G and G' are countable discrete, then a finite-index subgroup of G or G' is open and a compact normal subgroup of G or G' is finite. The proof of the following easy lemma is left to the reader.

LEMMA 2.3

Being commensurated is preserved in the following situations:

(1) *Preimages of commensurated subgroups under homomorphisms are commensurated.*

(2) *Intersections of finitely many commensurated subgroups are commensurated.*

(3) *A finite-index subgroup of a commensurated subgroup is itself commensurated.*

Both (CAF) and (NbC) are not necessarily preserved if one passes to a weakly commensurable group. They are, however, preserved by passing to quotients by finite kernels.

LEMMA 2.4
Let $K \lhd \Gamma$ be a finite normal subgroup. If Γ has property (CAF)*, then Γ/K has* (CAF).

Proof
Let $p\colon \Gamma \to \Gamma/K$ be the projection. Let $A < \Gamma/K$ be an amenable, commensurated subgroup. Then $p^{-1}(A) < \Gamma$ is commensurated by Lemma 2.3. Since $p^{-1}(A)$ is a finite extension of A, it is amenable too. By the property (CAF), the group $p^{-1}(A)$ is finite; thus, A is finite. □

LEMMA 2.5
Let $K \lhd \Gamma$ be a finite normal subgroup. If Γ has property (NbC)*, then Γ/K has* (NbC).

Proof
The stated property is obvious and built into the definition of (NbC). □

2.2. Groups in the class $\mathcal{D}_{\mathrm{reg}}$ and acylindrically hyperbolic groups
The class $\mathcal{D}_{\mathrm{reg}}$ was introduced by Thom in [71]. It is closely related to the class $\mathcal{C}_{\mathrm{reg}}$ of Monod and Shalom. Both $\mathcal{D}_{\mathrm{reg}}$ and $\mathcal{C}_{\mathrm{reg}}$ are an attempt to define negative curvature for groups in a cohomological way. The definition of $\mathcal{D}_{\mathrm{reg}}$ is analytical.

Let $\pi\colon \Gamma \to U(\mathcal{H})$ be a unitary representation. A map $c\colon \Gamma \to \mathcal{H}$ is a *quasi*-1-*cocycle* for π if

$$\Gamma \times \Gamma \ni (g,h) \mapsto \pi(g)c(h) - c(gh) + c(g) \in \mathcal{H}$$

is uniformly bounded on $\Gamma \times \Gamma$. The vector space of quasi-1-cocycles modulo the bounded ones forms a group $QH^1(\Gamma, \mathcal{H})$. The class $\mathcal{D}_{\mathrm{reg}}$ is the class of groups for which $QH^1(\Gamma, \ell^2(\Gamma))$ has nonzero $L(\Gamma)$-dimension in the sense of Lück [47, Section 6.1]. More concrete is the following description.

PROPOSITION 2.6 ([71, Corollary 2.5, Lemma 2.8])
A group Γ is in $\mathcal{D}_{\mathrm{reg}}$ if and only if there is an unbounded quasi-1-*cocycle $\Gamma \to \ell^2(\Gamma)$.*

The following theorem was proved by Bestvina and Fujiwara in [10] and Hamenstädt in [41] for classes of groups that were later identified as acylindrically hyperbolic groups by Osin in [57].

THEOREM 2.7 ([57, Theorem 8.3])
The class $\mathcal{D}_{\mathrm{reg}}$ strictly contains the class of acylindrically hyperbolic groups.

We also mention the following result of Sun.

THEOREM 2.8 ([70, Corollary 1.3])
Nonelementary convergence groups are acylindrically hyperbolic.

Remark 2.9
In the following, we show properties (NbC) and (CAF) for groups in $\mathcal{D}_{\mathrm{reg}}$. By using the difficult Theorem 2.7, this implies (NbC) and (CAF) for acylindrically hyperbolic groups. We emphasize, however, that one can deduce (CAF) and (NbC) directly and quite easily for acylindrically hyperbolic groups. For example, (CAF) follows directly from [57, Corollary 1.5] and the nonamenability of acylindrically hyperbolic groups.

We show that groups in $\mathcal{D}_{\mathrm{reg}}$ satisfy the (wNbC) property for the trivial reason that the situation in which to apply (wNbC) cannot happen. That means we show the following stronger property for groups in $\mathcal{D}_{\mathrm{reg}}$. Since $\mathcal{D}_{\mathrm{reg}}$ is closed under passing to quotients by finite normal subgroups, this will imply that groups in $\mathcal{D}_{\mathrm{reg}}$ satisfy (NbC).

Definition 2.10
A group Γ has property $(\mathrm{wNbC})_1$ if it satisfies the following. If $N \lhd \Gamma$ is a normal subgroup and $M < \Gamma$ is a commensurated subgroup and $N \cap M = \{1\}$, then N is finite or M is finite.

It is obvious that $(\mathrm{wNbC})_1$ implies (wNbC).

LEMMA 2.11
Let Λ be a group with a normal subgroup $N \lhd \Lambda$ and a commensurated subgroup $M < \Lambda$ such that $\Lambda = N \cdot M$ and $N \cap M = \{1\}$. To every finite subset $F \subset \Lambda$ we can assign subgroups $N_F < N$ and $M_F < M$ in such a way that the following holds:
(1) *N_F is normalized by M,*
(2) *M_F is a normal subgroup of finite index in M,*
(3) *$F \subset \Lambda_F := N_F M$ and $N_F M_F < \Lambda_F$ has finite index,*

(4) *M_F commutes with N_F, and thus, $N_F M_F \cong N_F \times M_F$,*
(5) *$F \subset F'$ implies that $N_F \subset N_{F'}$ and $M_F \supset M_{F'}$.*

In particular, if N is finitely generated, then M has a finite-index subgroup that commutes with N.

Proof

For every $F \subset \Lambda$ let M_F be the normal core within M of the subgroup

$$M'_F := \bigcap_{\lambda \in F \cup \{1\}} \lambda M \lambda^{-1} \subset M.$$

Since M is commensurated, $M'_F < M$ (hence, $M_F < M$) are of finite index. In other words, M_F is the largest normal subgroup of M with the property $\lambda M_F \lambda^{-1} \subset M$ for all $\lambda \in F$.

Let $q\colon \Lambda \to N$ be the map $q(nm) = n$; it is well defined because of $N \cap M = \{1\}$. Clearly, $M'_F = M'_{q(F)}$; thus, $M_F = M_{q(F)}$. Since $N \lhd \Lambda$ is normal, the commutator $[n,m] = n(mn^{-1}m^{-1})$ lies in N. For $m \in M_F = M_{q(F)}$ and $n \in q(F)$ we also have $[n,m] = (nmn^{-1})m^{-1} \in M$; hence, $[n,m] \in M \cap N = \{1\}$. So M_F commutes with the subset $q(F)$. Let

$$S := \bigcup_{m \in M} mq(F)m^{-1} \subset N.$$

Because of normality $M_F \lhd M$ the group M_F commutes with the subset S as well. The subgroups $N_F := \langle S \rangle < N$ and M_F satisfy all the required properties. □

THEOREM 2.12

Every group in $\mathcal{D}_{\mathrm{reg}}$ is (Irr)*,* (CAF)*, and* $(\mathrm{wNbC})_1$.

Proof

The properties (Irr) and (CAF) are shown in [71, Theorem 3.4]. Let $\Gamma \in \mathcal{D}_{\mathrm{reg}}$, and let $N \lhd \Gamma$ and $M < \Gamma$ be normal and commensurated subgroups, respectively, with trivial intersection. Assume by contradiction that both subgroups are infinite. According to [71, Theorem 3.4], N is nonamenable. Let $N = \{n_1, n_2, \ldots\}$ be an enumeration of N. We refer to the notation of Lemma 2.11 applied to our situation. The group N is the increasing union of subgroups $N_{F_i} < N$ for $F_i = \{n_1, \ldots, n_i\}$. Hence, there is some $k \in \mathbb{N}$ such that $N' := N_{F_k}$ is nonamenable and, thus, infinite.

The group N' commutes with the finite-index subgroup $M' := M_{F_k}$ of M. Since $M < \Gamma$ is commensurated, also $M' < \Gamma$ is commensurated. By [71, Lemma 3.3] the restriction maps $QH^1(\Gamma, \ell^2(\Gamma)) \to QH^1(M', \ell^2(\Gamma))$ and $QH^1(N'M', \ell^2(\Gamma)) \to QH^1(M', \ell^2(\Gamma))$ to commensurated subgroups are injective. Hence, the restriction

map

$$QH^1\big(\Gamma, \ell^2(\Gamma)\big) \to QH^1\big(N'M', \ell^2(\Gamma)\big)$$

is injective. We reach a contradiction as the latter module vanishes by [71, Theorem 3.4] and the fact that $N'M'$ is a product of two infinite groups. □

2.3. Conclusion of the proof of Theorem 1.4

Proof of property (CAF)
We refer to the cases in the statement of Theorem 1.4.

(1) Since all ℓ^2-Betti numbers of an infinite amenable group vanish by a result of Cheeger and Gromov from [25, Theorem 0.2], every group with some nonvanishing ℓ^2-Betti number is (CAF) according to [6, Corollary 1.4].
(2) This is exactly [15, Theorem 6.8].
(3) See Theorem 2.12. For the classes included in $\mathcal{D}_{\text{reg}}$ see Theorems 2.7 and 2.8. □

Proof of property (NbC)
We refer to the classes of groups in the statement of Theorem 1.4. All these classes are closed under passing to quotients by finite normal subgroups.

For (1′) the closure property follows from the fact that the vanishing of ℓ^2-Betti numbers is an invariant of virtual isomorphism which can be easily deduced from their basic properties (see [47]). Alternatively, one may cite the much more general result that the vanishing of ℓ^2-Betti numbers of lcsc groups is a coarse invariant (see [64]).

For (2′) one argues as follows: Let $\Gamma < \mathrm{GL}_n(k)$ be a linear group with finite normal subgroup $F < \Gamma$. A linear representation of the algebraic group $\bar{\Gamma}/K$ where $\bar{\Gamma}$ is the Zariski closure of $\mathrm{GL}_n(k)$ yields a linear embedding of Γ/K.

For (3′) the closure property is elementary and left to the reader.

For (4′) we argue as follows: In view of Proposition 2.6 let $c\colon \Gamma \to \ell^2(\Gamma)$ be an unbounded quasi-1-cocycle. Let $F < \Gamma$ be a finite normal subgroup. Let $\pi\colon \ell^2(\Gamma) \to \ell^2(\Gamma/F)$ be the map of unitary Γ-representation that sends the vector $\gamma \in \ell^2(\Gamma)$ to $\gamma F \in \ell^2(\Gamma/F)$. Let $s\colon \Gamma/F \to \Gamma$ be a set-theoretic section of the projection $\Gamma \to \Gamma/F$. Then $\pi \circ c \circ s\colon \Gamma/F \to \ell^2(\Gamma/F)$ is easily seen to be an unbounded quasi-1-cocycle.

Thus, it suffices to show property (wNbC) for the classes in Theorem 1.4, which we do next.

(1′) Let Γ be a group with positive first ℓ^2-Betti number. Assume by contradiction that Γ does not have $(\mathrm{wNbC})_1$. Let $M, N < \Gamma$ be subgroups with $M \cap N = \{1\}$ such that $N \lhd \Gamma$ is normal and M is commensurated by Γ. Assume that neither M nor N are finite. We have to show that $\beta_1^{(2)}(\Gamma) = 0$. Since $\Lambda := NM$ is commensurated

by Γ, the vanishing of $\beta_1^{(2)}(\Lambda)$ would imply this according to [6, Corollary 1.4]. By Lemma 2.11 the group Λ is an ascending union of groups Λ_n such that Λ_n is virtually a product of infinite groups. A product of infinite groups has vanishing first ℓ^2-Betti number because of the Künneth formula for ℓ^2-Betti numbers and the fact that the zeroth ℓ^2-Betti number of an infinite group vanishes. The same is true for a virtual product, so $\beta_1^{(2)}(\Lambda_n) = 0$ for large n (see the remark about ℓ^2-Betti numbers of virtually isomorphic groups above). Finally, $\beta_1^{(2)}(\Lambda_n) = 0$ for all n implies that $\beta_1^{(2)}(\Lambda) = 0$ by [47, Theorem 7.2(2)].

(2′) Let $\Gamma < \mathrm{GL}_n(k)$ be a linear group, and let $N, M < \Gamma$ be subgroups with trivial intersection such that $N \lhd \Gamma$ is normal and $M < \Gamma$ is commensurated by Γ. Let $F \mapsto N_F, M_F$ be an assignment of finite subsets to subgroups as in Lemma 2.11. By the Noetherian property of the algebraic group $\mathrm{GL}_n(k)$ there exists some finite set $F_1 \subset N$ such that the Zariski closure $\bar{M}_1$ of M_{F_1} is minimal among all finite subsets of N. Thus, for every $n \in N$, $M_{F_0 \cup \{n\}}$ is Zariski-dense in $\bar{M}_1$. Since

$$[n, M_{F_0 \cup \{n\}}] = 1$$

we get that $[n, \bar{M}_1] = 1$. In particular, the finite-index subgroup M_{F_1} of M commutes with N. This implies (wNbC) for linear groups.

(3′) As mentioned before, this is [20, Corollary 18].

(4′) See Theorem 2.12. □

Proof of property (Irr)

By the Künneth formula for ℓ^2-Betti numbers and the fact from the preceding proof that their vanishing is unaffected by passing to virtually isomorphic groups, it follows that groups with positive first ℓ^2-Betti numbers have property (Irr). That groups in $\mathcal{D}_{\mathrm{reg}}$ have (Irr) is contained in Theorem 2.12. □

2.4. *Examples of groups without property* (NbC)

Among the conditions required for our main result, the (NbC) property is the most opaque. We present two examples of groups that do not have property (NbC). The first example is a wreath product. The second one comes with a lattice envelope that shows the necessity of the (NbC) condition in Theorem A.

Example 2.13

For an infinite residually finite group M and a countable group H we consider the product group $H^M = \prod_M H$ endowed with the natural (shift) action of M. Let N be the subgroup in H^M consisting of *periodic* elements—that is, elements having an M-stabilizer of finite index in M. Clearly, N is a countable subgroup on which M acts. Let $\Gamma = M \ltimes N$. Observe that $\Gamma = NM$, $N \lhd \Gamma$, M is commensurated by Γ, and

$N \cap M = \{e\}$. The subgroup N does not commute with any finite-index subgroup of M, yet any finitely generated subgroup of N does so (cf. Lemma 2.11).

Example 2.14
Let Λ be an irreducible lattice in $\mathrm{SL}_n(\mathbb{Q}_p) \times \mathrm{SL}_n(\mathbb{R})$. Let X be the symmetric space of $\mathrm{SL}_n(\mathbb{R})$, and let B be the 1-skeleton of the Bruhat–Tits building associated with $\mathrm{SL}_n(\mathbb{Q}_p)$. Then the action of Λ on $B \times X$ can be extended to an action of an extension, denoted by Γ, of Λ by the fundamental group $\pi_1(B)$, which is an infinitely generated free group, on the universal covering $\tilde{B} \times X$. Note that $T := \tilde{B}$ is a tree. Moreover, Γ is an irreducible lattice in $\mathrm{Aut}(T) \times \mathrm{SL}_n(\mathbb{R})$. This type of example originates from the work of Burger and Mozes [18, Section 1.8]. More general constructions in this direction are discussed in Caprace and Monod [21, Section 6.C].

The group Γ has the properties (BT), (Irr), and (CAF), but it lacks property (NbC). Let us show property (CAF) first. Let $A < \Gamma$ be a commensurated subgroup. The image $p(A)$ under the projection $p : \Gamma \to \Lambda$ is an amenable commensurated subgroup of Γ. It follows from Theorem 1.4 that Λ has (CAF). Thus, $p(A)$ is finite. Since $\pi_1(B)$ satisfies (CAF) by [21, Section 6.C], the intersection $A \cap \pi_1(B)$ is finite as well, which implies that A is finite.

Let $U \subset \mathrm{Aut}(T)$ be the stabilizer of a vertex; it is a compact open subgroup. Then it is easy to verify that the normal subgroup $N = \pi_1(B) \lhd \Gamma$ and the subgroup $M = \Gamma \cap (U \times \mathrm{SL}_n(\mathbb{R}))$, which is commensurated by Lemma 3.6, violate property (NbC): no finite-index subgroup of M commutes with N.

3. General facts about locally compact groups and their lattices

In Section 3.1 we introduce the analogue of weak commensurability or virtual isomorphism of groups for lattice embeddings. In Sections 3.2 and 3.3 we have a look at broader classes of subgroups in lcsc groups than lattices, discrete subgroups, and closed subgroups of finite covolume. In Section 3.4 we recall the definition of the amenable radical and study its behavior under passage to closed subgroups of finite covolume. Finally, in Sections 3.5 and 3.6 we study automorphisms of lcsc groups and their lattices in different categories: outer automorphism groups and quasi-isometry groups.

3.1. Virtual isomorphism of lattice embeddings

We introduce a relative notion of virtual isomorphism (see Definition 2.2).

Definition 3.1
Two lattice embeddings $\Gamma \hookrightarrow G$ and $\Gamma' \hookrightarrow G'$ are *virtually isomorphic* if there is a virtual isomorphism from G to G' that restricts to a virtual isomorphism from Γ

to Γ'. A lattice embedding is *trivial* if it is virtually isomorphic to the identity homomorphism of a countable discrete group.

Remark 3.2
The above definition generalizes Definition 2.2 in the sense that two countable discrete groups Γ and Γ' are virtually isomorphic if and only if the lattice embeddings $\mathrm{id}\colon \Gamma \to \Gamma$ and $\mathrm{id}\colon \Gamma' \to \Gamma'$ are virtually isomorphic.

To make the definition above more explicit, if $\Gamma \hookrightarrow G$ and $\Gamma' \hookrightarrow G'$ are virtually isomorphic, then there are open finite-index subgroups $H < G$ and $H' < G'$ and compact normal subgroups $K \lhd H$ and $K' \lhd H'$ and a commutative square

$$\begin{array}{ccc} (\Gamma \cap H)/(\Gamma \cap K) & \hookrightarrow & H/K \\ \downarrow \cong & & \downarrow \cong \\ (\Gamma' \cap H')/(\Gamma' \cap K') & \hookrightarrow & H'/K' \end{array}$$

with vertical isomorphisms. The horizontal maps in the above diagram are lattice embeddings. This follows from Lemmas 3.4 and 3.5.

PROPOSITION 3.3
Virtual isomorphism of lattice embeddings is an equivalence relation.

Proof
The only nonobvious claim is transitivity. Its proof is similar to the proof that virtual isomorphism of groups is transitive. Let $j_l\colon \Gamma_l \hookrightarrow G_l$ for $l \in \{1,2,3\}$ be lattice embeddings such that j_1, j_2 and j_2, j_3 are virtually isomorphic. So for $l \in \{1,2,3\}$ there are finite-index subgroups $H_l < G_l$ and $H_2' < G_2$ and compact normal subgroups $K_l \lhd H_l$ and $K_2' \lhd H_2'$ and isomorphisms

$$H_1/K_1 \xrightarrow{\cong} H_2/K_2 \qquad \text{and} \qquad H_2'/K_2' \xrightarrow{\cong} H_3/K_3$$

that restrict to isomorphisms

$$(\Gamma_1 \cap H_1)/(\Gamma_1 \cap K_1) \cong (\Gamma_2 \cap H_2)/(\Gamma_2 \cap K_2) \qquad \text{and}$$
$$(\Gamma_2 \cap H_2')/(\Gamma_2 \cap K_2') \cong (\Gamma_3 \cap H_3)/(\Gamma_3 \cap K_3),$$

respectively. The subgroup $H_2'' := H_2 \cap H_2' < G_2$ has finite index in G_2. Let

$$\mathrm{pr}\colon H_2'' \to Q_2 := H_2''/(K_2 \cap H_2')$$

be the projection. Let K_2'' be the kernel of the composition of quotient maps

$$H_2'' \to Q_2 \to Q_2/\operatorname{pr}(K_2' \cap H_2).$$

Note that K_2'' contains $K_2 \cap H_2' = K_2 \cap H_2''$ and $K_2' \cap H_2 = K_2' \cap H_2''$. Since K_2 and $\operatorname{pr}(K_2' \cap H_2)$ are compact, $K_2'' \lhd H_2''$ is a compact normal subgroup of H_2''. Consider the compositions of quotient maps and isomorphisms as above:

$$f_1\colon H_1 \to H_1/K_1 \xrightarrow{\cong} H_2/K_2 \qquad \text{and} \qquad g_3\colon H_2' \to H_2'/K_2' \xrightarrow{\cong} H_3/K_3.$$

Similarly, using the inverses of the above isomorphisms we obtain homomorphisms

$$g_1\colon H_2 \to H_2/K_2 \xrightarrow{\cong} H_1/K_1 \qquad \text{and} \qquad f_3\colon H_3 \to H_3/K_3 \xrightarrow{\cong} H_2'/K_2'.$$

We set $C_i := g_i(H_2'')$ and $E_i := g_i(K_2'') \lhd C_i$ for $i \in \{1,3\}$; further, let $\tilde{C}_i < H_i$ and $\tilde{E}_i < H_i$ be the preimages of C_i and E_i under the quotient maps. Then $\tilde{C}_i$ has finite index in H_i and, thus, also in G_i, and $\tilde{E}_i$ is a compact normal subgroup of $\tilde{C}_i$ for $i \in \{1,3\}$. The homomorphism

$$\tilde{C}_i/\tilde{E}_i \to H_2''/K_2'', \qquad [x] \mapsto \big[f_i(x)\big]$$

is an isomorphism for $i \in \{1,3\}$ with inverse map $[y] \mapsto [g_i(y)]$. Since f_i and g_i preserve the lattices, this isomorphism restricts to an isomorphism

$$(\Gamma_i \cap \tilde{C}_i)/(\Gamma_i \cap \tilde{E}_i) \cong (\Gamma_2 \cap H_2'')/(\Gamma_2 \cap K_2'')$$

for $i \in \{1,3\}$. This finishes the proof of transitivity. □

3.2. Discrete subgroups of locally compact groups

We collect some important, albeit easy facts used in the proof of the main result.

LEMMA 3.4 ([23, Section 2.C])
Let $\Gamma < G$ be a lattice in an lcsc group. Let $U < G$ be an open subgroup. Then $\Gamma \cap U$ is a lattice in U.

LEMMA 3.5 ([62, Theorem 1.13, p. 23])
Let G be an lcsc group, and let $N \lhd G$ be a normal closed subgroup. The projection of a lattice $\Gamma < G$ to G/N is discrete if and only if $\Gamma \cap N$ is a lattice in N. If so, then the projection of Γ to G/N is also a lattice.

The following statement is our main source of commensurated subgroups.

LEMMA 3.6
Let G be an lcsc group, and let $U < G$ be a compact open subgroup. Then U is commensurated by G. For any subgroup $\Lambda < G$ the intersection $\Lambda \cap U$ is commensurated in Λ.

Proof
For any $g \in G$, $U \cap gUg^{-1}$ is an open subgroup of the compact group U (resp., in gUg^{-1}); hence, it has finite index in U (resp., in gUg^{-1}). The second statement follows by taking $g \in \Lambda$ and taking intersections with Λ. □

LEMMA 3.7 ([22, Lemma 2.12])
Any lattice envelope of a finitely generated group is compactly generated.

3.3. Closed subgroups of finite covolume
The notion of a lattice, that is, a discrete subgroup of finite covolume, can be generalized to closed, but not necessarily discrete, subgroups as follows. A closed subgroup $H < G$ of an lcsc group G is said to have *finite covolume* in G if G/H carries a finite G-invariant Borel measure.

If H has finite covolume in G, then G is unimodular if and only if H is unimodular (see [62, Lemma 1.4, p. 18]).

LEMMA 3.8 ([62, Lemma 1.6, p. 20])
Let G be an lcsc group, and let $H < L < G$ be closed subgroups. Then $H < G$ has finite covolume if and only if both $L < G$ and $H < L$ have finite covolumes.

It is well known that a torsion-free lattice in a tdlc group is uniform. A mild generalization is given by the following.

LEMMA 3.9 ([3, Corollary 4.11])
Let G be a unimodular tdlc group. Let $H < G$ be a closed subgroup of finite covolume. Let m_H be the Haar measure of H. If there is an upper bound on the Haar measures of compact open subgroups in H, that is,

$$\sup\{m_H(U) \mid U \text{ is a compact open subgroup in } H\} < \infty,$$

then G/H is compact.

LEMMA 3.10
Let G be a connected, semisimple, center-free, real Lie group without compact factors. Every closed subgroup $H < G$ of finite covolume has the form $H = \Delta \times G_2$, where G_2 is a direct factor of G, $G = G_1 \times G_2$, and $\Delta < G_1$ is a lattice.

Proof
The Lie algebra $\mathfrak{h}$ of the connected component H^0 is $\mathrm{Ad}(H)$-invariant. Since H is Zariski-dense in G by Borel's density theorem [11, Theorem 1], it follows that $\mathfrak{h}$ is an ideal in the semisimple Lie algebra $\mathfrak{g}$ of G. Let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$$

be the decomposition into simple Lie algebras. Let

$$I_1 = \left\{ i \in \{1, \ldots, n\} \;\middle|\; \mathfrak{g}_i \cap \mathfrak{h} = \{0\} \right\},$$

and let $I_2 = \{1, \ldots, n\} \backslash I_1$. Then $G = G_1 \times G_2$ with G_k being the Lie subgroup with Lie algebra

$$\bigoplus_{i \in I_k} \mathfrak{g}_i .$$

Since $\mathfrak{g}_i$ is simple we either have $\mathfrak{g}_i \cap \mathfrak{h} = \{0\}$ or $\mathfrak{g}_i \cap \mathfrak{h} = \mathfrak{g}_i$. This implies that $H = (G_1 \times \{1\} \cap H) \times G_2$, where $\Delta := G_1 \times \{1\} \cap H$ is discrete. Since H has finite covolume, Δ is a lattice. □

3.4. The amenable radical

Definition 3.11
The *amenable radical* of an lcsc group G is the maximal closed normal amenable subgroup. We denote it by $\mathrm{Rad}_{\mathrm{am}}(G) \lhd G$.

LEMMA 3.12
Let G be an lcsc group, and let $H < G$ be a closed subgroup of finite covolume. Then $\mathrm{Rad}_{\mathrm{am}}(H) = \mathrm{Rad}_{\mathrm{am}}(G) \cap H$.

This lemma can be deduced from the following result.

LEMMA 3.13 ([37, Proposition 4.4, Theorem 4.5])
Let G be an lcsc group, and let $G \curvearrowright M$ be a continuous action that is minimal and strongly proximal. Let $H < G$ be a closed subgroup of finite covolume. Then the restriction of the G-action to H is also minimal and strongly proximal.

In [37] Furstenberg was interested in the restriction to lattices, but the proof applies to general closed subgroups of finite covolume.

Proof of Lemma 3.12 from Lemma 3.13
The very definition of $\mathrm{Rad}_{\mathrm{am}}(G)$ yields the inclusion

$$\mathrm{Rad}_{\mathrm{am}}(G) \cap H < \mathrm{Rad}_{\mathrm{am}}(H).$$

For the converse inclusion, we use an equivalent characterization of the amenable radical as the common kernel of all *minimal, strongly proximal* actions on compact metrizable spaces (see [36, Proposition 7]). Thus, for the converse inclusion it suffices to show that, given an arbitrary minimal and strongly proximal G-action $G \to \mathrm{Homeo}(M)$, its restriction to H is also minimal and strongly proximal, because it would show that $\mathrm{Rad}_{\mathrm{am}}(H)$ acts trivially in every minimal strongly proximal G-action, which is the content of Lemma 3.13. □

3.5. (Outer) automorphism groups
We describe various results about groups of automorphisms and groups of outer automorphisms of certain groups. We are confident that the main result of this section, Theorem 3.14, is known to experts, but it does not seem to be in the literature in the appropriate generality.

THEOREM 3.14
Let $K_1, \dots, K_n$ be number fields, and let $\mathbf{H}_i$ be connected, noncommutative, adjoint, absolutely simple K_i-groups. Let $S_i \subset \mathcal{V}_i$ be finite compatible sets of places of K_i as in Section 4.1. Then any group that is (abstractly) commensurable with

$$\prod_{i=1}^{n} \mathbf{H}_i\left(\mathcal{O}_i[S_i]\right)$$

has a finite outer automorphism group.

(The proof of Theorem 3.14 will be given by the end of this section. This theorem will be used in Section 5 together with Lemma 3.15 to split certain extensions of groups.)

For a subgroup $B < A$ of a group A, let $\mathrm{Aut}(A)$, $\mathrm{Aut}_B(A)$, and $\mathrm{Out}(A)$ denote the automorphism group of A, the subgroup of $\mathrm{Aut}(A)$ preserving B setwise, and the outer automorphism group of A. For a σ-compact lcsc group G we denote the group of continuous automorphisms by $\mathrm{Aut}^c(G)$ and the group of continuous outer automorphisms by $\mathrm{Out}^c(G) = \mathrm{Aut}^c(G)/\operatorname{Inn} G$. Note that a continuous automorphism of G is a homeomorphism by the open mapping theorem from [13, IX, Section 5.3].

We begin by stating two auxiliary results which are elementary. We leave the first one to the reader and provide a reference for the second one.

LEMMA 3.15
Let $B \lhd A$ be a normal subgroup in a group A. Then the kernel of the natural homomorphism $A \to \mathrm{Out}(B)$ is $B\mathcal{Z}_A(B)$, where $\mathcal{Z}_A(B)$ is the centralizer of B in A. If, in addition, B has trivial center, then $B\mathcal{Z}_A(B)$ is a direct product $B \times \mathcal{Z}_A(B)$. In particular, the short exact sequence $B \hookrightarrow A \twoheadrightarrow A/B$ splits as a direct product provided B has trivial center and $A \to \mathrm{Out}(B)$ has trivial image.

LEMMA 3.16 ([73, Theorem, p.90])
Let $B \lhd A$ be a normal subgroup in a group A. Let C be the center of B. Then the sequence of groups

$$1 \to Z^1(A/B; C) \xrightarrow{j} \mathrm{Aut}_B(A) \xrightarrow{\pi} \mathrm{Aut}(B) \times \mathrm{Aut}(A/B) \tag{3.1}$$

is exact. Here the left group is the Abelian group of 1-cocycles of A/B in C, j maps a cocycle c to $a \mapsto c([a])a$, and $\pi(f) = (f|_B, [f])$.

The following three results are immediate consequences of Lemma 3.16.

LEMMA 3.17
Every topological automorphism of a product $G = G^0 \times G^{\mathrm{td}}$ of a center-free connected group G^0 and a tdlc group G^{td} is a product of a topological automorphism of G^0 and a topological automorphism of G^{td}.

Proof
We apply Lemma 3.16 to $A = G$ and $B = G^0$. In this situation, the map π in (3.1) has an obvious right inverse s. Both π and s preserve continuity of automorphisms. Let $\alpha\colon G \to G$ be a continuous automorphism. Then $\alpha(G^0) = G^0$. Since G^0 has trivial center, π is injective. So $\alpha = s(\pi(\alpha))$. □

LEMMA 3.18
Let A be a group, and let $B \lhd A$ be a characteristic subgroup of finite index. If B has a finite center, then the restriction map $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$ has a finite kernel.

Proof
Denote by K the kernel of $\mathrm{Aut}(A) = \mathrm{Aut}_B(A) \to \mathrm{Aut}(B)$, and denote by C the center of B. Since $\mathrm{Aut}(A/B)$ is finite, it is enough to show that the kernel K' of $K \to \mathrm{Aut}(A/B)$ is finite. By Lemma 3.16, K' is isomorphic to $Z^1(A/B; C)$, which is finite since A/B and C are finite. □

COROLLARY 3.19
If a group A has a characteristic subgroup of finite index that has a finite center and a finite outer automorphism group, then also A has a finite outer automorphism group.

Proof
Let $B \lhd A$ be characteristic with finite center and finite outer automorphism group. Since the group $\mathrm{Aut}(A)/B$ surjects onto $\mathrm{Aut}(A)/A = \mathrm{Out}(A)$, it is enough to show that the former is finite. This follows by the exactness of

$$1 \to \ker\big(\mathrm{Aut}(A) \to \mathrm{Aut}(B)\big) \to \mathrm{Aut}(A)/B \to \mathrm{Out}(B),$$

as $\mathrm{Out}(B)$ is finite by assumption, and the kernel is finite by Lemma 3.18. □

For the purposes of this section, we introduce the following definition.

Definition 3.20
An infinite group A is said to be *strongly irreducible* if, for every homomorphism $\pi : B_1 \times B_2 \to A$ with finite cokernel, either $B_1 < \ker\pi$ or $B_2 < \ker\pi$. A group is said to be *strongly outer finite* if the outer automorphism group of any of its finite-index subgroups is finite.

LEMMA 3.21
A Zariski-dense subgroup of the K-points of a connected, noncommutative, adjoint, K-simple K-algebraic group is strongly irreducible.

Proof
Let $\mathbf{H}$ be a connected, noncommutative, adjoint, K-simple K-group, and let $\Gamma < \mathbf{H}(K)$ be a Zariski-dense subgroup. Let $\pi\colon B_1 \times B_2 \to \Gamma$ be a homomorphism with finite cokernel. Since $\Gamma < \mathbf{H}(K)$ is Zariski-dense, the Zariski closure of $\mathrm{im}(\pi)$ has to be $\mathbf{H}$ as a finite-index subgroup of the connected group $\mathbf{H}$. The Zariski closures of $\pi(B_i)$ define K-subgroups $\mathbf{H}_i$, $i \in \{1,2\}$, that commute in $\mathbf{H}$ and together generate $\mathbf{H}$. By the simplicity of $\mathbf{H}$ we conclude that, for some $i \in \{1,2\}$, $\mathbf{H}_i$ is trivial. It follows that $B_i < \ker\pi$. □

THEOREM 3.22
Let B be a group that is (abstractly) commensurable to a product of a finite family of finitely generated, strongly irreducible, strongly outer finite groups. Then B has a finite outer automorphism group.

The proof of Theorem 3.22 will be preceded by some preparation. First, we make the following ad hoc definition, which is not standard.

Definition 3.23
A *standard subgroup* of a product $\prod_{i\in I} A_i$ of groups is a subgroup of the form $\prod_{i\in I} B_i$ with each $B_i < A_i$ being a finite-index subgroup.

LEMMA 3.24
Let $(A_i)_{i\in I}$ be a finite family of strongly irreducible groups, and let $(B_j)_{j\in J}$ be a finite family of nontrivial groups with $|J| \geq |I|$. If

$$\prod_{j\in J} B_j \xrightarrow{\phi} \prod_{i\in I} A_i$$

is a monomorphism with finite cokernel, then there are a bijection $f : J \to I$ and, for each $j \in J$, a monomorphism $\phi_j : B_j \to A_{f(j)}$ with finite cokernel such that ϕ is the product of the maps ϕ_j. In particular, the image $\mathrm{im}(\phi)$ *is a standard subgroup.*

Proof
Let A and B be the product of $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$, respectively. The proof is by induction on $|I|$. If $|I| = 1$, there is nothing to prove. Let $|I| > 1$. Then $|J| > 1$, and we can decompose it nontrivially as $J = J_1 \sqcup J_2$. For $k \in \{1, 2\}$ we set $B^{(k)} = \prod_{j\in J_k} B_j$. We let

$$I_k = \{i \in I \mid B_{3-k} < \ker \pi_i\} \qquad \text{and} \qquad A^{(k)} = \prod_{i\in I_k} A_i,$$

where $\pi_i : A \to A_i$ is the ith projection. By the strong irreducibility of the A_i's, $I = I_1 \sqcup I_2$ is a partition. Hence, the composition of $\phi : B \to A$ with the projection $A \to A^{(k)}$ factors through $B^{(k)} \cong B/B^{(3-k)}$, thus giving a homomorphism $\phi_k : B^{(k)} \to A^{(k)}$. We observe that these ϕ_k's are injective homomorphisms with finite cokernels. Without loss of generality, $|I_1| \leq |J_1|$. By the induction hypothesis, $|I_1| = |J_1|$ and therefore $|I_2| \leq |J_2|$. By applying the induction hypothesis to both ϕ_k's, the statement follows. □

Applying Lemma 3.24 to automorphisms ϕ, we obtain the following.

COROLLARY 3.25
Let $(C_i)_{i\in I}$ be a finite family of strongly irreducible groups. Then the obvious embedding

$$\prod_{i\in I} \mathrm{Aut}(C_i) \to \mathrm{Aut}\Big(\prod_{i\in I} C_i\Big)$$

has a finite cokernel.

LEMMA 3.26
Let $(A_i)_{i\in I}$ be a finite family of strongly irreducible groups, and let A be its product. Let $B < A$ be a subgroup of finite index. Then $\prod_{i\in I}(B \cap A_i)$ is a characteristic subgroup of B of finite index.

Proof
Let us call a subgroup of B that is simultaneously a standard subgroup of A a *substandard subgroup*. By Lemma 3.24, every $\phi \in \mathrm{Aut}(B)$ preserves substandard subgroups. Hence, $\prod_{i\in I}(B \cap A_i)$ is characteristic in B as it is the unique maximal element in the collection of substandard subgroups. □

Proof of Theorem 3.22
By assumption, B has a finite-index subgroup $C' < B$ that is isomorphic to a finite-index subgroup of the product A of a finite family $(A_i)_{i\in I}$ of finitely generated, strongly irreducible, strongly outer finite groups. From now on we will identify C' and its various subgroups with their images in A under this isomorphism. As each A_i is finitely generated, A, C', and thus B are finitely generated too. Thus, C' has a finite-index subgroup C'' that is characteristic in B (e.g., the intersection of all subgroups of B with index $[B : C']$). By Lemma 3.26, C'' has a characteristic subgroup C of finite index which is standard in A; that is, $C = \prod_{i\in I} C_i$, where $C_i < A_i$ are of finite index. Strong irreducibility passes to finite-index subgroups. So each C_i is strongly irreducible. Being characteristic is transitive, so C is characteristic in B.

By Corollary 3.25, the obvious embedding $\prod_{i\in I} \mathrm{Aut}(C_i) \to \mathrm{Aut}(C)$ has a finite cokernel. By the strong outer finite assumption, $\mathrm{Out}(C_i)$ is finite for each $i \in I$. Hence, $\mathrm{Out}(C)$ is finite. By the strong irreducibility assumption C has a finite center. Thus, Corollary 3.19 implies that $\mathrm{Out}(B)$ is finite. □

The following proposition is well known.

PROPOSITION 3.27
Let K be a number field, and let $\mathbf{H}$ be a noncommutative, connected, adjoint, absolutely simple K-group. Let $S \subset \mathcal{V}$ be a finite compatible set of places of K as in Section 4.1. Then $\prod_{v\in S} \mathbf{H}(K_v)$ has a finite group of continuous outer automorphisms.

Proof
By Lemma 3.21, each group $\mathbf{H}(K_v)$ is strongly irreducible. By Corollary 3.25, the natural embedding

$$\prod_{v\in S} \mathrm{Aut}\big(\mathbf{H}(K_v)\big) \xrightarrow{j} \mathrm{Aut}\Big(\prod_{v\in S} \mathbf{H}(K_v)\Big)$$

has a finite cokernel. Further, if $f \in \mathrm{im}(j)$ is a continuous automorphism, then f is a product of continuous automorphisms. This implies that the restriction of j

$$\prod_{v\in S} \mathrm{Aut}^c\big(\mathbf{H}(K_v)\big) \to \mathrm{Aut}^c\Big(\prod_{v\in S} \mathbf{H}(K_v)\Big)$$

has a finite cokernel, too. It is thus enough to show that $\mathrm{Out}^c(\mathbf{H}(K_v))$ is finite for every $v \in S$. Since every continuous field automorphism of K_v is trivial on the closure of $\mathbb{Q}$ over which K_v is finite, the field K_v has a finite group of continuous field automorphisms. Moreover, the group $\mathbf{H}$ has a finite group of algebraic outer automorphisms, given by Dynkin diagram automorphisms. Thus, $\mathrm{Out}^c(\mathbf{H}(K_v))$ is indeed finite by [50, Chapter I, Section 1.8.2(IV)]. □

PROPOSITION 3.28

Let K be a number field, and let $\mathbf{H}$ be a noncommutative, connected, adjoint, absolutely simple K-group. Let $S \subset \mathcal{V}$ be a finite compatible set of places of K as in Section 4.1. Then $\mathbf{H}(\mathcal{O}[S])$ is finitely generated, strongly irreducible, and strongly outer finite.

Proof

By [50, Chapter IX, Theorem 3.2] $\mathbf{H}(\mathcal{O}[S])$ is finitely generated, and by Lemma 3.21 $\mathbf{H}(\mathcal{O}[S])$ is strongly irreducible. We are left to prove that $\mathbf{H}(\mathcal{O}[S])$ is strongly outer finite. To this end, consider a subgroup $\Gamma < \mathbf{H}(\mathcal{O}[S])$ of finite index.

The group Γ is an irreducible lattice in $H = \prod_{v\in S} \mathbf{H}(K_v)$. Let $A = \mathrm{Aut}^c(H)$. The conjugation homomorphism $H \to A$ is injective as H is center-free and has a finite cokernel by Proposition 3.27. Then $\Gamma \hookrightarrow A$ embeds as a lattice. We claim the following:

(1) The normalizer $N = \mathcal{N}_A(\Gamma)$ of Γ in A is discrete in A.

(2) The conjugation action $N \to \mathrm{Aut}(\Gamma)$ is an isomorphism.

Let us first conclude the finiteness of $\mathrm{Out}(\Gamma)$ from (1) and (2). Since N contains the lattice Γ and N is discrete in A, $N < A$ is also a lattice. Hence, $[N : \Gamma] < \infty$. By (2), $\mathrm{Out}(\Gamma)$ is isomorphic to N/Γ and, hence, finite.

Regarding (1), discreteness will follow once we show that N is countable, because $N < A$ is closed. Since Γ is finitely generated, $\mathrm{Aut}(\Gamma)$ is countable. Hence, $N/\mathcal{Z}_A(\Gamma)$ is countable, where $\mathcal{Z}_A(\Gamma)$ denotes the centralizer of Γ in A. Each factor of H is center-free, so $\mathcal{Z}_H(\Gamma)$ is trivial by the density of the projection of Γ in each factor. Since $H < A$ is of finite index, $\mathcal{Z}_A(\Gamma)$ is finite, and the countability of N follows.

To verify (2), it is enough to show that every automorphism of Γ induces a continuous automorphism of H. This is a consequence of the Mostow–Prasad strong rigidity theorem for H (see [50, Theorem VII 7.1]). □

Proof of Theorem 3.14
By Proposition 3.28, the groups $\mathbf{H}_i(\mathcal{O}_i[S_i])$ are finitely generated, strongly irreducible, and strongly outer finite. Thus, the proof follows at once from Theorem 3.22. □

Finally, we need a classical result from the work of Tits and of Borel and Tits.

THEOREM 3.29
Let K be a number field, and let v be a finite place of K. Let $\mathbf{H}$ be a noncommutative, connected, adjoint, absolutely simple K-group of K_v-rank at least 2. Then the automorphism group of the Bruhat–Tits building associated to $\mathbf{H}$ and v contains $\mathbf{H}(K_v)^+$ as a subgroup of finite index.

Proof
In higher rank, the automorphism group of the associated Bruhat–Tits building X is isomorphic to $\mathrm{Aut}(\mathbf{H}(K_v))$, which is isomorphic to an extension of the group $\mathrm{Aut}(\mathbf{H})(K_v)$ of algebraic automorphisms over K_v by a subgroup of field automorphisms of K_v. This follows for example from [4, Proposition C.1] and the fact that a local field of characteristic 0 has finitely many continuous field automorphisms. □

3.6. Quasi-isometries of locally compact groups
Recall that a map between metric spaces $f\colon (X, d_X) \to (Y, d_Y)$ is an (L, C)-*quasi-isometry* if

$$\frac{1}{L}\cdot d_X(x,x') - C < d_Y\big(f(x), f(x')\big) < L\cdot d_X(x,x') + C$$

for all $x, x' \in X$ and for every $y \in Y$ there is $x \in X$ so that $d_Y(y, f(x)) < C$. Two maps $f, g\colon (X, d_X) \to (Y, d_Y)$ have *bounded distance* if

$$\sup_{x\in X} d_Y\big(f(x), g(x)\big) < \infty.$$

The *quasi-isometry group* $\mathrm{QI}(X, d_X)$ of (X, d_X) is the group (under composition) of equivalence classes of quasi-isometries $(X, d_X) \to (X, d_X)$ under bounded distance.

LEMMA 3.30
Let G be an lcsc group containing a closed subgroup $H < G$ such that G/H is compact, the projection $G \to H\backslash G$ has locally a continuous cross section, and H is compactly generated. Let $H \to \mathrm{Isom}(X, d)$ be a properly discontinuous cocompact action on some proper geodesic metric space (X, d). Then the natural homomorphism

$$\phi\colon H \to \mathrm{Isom}(X, d) \to \mathrm{QI}(X, d)$$

extends to a homomorphism $\bar{\phi}\colon G \to \mathrm{QI}(X,d)$ *with the following property. There exist constants* $L, C > 0$ *so that each* $\bar{\phi}(g)$, $g \in G$, *is represented by an* (L,C)*-quasi-isometry* $f_g\colon X \to X$ *such that for every bounded set* $B \subset X$ *there is a neighborhood of the identity* $V \subset G$ *with*

$$\forall_{g\in V}\forall_{x\in B} \quad d\big(f_g(x),x\big) \le C. \tag{3.2}$$

In particular, the isometric action of H *on* X *extends to a quasiaction of* G *on* X.

Remark 3.31
The projection $G \to H\backslash G$ of an lcsc group G to the coset space of a closed subgroup $H < G$ has locally a continuous cross section if G is finite-dimensional with respect to covering dimension (see [42, Theorem 2]).

Proof of Lemma 3.30
Since H is compactly generated and $H\backslash G$ is compact, G is also compactly generated. Thus, G and H possess left-invariant word metrics that are unique up to quasi-isometry. The inclusion $j\colon H \to G$ is a quasi-isometry. We construct a quasi-inverse $j'\colon G \to H$ of j, which is unique up to bounded distance, in the following way.

Let $\mathrm{pr}\colon G \to H\backslash G$ be the projection. Let $s\colon U_0 \to G$ be a local continuous cross section of pr defined on some compact neighborhood $U_0 \subset H\backslash G$ of $H1$ such that $s(H1) = 1$. We can extend s to a global cross section $s\colon H\backslash G \to G$ with relatively compact image $s(H\backslash G)$, which is not necessarily continuous. Then

$$j'\colon G \to H, \qquad g \mapsto g\cdot s\big(\mathrm{pr}(g)\big)^{-1}$$

is a quasi-inverse of j.

The assignment $f \mapsto j\circ f\circ j'$ yields an isomorphism $j_*\colon \mathrm{QI}(H) \to \mathrm{QI}(G)$. Let $x_0 \in X$. By the Švarc–Milnor theorem, the map $h \to hx_0$ is a quasi-isometry $f\colon H \to X$. We pick a quasi-inverse $f'\colon X \to H$ of f. Similarly, we obtain an isomorphism $f_*\colon \mathrm{QI}(H) \to \mathrm{QI}(X)$. Let $H \to \mathrm{QI}(H)$ be the homomorphism that sends a group element to its left translation. Similarly, $G \to \mathrm{QI}(G)$.

It is easy to see that the map ϕ coincides with the composition

$$H \to \mathrm{QI}(H) \xrightarrow{f_*} \mathrm{QI}(X).$$

We define $\bar{\phi}$ to be the composition

$$G \to \mathrm{QI}(G) \xrightarrow{j_*^{-1}} \mathrm{QI}(H) \xrightarrow{f_*} \mathrm{QI}(X).$$

Clearly, $\bar{\phi}$ extends ϕ. Let $l_g^G\colon G \to G$ denote left translation by $g \in G$ in G. Similarly, let $l_h^H\colon H \to H$ be left translation by $h \in H$. Then $\bar{\phi}(g)$ is represented by the quasi-

isometry

$$f_g := f \circ j' \circ l_g^G \circ j \circ f'.$$

Clearly, there are constants $C, L > 0$ only depending on j, j', f, and f' such that f_g is an (L, C)-quasi-isometry.

Let $B \subset X$ be a bounded subset. Let

$$D := \sup_{x \in X} d\big(x, f\big(f'(x)\big)\big) = \sup_{x \in X} d\big(x, f'(x)x_0\big) < \infty.$$

The set

$$\big\{ f'(x) \,\big|\, x \in B \big\} \subset H$$

is bounded and, thus, relatively compact. Let $K \subset H$ be its closure. The continuous map

$$H \times G \to H \backslash G, \qquad (h, g) \mapsto \mathrm{pr}(gh)$$

sends $H \times \{1\}$ to $H1$. By the continuity and compactness of K there is an identity neighborhood $V_0 \subset G$ such that $K \times V_0$ is mapped to $U_0 \subset H \backslash G$. Then the function

$$\psi \colon K \times V_0 \to \mathbb{R}, \qquad (h, g) \mapsto d\big(ghs\big(\mathrm{pr}(gh)\big)^{-1} x_0, hx_0\big)$$

is continuous and maps $K \times \{1\}$ to 0. By the continuity and compactness of K there is an identity neighborhood $V \subset V_0$ such that $\psi(K \times V) \subset [0, 1]$.

Now let $x \in B$, and let $g \in V$. Let $h = f'(x) \in K$. Then

$$\begin{aligned} d\big(f_g(x), x\big) = d\big(f\big(j'\big(gj\big(f'(x)\big)\big)\big), x\big) &= d\big(j'\big(gf'(x)\big)x_0, x\big) \\ &\le d\big(j'(gh)x_0, hx_0\big) + D \\ &= d\big(ghs\big(\mathrm{pr}(gh)\big)^{-1} x_0, hx_0\big) + D \\ &\le D + 1. \end{aligned}$$

Upon replacing the constant C by $\max\{C, D + 1\}$, the statement follows. □

4. Arithmetic lattices, tree extensions, and the arithmetic core

In Section 4.1 we introduce the notion of tree extensions of S-arithmetic lattices, which appears in our main result, Theorem A. In Section 5 we discuss an arithmeticity result for lattices in a product of a semisimple Lie group and a tdlc group, which identifies the second case in Theorem A.

4.1. Arithmetic lattices and tree extensions

Setup 4.1
Let K be a number field, let $\mathcal{O}$ be its ring of integers, and let $\mathbf{H}$ be a connected, noncommutative, absolutely simple adjoint K-group. Let $\mathcal{V}$ be the set of inequivalent valuations (places) of K, let $\mathcal{V}^\infty$ denote the Archimedean ones, and let $\mathcal{V}^{\text{fin}} = \mathcal{V} - \mathcal{V}^\infty$ be the non-Archimedean ones (finite places). For $v \in \mathcal{V}$ we denote by K_v the completion of K with respect to v; it is a local field. Let $S \subset \mathcal{V}$ be a finite subset of places that is *compatible with* $\mathbf{H}$ in the sense of [8]. Explicitly, this means the following:
(1) For every $v \in S$, $\mathbf{H}$ is K_v-isotropic.
(2) S contains all $v \in \mathcal{V}^\infty$ for which $\mathbf{H}$ is K_v-isotropic.
(3) S contains at least one finite and one infinite place.
Let $S^\infty = S \cap \mathcal{V}^\infty$ and $S^{\text{fin}} = S \cap \mathcal{V}^{\text{fin}}$. Let $\mathcal{O}[S] \subset K$ be the ring of S-integers.

For $v \in \mathcal{V}$ let $\mathbf{H}(K_v)^+ < \mathbf{H}(K_v)$ be the normal subgroup as defined in [12, Section 6]. Since K_v is perfect, the group $\mathbf{H}(K_v)^+$ is the subgroup generated by all unipotent elements and it has finite index in $\mathbf{H}(K_v)$. If $v \in \mathcal{V}^\infty$, then $\mathbf{H}(K_v)^+$ is just the connected component of the identity in the real Lie group $\mathbf{H}(K_v)$. Define

$$\mathbf{H}_{K,S} := \prod_{v \in S} \mathbf{H}(K_v), \qquad \mathbf{H}^+_{K,S} := \prod_{v \in S} \mathbf{H}(K_v)^+.$$

The quotient $\mathbf{H}_{K,S}/\mathbf{H}^+_{K,S}$ is finite.

By reduction theory after Borel and Harish-Chandra the diagonal embedding realizes $\mathbf{H}(\mathcal{O}[S])$ is a lattice in $\mathbf{H}_{K,S}$. Note that

$$\mathbf{H}_{K,S} = \mathbf{H}_{K,S^\infty} \times \mathbf{H}_{K,S^{\text{fin}}}$$

is the splitting into a semisimple real Lie group and a totally disconnected locally compact group.

Remark 4.2
If S is finite set of places compatible with $\mathbf{H}$, then $\mathbf{H}(\mathcal{O}[S])$ is a lattice in a product of at least two lcsc groups, one of them being a Lie group and one of them being a tdlc group.

We now exhibit a generalization of the previous type of lattices.

Definition 4.3
Let S be finite. Let S_1^{fin} denote the finite places $v \in S^{\text{fin}}$ such that the K_v-rank of $\mathbf{H}$ is 1; denote by T_v the associated Bruhat–Tits tree, and denote by $\text{Aut}(T_v)$ the tdlc group

of automorphisms of this simplicial tree. Any closed intermediate group

$$\mathbf{H}^+_{K,S} = \prod_{\nu\in S} \mathbf{H}(K_\nu)^+ < H^* < \prod_{\mu\in S-S_1^{\mathrm{fin}}} \mathbf{H}(K_\mu) \times \prod_{\nu\in S_1^{\mathrm{fin}}} \mathrm{Aut}(T_\nu)$$

is called a *tree extension* of $\mathbf{H}_{K,S}$.

Since $\mathbf{H}^+_{K,S} = \prod_{\nu\in S} \mathbf{H}(K_\nu)^+ < H^*$ is cocompact, the inclusion of a subgroup commensurable to $\mathbf{H}(\mathcal{O}[S])$ into a tree extension of $\mathbf{H}_{K,S}$ is a lattice embedding by Lemma 3.8.

Definition 4.4
Retaining Setup 4.1, we call the inclusion of a subgroup commensurable to $\mathbf{H}(\mathcal{O}[S])$ into $\mathbf{H}_{K,S}$ or into a tree extension of $\mathbf{H}_{K,S}$ an *S-arithmetic lattice (embedding)* or an *S-arithmetic lattice (embedding) up to tree extension*, respectively.

Remark 4.5
Denote by X_ν the Bruhat–Tits buildings or the symmetric space of $\mathbf{H}(K_\nu)$. For $\nu \in S - S_1^{\mathrm{fin}}$ the group $\mathrm{Isom}(X_\nu)$ contains $\mathbf{H}(K_\nu)^+$ as a subgroup of finite index by Theorem 3.29. Further, $\prod_{\nu\in S} \mathrm{Isom}(X_\nu)$ is a finite-index subgroup of $\mathrm{Isom}(\prod_{\nu\in S} X_\nu)$. This follows from a generalization of the de Rham decomposition (see [21, Theorem 1.9]). Thus, any closed intermediate subgroup

$$\mathbf{H}^+_{K,S} < H^* < \mathrm{Isom}\Big(\prod_{\nu\in S} X_\nu\Big)$$

is, up to passing to a finite-index subgroup, a tree extension of $\mathbf{H}_{K,S}$.

4.2. Arithmetic core theorem
Next we state an arithmeticity result for lattices in products that provides a key step in the proof of Theorem A.

THEOREM 4.6 (Arithmetic core theorem)
Let H be a connected, center-free, semisimple, real Lie group without compact factors, let D be a tdlc group, and let $\Gamma < H \times D$ be a lattice. Assume that
(1) *the projection $\Gamma \to D$ has a dense image,*
(2) *the projection $\Gamma \to H$ has a dense image,*
(3) *the projection $\Gamma \to H$ is injective,*
(4) *D is compactly generated.*
Then there exist number fields $K_1, \dots, K_n$, connected, noncommutative, adjoint, absolutely simple K_i-groups $\mathbf{H}_i$, and finite sets $S_i \subset \mathcal{V}_i$ of places of K_i compatible

with $\mathbf{H}_i$ *as in Section 4.1 with the following properties. There are a topological isomorphism*

$$H \cong \prod_{i=1}^{n} (\mathbf{H}_i)^0_{K_i, S_i^\infty}$$

and a continuous epimorphism $D \twoheadrightarrow Q$ *with compact kernel such that* Q *is a closed intermediate subgroup*

$$\prod_{i=1}^{n} (\mathbf{H}_i)^+_{K_i, S_i^{\mathrm{fin}}} < Q < \prod_{i=1}^{n} (\mathbf{H}_i)_{K_i, S_i^{\mathrm{fin}}}$$

and the image of Γ *in* $H \times Q$ *is commensurable with the image of*

$$\prod_{i=1}^{n} \mathbf{H}_i\big(\mathcal{O}_i[S_i]\big) \longrightarrow \prod_{i=1}^{n} (\mathbf{H}_i)_{K_i, S_i^\infty} \times \prod_{i=1}^{n} (\mathbf{H}_i)_{K_i, S_i^{\mathrm{fin}}}.$$

This result is quite close to Theorem 5.20 in the paper [22] of Caprace and Monod. In fact, if H is assumed to be simple instead of semisimple, then Theorem 4.6 is essentially their theorem. In our companion paper [8] we prove Theorem 4.6, deducing it from a more general statement, in which D is not assumed to be compactly generated. In this more general case the sets S_i of places of K_i might be infinite, and the gap between $\mathbf{H}^+_{K,S}$ and $\mathbf{H}_{K,S}$ becomes large.

In the current paper we use the above arithmetic core Theorem 4.6 in Section 5, Step 3, case (iii). At this point in the proof we do not know yet that Γ (called Γ_3 there) is finitely generated, but do know the compact generation of D (called G_3^{td} there) inherited from the compact generation assumption on G.

For the reader's convenience we sketch the idea of the proof of Theorem 4.6 as it appears in [8]. Apart from Step 1 below our approach differs from that taken by Caprace and Monod in [22]. In the following sketch we ignore some important details for the sake of transparency.

(1) Choose a compact open subgroup $U < D$, and observe that the projection Δ of $\Gamma_U := \Gamma \cap (H \times U)$ to H is a lattice in H, commensurated by the projection of all of Γ to H. If Δ is irreducible, Margulis's commensurator superrigidity and arithmeticity theorems provide a number field K and a simple K-algebraic group $\mathbf{H}$, so that the semisimple real Lie group H is locally isomorphic to $\mathbf{H}_{K,S^\infty}$ and Δ is commensurable to the arithmetic lattice $\mathbf{H}(\mathcal{O})$; here S^∞ is the set of infinite places $v \in \mathcal{V}^\infty$ for which $\mathbf{H}(K_v)$ is noncompact. If Δ is reducible, it is commensurable to a product

$$\Delta_1 \times \cdots \times \Delta_n < H_1 \times \cdots \times H_n,$$

where $\Delta_i < H_i$ are irreducible lattices, leading to fields K_i and K_i-simple groups $\mathbf{H}_i$, $i = 1, \dots, n$. For clarity we continue with the $n = 1$ case.

(2) We now view Γ as a subgroup of the commensurator $\mathrm{Commen}_H(\mathbf{H}(\mathcal{O}))$ of the arithmetic lattice $\mathbf{H}(\mathcal{O}) \simeq \Delta$. If one glosses over the difference between the simply connected and adjoint forms of $\mathbf{H}$ (such as SL_d and PGL_d), then the above commensurator is the subgroup of rational points $\mathbf{H}(K)$, and therefore, $\Gamma < \mathbf{H}(K)$. Define S^{fin} to be the set of those non-Archimedean places $v \in \mathcal{V}^{\mathrm{fin}}$ for which the image of Γ in $\mathbf{H}(K_v)$ is unbounded. (For example, for $\mathrm{SL}_d(\mathbb{Z}) < \Gamma < \mathrm{PSL}(\mathbb{Q})$ the set S^{fin} would consist of primes p that appear with arbitrarily high powers in denominators of entries of $\gamma \in \Gamma$). Then Γ is commensurable to a subgroup of $\mathbf{H}(\mathcal{O}[S])$.

(3) For $v \in S^{\mathrm{fin}}$, the image of Γ in the tdlc group $\mathbf{H}(K_v)$ is not precompact. It can be shown using Howe and Moore's theorem (alternatively, using [61, Theorem (T)]) that this implies that the image of Γ is dense in (an open subgroup of finite index in) $\mathbf{H}(K_v)$. One can even show that the image of Γ is dense in (an open subgroup of finite index in) $\mathbf{H}_{K,S^{\mathrm{fin}}}$.

(4) We now consider the closure L of the diagonal imbedding of Γ in $D \times \mathbf{H}_{K,S^{\mathrm{fin}}}$. Using the fact that the projections to both factors are dense and that the closure of $\Delta \simeq \mathbf{H}(\mathcal{O})$ in both projections is open compact, one shows that L is a graph of a continuous epimorphism $D \to \mathbf{H}_{K,S^{\mathrm{fin}}}$ with compact kernel.

(5) Finally, the fact that $\Gamma < H \times D$ is a lattice implies that its image, contained in the S-arithmetic lattice $\mathbf{H}(\mathcal{O}[S])$, is a lattice in $H \times \mathbf{H}_{K,S^{\mathrm{fin}}} \simeq \mathbf{H}_{K,S}$. Thus, $\Gamma \simeq \mathbf{H}(\mathcal{O}[S])$. If D is compactly generated, then so is $\mathbf{H}_{K,S^{\mathrm{fin}}}$, which implies that S^{fin} (hence, also all of S) is finite. It also follows that Γ is finitely generated.

5. Proof of Theorem A

The starting point of the proof of Theorem A is a consequence of Hilbert's fifth problem, which was observed by Burger and Monod in [17, Theorem 3.3.3].

THEOREM 5.1

Every locally compact group H contains an open, normal, finite-index subgroup H' containing $\mathrm{Rad}_{\mathrm{am}}(H)$ such that the quotient $H'/\mathrm{Rad}_{\mathrm{am}}(H)$ is topologically isomorphic to a direct product of a connected, center-free, semisimple, real Lie group without compact factors and a tdlc group with trivial amenable radical.

In fact, $H' = \ker[H \to \mathrm{Out}((H/\mathrm{Rad}_{\mathrm{am}}(H))^0)]$ is the kernel of the homomorphism to the (finite) outer automorphism group of the semisimple, center-free, real Lie group $(H/\mathrm{Rad}_{\mathrm{am}}(H))^0$—the connected component of the identity of the lcsc group $H/\mathrm{Rad}_{\mathrm{am}}(H)$.

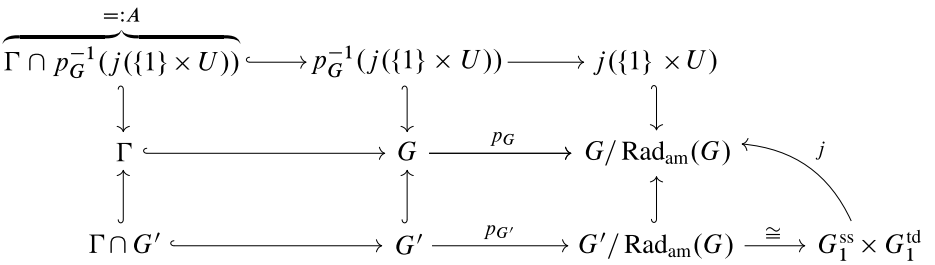
Step 1: Reduction to a lattice in a product
It is only the very first step of the proof where we will take advantage of the property (CAF).

THEOREM 5.2
The amenable radical of a lattice envelope of a group with property (CAF) *is compact.*

Proof
Let $G' < G$ be an open, normal, finite-index subgroup so that

$$G_1 := G'/\operatorname{Rad}_{\mathrm{am}}(G) \cong G_1^{\mathrm{ss}} \times G_1^{\mathrm{td}}$$

as in Theorem 5.1, namely, G_1^{ss} is a connected, center-free, semisimple, real Lie group without compact factors, and G_1^{td} is a tdlc group with $\operatorname{Rad}_{\mathrm{am}}(G_1^{\mathrm{td}}) = \{1\}$. Pick a compact open subgroup $U < G_1^{\mathrm{td}}$. We consider the following commutative diagram:



The arrows are the obvious inclusions and projections. Moreover, j is defined by requiring commutativity.

First, we show that A is commensurated by Γ, which is equivalent to $j(\{1\} \times U)$ being commensurated by $G/\operatorname{Rad}_{\mathrm{am}}(G)$ by Lemma 2.3. Let $g \in G/\operatorname{Rad}_{\mathrm{am}}(G)$. Since G' is normal in G and thus $G'/\operatorname{Rad}_{\mathrm{am}}(G)$ is normal in $G/\operatorname{Rad}_{\mathrm{am}}(G)$, conjugation by g is a continuous automorphism of $j(G_1^{\mathrm{ss}} \times G_1^{\mathrm{td}}) \cong G_1^{\mathrm{ss}} \times G_1^{\mathrm{td}}$. By Lemma 3.17, this automorphism is the product of continuous automorphisms $c_g^{\mathrm{ss}} \colon G_1^{\mathrm{ss}} \to G_1^{\mathrm{ss}}$ and $c_g^{\mathrm{td}} \colon G_1^{\mathrm{td}} \to G_1^{\mathrm{td}}$. Hence, conjugation by g maps $j(\{1\} \times U)$ to $j(\{1\} \times c_g^{\mathrm{td}}(U))$. Since the subgroup $c_g^{\mathrm{td}}(U) < G_1^{\mathrm{td}}$ is open and compact, the intersection $c_g^{\mathrm{td}}(U) \cap U$ has finite index in U. This implies that $j(\{1\} \times U)$ is commensurated by $G/\operatorname{Rad}_{\mathrm{am}}(G)$, from which we conclude that A is commensurated by Γ.

Let $G'_U < G'$ be the preimage of $G_1^{\mathrm{ss}} \times U$ under $p_{G'}$, and let

$$K := \ker(G'_U \xrightarrow{p_{G'}} G_1^{\mathrm{ss}} \times U \to G_1^{\mathrm{ss}}).$$

The group K is an extension of the amenable group $\operatorname{Rad}_{\mathrm{am}}(G)$ by the compact group U. Hence, K is amenable. The group $\Gamma \cap G'_U$ is a lattice in G'_U by Lemma 3.4.

We now invoke a deep result of Breuillard and Gelander from [14, Section 9]. It follows from [14, Theorem 9.5] and K being a closed amenable subgroup that the projection of the lattice $\Gamma \cap G'_U$ in G'_U/K is discrete. The result of Breuillard and Gelander is a generalization of Auslander's theorem, where K is a solvable Lie group.

By Lemma 3.5, the group

$$A = \Gamma \cap p_{G'}^{-1}\big(j\left(\{1\} \times U\right)\big) = K \cap (\Gamma \cap G'_U)$$

is a lattice in K and, thus, is amenable. Note that we used $p_G^{-1}(j(\{1\} \times U)) = p_{G'}^{-1}(j(\{1\} \times U))$ here, which follows from the diagram. On the other hand A is commensurated by Γ and, hence, finite. This implies that K and its closed subgroup $\mathrm{Rad}_{\mathrm{am}}(G)$ are compact. □

Let us summarize the situation after the first step.

PROPOSITION 5.3

Let Γ have property (CAF)*, and let $\Gamma < G$ be a lattice embedding. Then the amenable radical $K := \mathrm{Rad}_{\mathrm{am}}(G)$ of G is compact. By passing to an open normal finite-index subgroup $G' < G$ containing K and to the finite-index subgroup $\Gamma' := \Gamma \cap G' < \Gamma$ and taking quotients by compact or finite normal subgroups $G_1 = G'/K$ and $\Gamma_1 := \Gamma'/(\Gamma' \cap K)$, one obtains a lattice embedding*

$$\Gamma_1 < G_1 = G_1^{\mathrm{ss}} \times G_1^{\mathrm{td}},$$

virtually isomorphic to $\Gamma < G$, into a product of a connected, center-free, semisimple real Lie group without compact factors G_1^{ss} and a tdlc group with trivial amenable radical G_1^{td}. Furthermore, G is compactly generated if and only if G_1^{td} is compactly generated.

Step 2: Projection to the semisimple factor is irreducible

Let $\Gamma_1 < G_1 = G_1^{\mathrm{ss}} \times G_1^{\mathrm{td}}$ be a lattice in a product as in Proposition 5.3. Since compact generation of G is assumed in Theorem A, the group G_1 is compactly generated. The center-free, semisimple, real Lie group G_1^{ss} splits as a direct product of the simple factors:

$$G_1^{\mathrm{ss}} = S_1 \times \cdots \times S_n.$$

For a subset $J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ we denote by $S_J = S_{j_1} \times \cdots \times S_{j_k}$ the subproduct, which can be viewed both as a subgroup and as a factor of the semisimple group G_1^{ss}. Set $S_\emptyset = \{1\}$. Given a subset $J \subset \{1, \ldots, n\}$ consider the image of Γ_1

under the projection

$$\mathrm{pr}_J : G_1 \longrightarrow G_1^{\mathrm{ss}} \longrightarrow S_J.$$

Note that it is possible that $\mathrm{pr}_{\{i\}}(\Gamma_1)$ is dense in S_i for each $i \in \{1, \ldots, n\}$, but $\mathrm{pr}_J(\Gamma_1)$ is discrete for some nonempty proper subset $J \subset \{1, \ldots, n\}$.

LEMMA 5.4
There is a unique maximal subset $J \subset \{1, \ldots, n\}$ for which the projection $\mathrm{pr}_J(\Gamma_1)$ is discrete in S_J.

Proof
It suffices to show that the collection of all subsets $J \subset \{1, \ldots, n\}$ with discrete projection to S_J is closed under union. Let $J, K \subset \{1, \ldots, n\}$ be subsets such that $\mathrm{pr}_J(\Gamma_1)$ is discrete in S_J and $\mathrm{pr}_K(\Gamma_1)$ is discrete in S_K. Let $V \subset S_J$, $W \subset S_K$ be open neighborhoods of the identity with

$$\mathrm{pr}_J(\Gamma_1) \cap V = \{1\}, \qquad \mathrm{pr}_K(\Gamma_1) \cap W = \{1\}.$$

View $S_{J\cup K}$ as a subgroup in $S_J \times S_K$. Then $U = S_{J\cup K} \cap (V \times W)$ is an open neighborhood of the identity with $\mathrm{pr}_{J\cup K}(\Gamma_1) \cap U = \{1\}$. Therefore, $\mathrm{pr}_{J\cup K}(\Gamma_1)$ is discrete in $S_{J\cup K}$. This proves the lemma. □

Let $J \subset \{1, \ldots, n\}$ be the maximal subset as in Lemma 5.4. Denote

$$L = S_J = \prod_{j\in J} S_j, \qquad H = S_{J^c} = \prod_{i\notin J} S_i.$$

We have $G_1^{\mathrm{ss}} = L \times H$ and consider the projection

$$\mathrm{pr}_L : G_1 = L \times H \times G_1^{\mathrm{td}} \to L, \qquad \Gamma_0 = \mathrm{pr}_L(\Gamma_1).$$

Define Γ_2 to be the kernel of this projection

$$\Gamma_2 = \mathrm{Ker}(\mathrm{pr}_L : \Gamma_1 \longrightarrow \Gamma_0).$$

Then $\Gamma_0 < L$ and $\Gamma_2 < H \times G_1^{\mathrm{td}}$ are lattices (Lemma 3.5). Consider the projections

$$p : H \times G_1^{\mathrm{td}} \longrightarrow H, \qquad q : H \times G_1^{\mathrm{td}} \longrightarrow G_1^{\mathrm{td}}.$$

LEMMA 5.5
The projection $p(\Gamma_2)$ of Γ_2 to H is dense.

Proof
Let $H' = \overline{p(\Gamma_2)}$ be the closure of the projection of Γ_2 to H. Then $H' \times G_1^{\mathrm{td}}$ is a closed subgroup of $H \times G_1^{\mathrm{td}}$ containing the lattice Γ_2. By Lemma 3.8, $H' \times G_1^{\mathrm{td}}$ is a closed subgroup of finite covolume in $H \times G_1^{\mathrm{td}}$. It follows that H' has finite covolume in H. By Lemma 3.10, H' has the form $H' = \Delta \times H_2$, where H_2 is a direct factor of H, $H = H_1 \times H_2$, and $\Delta < H_1$ is a lattice. In our setting such a splitting has to be trivial:

$$H' = H_2 = H, \qquad H_1 = \Delta = \{1\}.$$

Indeed, otherwise the semisimple group G_1^{ss} splits as $G_1^{\mathrm{ss}} = (L \times H_1) \times H_2$, where the projection of Γ_1 to the $L \times H_1$-factor lies in $\Gamma_0 \times \Delta$, which is discrete. This contradicts the maximality of L as such a factor. This completes the proof of the lemma. □

We now define an lcsc group G_2 to be $G_2 = G_2^{\mathrm{ss}} \times G_2^{\mathrm{td}}$, where

$$G_2^{\mathrm{ss}} = H, \qquad G_2^{\mathrm{td}} = \overline{q(\Gamma_2)} < G_1^{\mathrm{td}}.$$

By being a closed subgroup of a tdlc group G_1^{td}, the group G_2^{td} itself is tdlc. The group $G_2^{\mathrm{ss}} = H$ is a connected, center-free, semisimple, real Lie group.

The group G_2 is a closed subgroup of $H \times G_1^{\mathrm{td}}$ containing a lattice $\Gamma_2 < H \times G_1^{\mathrm{td}}$. By Lemma 3.8, Γ_2 forms a lattice in G_2, and $G_2 = H \times G_2^{\mathrm{td}}$ is a finite covolume subgroup of $H \times G_1^{\mathrm{td}}$. Thus, G_2^{td} is a subgroup of finite covolume in G_1^{td}. Since $\mathrm{Rad}_{\mathrm{am}}(G_1^{\mathrm{td}}) = \{1\}$ we deduce that $\mathrm{Rad}_{\mathrm{am}}(G_2^{\mathrm{td}}) = \{1\}$ using Lemma 3.12. We summarize in the following.

PROPOSITION 5.6
Let $\Gamma_1 < G_1 = G_1^{\mathrm{ss}} \times G_1^{\mathrm{td}}$ be as in Proposition 5.3. Then there are a splitting $G_1^{\mathrm{ss}} = L \times H$ and a closed subgroup $G_2^{\mathrm{td}} < G_1^{\mathrm{td}}$ such that, by setting $G_2^{\mathrm{ss}} = H$ and $G_2 = G_2^{\mathrm{ss}} \times G_2^{\mathrm{td}}$, we have the following.

(1) *The projection $\Gamma_0 = \mathrm{pr}_L(\Gamma_1)$ of Γ_1 is a lattice in L.*
(2) *The kernel $\Gamma_2 = \mathrm{Ker}(\Gamma_1 \xrightarrow{\mathrm{pr}_L} \Gamma_0)$ is a lattice in $G_2 = G_2^{\mathrm{ss}} \times G_2^{\mathrm{td}}$.*
(3) *The projection $\mathrm{pr}_2^{\mathrm{ss}}(\Gamma_2)$ is dense in $G_2^{\mathrm{ss}} = H$.*
(4) *The projection $\mathrm{pr}_2^{\mathrm{td}}(\Gamma_2)$ is dense in G_2^{td}.*
(5) *$G_2^{\mathrm{td}} < G_1^{\mathrm{td}}$ is a closed subgroup of finite covolume.*
(6) *G_2^{td} has trivial amenable radical.*
(7) *G_2^{td} is compactly generated.*

Most of the conditions needed for an application of Theorem 4.6 are satisfied with the exception of injectivity of the projection

$$\mathrm{pr}_2^{\mathrm{ss}} : \Gamma_2 \longrightarrow G_2^{\mathrm{ss}} = H.$$

This is our next topic of concern.

Step 3: Identifying the lattice embedding
We still denote the projections of G_1 and G_2 to their Lie and tdlc factors by

$$\mathrm{pr}_i^{\mathrm{ss}} \colon G_i \to G_i^{\mathrm{ss}} \qquad \text{and} \qquad \mathrm{pr}_i^{\mathrm{td}} \colon G_i \to G_i^{\mathrm{td}} \quad \text{for } i \in \{1,2\}.$$

The images of a subgroup in G_i under $\mathrm{pr}_i^{\mathrm{ss}}$ and $\mathrm{pr}_i^{\mathrm{td}}$ will be indicated by superscripts ss and td, respectively. For a subgroup $H < G$ let

$$\mathcal{Z}_G(H) := \{g \in G \mid \forall_{h \in H}\ gh = hg\}$$

denote the centralizer of H in G.

Let $U < G_1^{\mathrm{td}}$ be a compact open subgroup in the tdlc group G_1^{td} such that $\Gamma_1 \cap (\{1\} \times U) = \{1\}$. We define the following groups:

$$\begin{aligned}
&N_i := \Gamma_i \cap \left(\{1\} \times G_i^{\mathrm{td}}\right) \quad \text{for } i \in \{1,2\},\\
&M_1 := \Gamma_1 \cap (G_1^{\mathrm{ss}} \times U),\\
&M_2 := \Gamma_2 \cap M_1,\\
&\Gamma_3 := \Gamma_2 / N_2,\\
&M_3 := p(M_2) \quad \text{where } p \colon \Gamma_2 \twoheadrightarrow \Gamma_3 \text{ is the projection,}\\
&G_3^{\mathrm{ss}} := G_2^{\mathrm{ss}}, \qquad G_3^{\mathrm{td}} := G_2^{\mathrm{td}} / N_2^{\mathrm{td}}, \qquad G_3 := G_3^{\mathrm{ss}} \times G_3^{\mathrm{td}}.
\end{aligned} \tag{5.1}$$

Remark 5.7
Since N_2^{td} is normalized by the dense image of Γ_2 in G_2^{td} and the normalizer of a closed subgroup is closed, the subgroup N_2^{td} is normal in G_2^{td}. This justifies the last definition. Moreover, since G_1^{ss} commutes with N_2 it follows that N_2 is normal in G_2. Note that, since compact generation passes to quotients, G_3^{td} is compactly generated.

Remark 5.8
At this point, we do not claim triviality of the amenable radical of G_3^{td}, even though it will follow from later analysis.

LEMMA 5.9
The group N_1 commutes with some finite-index subgroup of M_1.

Proof
At this point we want to apply property (NbC) of the original group Γ. Note that property (NbC) does not pass to finite-index subgroups in general, so we have to

argue more specifically. We refer to the notation of Proposition 5.3. Note that $\Gamma_1 = (\Gamma \cap G')/(\Gamma \cap K)$ is a subgroup of $\Gamma/(\Gamma \cap K)$. We show that N_1 is normal in $\Gamma/(\Gamma \cap K)$ and M_1 is commensurated by $\Gamma/(\Gamma \cap K)$. From this, the claim follows since $\Gamma/(\Gamma \cap K)$ has (NbC) by Lemma 2.5.

The group $\{1\} \times G_1^{\mathrm{td}}$ is a topologically characteristic subgroup of $G_1 = G_1^{\mathrm{ss}} \times G_1^{\mathrm{td}} = G'/K$ by Lemma 3.17. Since $G'/K < G/K$ is normal, $\{1\} \times G_1^{\mathrm{td}}$ is normal in G/K. Since $\Gamma_1 = (\Gamma \cap G')/(\Gamma \cap K)$ is a normal subgroup of $\Gamma/(\Gamma \cap K)$, the group N_1 is normal in $\Gamma/(\Gamma \cap K)$. The subgroup $G_1^{\mathrm{ss}} \times U < G_1 = G'/K$ is commensurated by G/K since $G'/K \lhd G/K$ is normal and every (topological) automorphism of G_1 is a product of an automorphism of G_1^{td} and one of G_1^{ss} by Lemma 3.17. Since $\Gamma_1 \lhd \Gamma/(\Gamma \cap K)$ is normal, M_1 is commensurated by $\Gamma/(\Gamma \cap K)$. □

By Lemma 5.9 and upon making U smaller, we may and will assume that M_1 itself centralizes N_1 and $M_1 \cap N_1 = \{1\}$. It follows that M_2 centralizes N_2. We record that

$$M_i \cap N_i = \{1\} \qquad \text{and} \qquad [M_i, N_i] = \{1\} \quad \text{for } i \in \{1, 2\}. \tag{5.2}$$

In particular, the subgroup $M_i N_i < \Gamma_i$ is isomorphic to $M_i \times N_i$.

LEMMA 5.10
The inclusions $M_i^{\mathrm{ss}} \hookrightarrow G_i^{\mathrm{ss}}$ for $i \in \{1, 2, 3\}$ and $\Gamma_3 \hookrightarrow G_3$ are lattice embeddings.

Proof
By Lemma 3.4, M_1 is a lattice in $G_1^{\mathrm{ss}} \times U$, since $U < G_1^{\mathrm{td}}$ is an open subgroup. Since U is also compact, the image M_1^{ss} of M_1 in G_1^{ss} is a lattice of G_1^{ss} as well. Similarly, $M_2 < G_2^{\mathrm{ss}} \times U$ and $M_2^{\mathrm{ss}} < G_2^{\mathrm{ss}}$ are lattices. Since $G_2^{\mathrm{ss}} = G_3^{\mathrm{ss}}$ by definition one easily sees that the images of M_2 and M_3 in G_2^{ss} coincide, so $M_3^{\mathrm{ss}} = M_2^{\mathrm{ss}}$. In particular, $M_3^{\mathrm{ss}} < G_3^{\mathrm{ss}}$ is a lattice. Since $N_2 \lhd \Gamma_2$ is also normal in G_2 (see the remark below (5.1)), the quotient Γ_3 is a lattice in $G_3 = G_2^{\mathrm{ss}} \times G_2^{\mathrm{td}}/N_2^{\mathrm{td}} \cong G_2/N_2$ by Lemma 3.5. □

We are finally in a position to identify the lattice embedding $\Gamma_1 < G_1$, which is virtually isomorphic to the original lattice embedding $\Gamma < G$. We distinguish three cases depending on the finiteness of the groups M_1 and M_2.

Case (i): M_1 is finite (Γ_1 is a lattice in a tdlc group)
In this case, the connected real Lie group G_1^{ss} without compact factors has a finite group M_1^{ss} as a lattice; thus, G_1^{ss} must be trivial. Thus, $G_1 = G_1^{\mathrm{td}}$ is a tdlc group with trivial amenable radical that contains Γ_1 as a lattice. If Γ is assumed to have property

(BT), then the same applies to Γ_1, and by Lemma 3.9, $\Gamma_1 < G_1$ is a uniform lattice. Either G_1 is discrete, which means that the original lattice embedding $\Gamma \hookrightarrow G$ was trivial or we are in case (3) of the main theorem.

Case (ii): M_1 is infinite, but M_2 is finite (Γ_1 is a lattice in a semisimple Lie group)
In this case, M_2^{ss} is finite and a lattice in G_2^{ss}. As above, we conclude that $G_2^{\mathrm{ss}} = H$ is trivial. Therefore, $\Gamma_2 = N_2$, and

$$G_2^{\mathrm{td}} = \overline{\Gamma_2^{\mathrm{td}}} = \overline{N_2^{\mathrm{td}}} = N_2^{\mathrm{td}}.$$

The last equality follows from the discreteness of N_2^{td}.

Since $M_1^{\mathrm{ss}} = \mathrm{pr}_1^{\mathrm{ss}}(M_1)$ is a lattice in $G_1^{\mathrm{ss}} = L \times H = L$ and a subgroup of the lattice Γ_0 (see Proposition 5.6), $M_1^{\mathrm{ss}} < \Gamma_0$ must have finite index. We have a short exact sequence

$$1 \longrightarrow \Gamma_2 \longrightarrow \Gamma_1 \xrightarrow{\mathrm{pr}_L} \Gamma_0 \longrightarrow 1.$$

The pr_L-preimage of M_1^{ss} which equals $\Gamma_2 \cdot M_1 = N_2 M_1 \cong N_2 \times M_1$ is a finite-index subgroup of Γ_1. Since M_1 is infinite, the (Irr) condition forces $\Gamma_2 = N_2$ to be finite. So G_2^{td} is finite. As G_2^{td} has finite covolume in G_1^{td}, the latter is compact. In fact, G_1^{td} has to be trivial, because it has a trivial amenable radical. We conclude that Γ_1 is a classical lattice in a connected, center-free, semisimple, real Lie group $L = G_1^{\mathrm{ss}} = G_1$. This lattice is irreducible due to assumption (Irr). This corresponds to case (1) in the main theorem.

Case (iii): M_2 is infinite (Γ_1 is an S-arithmetic lattice)
Recall that the projection M_2^{ss} of M_2 to $G_2^{\mathrm{ss}} = G_3^{\mathrm{ss}} = H$ is a lattice there. The assumption that M_2 is infinite means that G_3^{ss} is nontrivial. The inclusion

$$\Gamma_3 < G_3 = G_3^{\mathrm{ss}} \times G_3^{\mathrm{td}}$$

is a lattice embedding by Lemma 5.10. At this point, we may and will apply the arithmetic core Theorem 4.6 to $\Gamma_3 < G_3$ to deduce that, up to dividing G_3^{td} by a compact normal subgroup C, one has a product of S-arithmetic lattices with nontrivial semisimple and totally disconnected factors. (It will become clear below that C is trivial and that there is only one irreducible S-arithmetic lattice). More precisely, there are number fields $K_1, \ldots, K_n$, absolutely simple K_i-algebraic groups $\mathbf{H}_i$, and finite sets $S_i \subset \mathcal{V}_i$ of places of K_i compatible with $\mathbf{H}_i$ such that, denoting the connected real Lie groups by $H_i = (\mathbf{H}_i)^+_{K_i, S_i^\infty} = (\mathbf{H}_i)^\circ_{K_i, S_i^\infty}$, we have

$$G_3^{\mathrm{ss}} \times G_3^{\mathrm{td}}/C = \prod_{i=1}^{n} H_i \times \prod_{i=1}^{n} Q_i$$

for certain closed intermediate groups

$$(\mathbf{H}_i)^+_{K_i,S_i^{\mathrm{fin}}} < Q_i < (\mathbf{H}_i)_{K_i,S_i^{\mathrm{fin}}}.$$

Moreover, Γ_3 is commensurable to the product of the S_i-arithmetic lattices. There are finite-index subgroups $\Gamma_{3,i} < \mathbf{H}_i(\mathcal{O}_i[S_i])$ such that

$$\Gamma_{3,1} \times \cdots \times \Gamma_{3,n} < \Gamma_3, \quad [\Gamma_3 : \Gamma_{3,1} \times \cdots \times \Gamma_{3,n}] < \infty \tag{5.3}$$

has finite index. Upon passing to a smaller finite-index subgroup, we may assume that (5.3) is the inclusion of a normal subgroup. By Lemma 5.10,

$$M_3^{\mathrm{ss}} < G_3^{\mathrm{ss}} = H_1 \times \cdots \times H_n$$

is a lattice. It contains a product of irreducible lattices $\Delta_i < H_i$ as a finite-index subgroup:

$$\Delta_1 \times \cdots \times \Delta_n < M_3^{\mathrm{ss}}, \quad [M_3^{\mathrm{ss}}, \Delta_1 \times \cdots \times \Delta_n] < \infty.$$

Relying on the (NbC) condition we showed (see (5.2)) that $M_2 < \Gamma_2$ commutes with N_2, so

$$M_2 < \mathcal{Z}_{\Gamma_2}(N_2). \tag{5.4}$$

The Abelian subgroup $\mathcal{Z}_{\Gamma_2}(N_2) \cap N_2 = \mathcal{Z}(N_2)$ is, being a characteristic subgroup of $N_2^{\mathrm{td}} \lhd G_2^{\mathrm{td}}$, normal in G_2^{td}. Since G_2^{td} has a trivial amenable radical,

$$\mathcal{Z}_{\Gamma_2}(N_2) \cap N_2 = \{1\}. \tag{5.5}$$

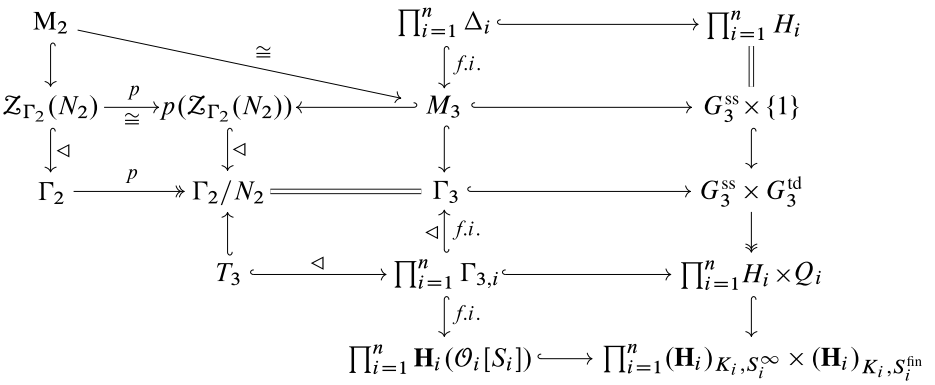
Hence, the restriction $p|_{\mathcal{Z}_{\Gamma_2}(N_2)}$ of the quotient map $p\colon \Gamma_2 \to \Gamma_3 = \Gamma_2/N_2$ is injective. The image $p(\mathcal{Z}_{\Gamma_2}(N_2))$ of the normal subgroup $\mathcal{Z}_{\Gamma_2}(N_2)$ is normal in Γ_3 and contains M_3. The subgroup

$$T_3 := p\big(\mathcal{Z}_{\Gamma_2}(N_2)\big) \cap \prod_{i=1}^{n} \Gamma_{3,i}$$

has finite index in $p(\mathcal{Z}_{\Gamma_2}(N_2))$ and is normal in $\Gamma_{3,1} \times \cdots \times \Gamma_{3,n}$. The commutative diagram below summarizes the various relations between the groups.

Since Δ_i and $M_3 \cap T_3 \cap \Gamma_{3,i}$ are commensurable, the normal subgroup $T_3 \cap \Gamma_{3,i} \lhd \Gamma_{3,i}$ is infinite. Since each $\Gamma_{3,i}$ is an irreducible S_i-arithmetic lattice, Margulis's normal subgroup theorem implies that each $T_3 \cap \Gamma_{3,i} < \Gamma_{3,i}$ is of finite index; thus, $p(\mathcal{Z}_{\Gamma_2}(N_2)) < \Gamma_3$ is of finite index. By the injectivity of $p|_{\mathcal{Z}_{\Gamma_2}(N_2)}$, the group $\mathcal{Z}_{\Gamma_2}(N_2)$ is thus (abstractly) commensurable with a product of S-arithmetic lattices.

By Theorem 3.14, $\mathcal{Z}_{\Gamma_2}(N_2)$ has a finite outer automorphism group. Further, the center of $\mathcal{Z}_{\Gamma_2}(N_2) \lhd \Gamma_2$ is trivial, since it lies in the amenable radical of Γ_2, which is trivial by Proposition 5.6 and Lemma 3.12. We have the following diagram:



Since N_2 is the intersection of $\Gamma_2 \lhd \Gamma_1$ and $N_1 \lhd \Gamma_1$ the subgroup N_2 is normal in Γ_1, which implies that $\mathcal{Z}_{\Gamma_2}(N_2)$ is normal in Γ_1. By Lemma 3.15 the finite-index subgroup $\Gamma_1' = \ker(\Gamma_1 \to \mathrm{Out}(\mathcal{Z}_{\Gamma_2}(N_2))) < \Gamma_1$ splits as a direct product:

$$\Gamma_1' \cong \mathcal{Z}_{\Gamma_2}(N_2) \times \big(\Gamma_1'/\mathcal{Z}_{\Gamma_2}(N_2)\big).$$

Assumption (Irr) implies that one of the factors is finite. By M_2 being infinite and (5.4), we obtain that $\mathcal{Z}_{\Gamma_2}(N_2)$ is of finite index in Γ_1; hence, $\Gamma_2 < \Gamma_1$ is of finite index. So Γ_0 is finite. Because of $[\Gamma_1 : \mathcal{Z}_{\Gamma_2}(N_2)] < \infty$ and (5.5) we obtain that N_1 and N_2 are finite. The group L, being a lattice envelope of Γ_0, is compact; thus, L and Γ_0 are trivial.

So $G_1^{\mathrm{ss}} = H = G_2^{\mathrm{ss}} = G_3^{\mathrm{ss}}$. Using condition (Irr) we can now also deduce that there was only one irreducible factor; hence, $n = 1$ in (5.3).

Furthermore, the group $N_2 \cong N_2^{\mathrm{td}}$, which is now known to be finite, has to be trivial, because N_2^{td} is normal in G_2^{td}, while $\mathrm{Rad}_{\mathrm{am}}(G_2^{\mathrm{td}}) = \{1\}$. It follows that $G_3 = G_2$ and $\mathrm{Rad}_{\mathrm{am}}(G_3^{\mathrm{td}}) = \{1\}$. In particular, the compact normal subgroup $C \lhd G_3^{\mathrm{td}}$ is actually trivial.

We deduce that there is a number field K, a connected, noncommutative, adjoint, absolutely simple K-group $\mathbf{H}$, and a finite set $S = S^\infty \cup S^{\mathrm{fin}}$ of places as in Section 4.1 so that

$$\Gamma_1 = \Gamma_2 = \Gamma_3 \simeq \mathbf{H}\big(\mathcal{O}[S]\big)^+$$

is an irreducible S-arithmetic lattice in the lcsc group

$$G_2 = G_2^{\mathrm{ss}} \times G_2^{\mathrm{td}} = \mathbf{H}^+_{K,S^\infty} \times \mathbf{H}^+_{K,S^{\mathrm{fin}}}.$$

It remains to identify the lcsc group G_1. We have established the triviality of Γ_0 and of L; hence,

$$G_1^{\mathrm{ss}} = H \times L = H = G_2^{\mathrm{ss}} = G_3^{\mathrm{ss}} = \mathbf{H}^+_{K,S^\infty}.$$

It remains to identify the totally disconnected component G_1^{td} that contains

$$G_2^{\mathrm{td}} = \mathbf{H}^+_{K,S^{\mathrm{fin}}} = \prod_{v \in S^{\mathrm{fin}}} \mathbf{H}(K_v)^+$$

as a closed subgroup of finite covolume.

Step 4: Lattice envelope G_1^{td} of $G_2^{\mathrm{td}} = \mathbf{H}^+_{K,S^{\mathrm{fin}}}$
Let us enumerate the elements in $S^{\mathrm{fin}} = \{v_1, \dots, v_n\}$ in such a way that $\mathbf{H}$ has K_{v_i}-rank at least 2 for $1 \le i \le k$ and has K_{v_i}-rank 1 for $k < i \le n$. The extreme cases $k = n$ or $k = 0$ are, of course, possible.

Let X_i be the Bruhat and Tits building associated to $\mathbf{H}(K_{v_i})$, and let $H_i := \mathbf{H}(K_{v_i})^+$. We write

$$H = H_1 \times \cdots \times H_n \qquad \text{and} \qquad X = X_1 \times \cdots \times X_n.$$

With that notation, $H = G_2^{\mathrm{td}}$. The spaces X_i are irreducible Euclidean buildings with cocompact affine Weyl group. The group H acts by automorphisms on the simplicial complex X, and this action is *strongly transitive* in the sense that H acts transitively on pairs (C, A), where $C \subset A$ is a chamber in an apartment.

The proof of Theorem A is completed by applying the following general theorem to the subgroup $H = G_2^{\mathrm{td}}$ in $G = G_1^{\mathrm{td}}$, using Remark 4.5.

THEOREM 5.11
Let X be a locally finite affine building without Euclidean factors. Let H be a lcsc group acting isometrically on X strongly transitively and with a compact kernel. Assume that $\phi\colon H \to G$ is a continuous homomorphism with compact kernel and a closed image of cofinite volume. Then the action of H on X extends via ϕ to an isometric action with a compact kernel of G on X.

The proof of Theorem 5.11 proceeds in several steps: reduction to the case where G is tdlc (already given in our main application), proof that the image of H is cocompact in G, and finally the use of techniques from geometric group theory, similar to the approach taken in [35, Section 3].

Reduction to G being tdlc with trivial amenable radical
This reduction is not needed for our application, where G is G_1^{td}. We will first show that both the connected component G^0 and $\mathrm{Rad}_{\mathrm{am}}(G)$ are compact normal subgroups,

so that dividing by them we may assume G to be tdlc with trivial amenable radical. A group acting strongly transitively on a thick Euclidean building has trivial amenable radical (cf. [21, Theorem 1.10], where this is deduced from [2]). Therefore, $\mathrm{Rad}_{\mathrm{am}}(H)$ is contained in $\ker(H \to \mathrm{Aut}(X))$, which is compact by assumption. It follows from Lemma 3.12 that $\mathrm{Rad}_{\mathrm{am}}(G) \cap \phi(H)$ is a compact group. Let $G' = G/\mathrm{Rad}_{\mathrm{am}}(G)$, and let H' be the image of H in G'. By [9, Theorem 5.1] the image of $H/\mathrm{Rad}_{\mathrm{am}}(H)$ in G' is closed. (Note that $H/\mathrm{Rad}_{\mathrm{am}}(H)$ is *quasisemisimple (qss)* by [9, Theorem 3.7].) It follows that $M = H \cdot \mathrm{Rad}_{\mathrm{am}}(G)$ is a closed subgroup of G, and applying Lemma 3.8 to the closed subgroups $H < M < G$ we conclude that $M/H \cong \mathrm{Rad}_{\mathrm{am}}(G)/\mathrm{Rad}_{\mathrm{am}}(H)$ has finite volume. Since $\mathrm{Rad}_{\mathrm{am}}(H)$ is compact, it follows that $\mathrm{Rad}_{\mathrm{am}}(G)$ has finite Haar measure and, therefore, is a compact group.

Dividing by the compact amenable radical, hereafter we assume that G and H have trivial amenable radicals. The connected component G^0 of the identity in G is a connected, center-free, semisimple real Lie group, and we consider the action of H on G^0 by conjugation, providing a homomorphism $\rho : H \to \mathrm{Aut}(G^0)$. Note that $\mathrm{Aut}(G^0)$ is a Lie group, and the image of H is closed (cf. [9, Theorem 5.1]). Since H is tdlc, the image $\rho(H)$ is discrete. Considering the isometric continuous action of H on the discrete space $\rho(H)$ we conclude by [9, Theorem 6.1] that $H_0 = \ker\rho$ has finite index in H. As before, the image of H in G/G^0 is closed, so $H \cdot G^0$ is closed in G, and the same applies to the finite-index subgroup $H_0 \cdot G^0$. Lemma 3.8 applied to $H_0 < H_0 \cdot G^0 < G$ shows that $H_0 \cdot G^0/H_0 \cong G^0/(H_0 \cap G^0)$ has finite Haar measure. But $H_0 \cap G^0 = \{1\}$ because G^0 is center-free. We conclude that G^0 has finite Haar measure and, therefore, is a compact group. Thus, for the rest of the proof we assume G to be tdlc with trivial amenable radical.

Compactness of G/H

By the Bruhat–Tits fixed-point theorem (see [16, Corollary 2.8, p. 179] for a more general result on CAT(0)-spaces) every compact subgroup of H fixes a vertex of X. But since there are only finitely many H-orbits of vertices, there are only finitely many vertex stabilizer groups up to conjugation. In particular, H endowed with Haar measure m has an upper bound on the Haar measure of its open compact subgroups:

$$\sup\{m(U) \mid U \text{ is an open compact subgroup of } H\} < +\infty.$$

Thus, Lemma 3.9 applies to $H < G$ and implies the compactness of G/H.

Therefore, we are in a position to apply Lemma 3.30 in combination with Remark 3.31 to H acting on X to obtain a homomorphism of G to the quasi-isometry group of X, $\phi\colon G \to \mathrm{QI}(X)$. In fact, we have a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Isom}(X) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\phi} & \mathrm{QI}(X) \end{array}$$

Further, there are constants $C, L > 0$ such that for every $g \in G$ there is an (L, C)-quasi-isometry $f_g : X \to X$ that represents the class $\phi(g) \in \mathrm{QI}(X)$ and has the following property: for every bounded set $B \subset X$ there is a neighborhood of the identity $V \subset G$ so that

$$\forall_{g \in V} \forall_{x \in B} \quad d_X\big(f_g(x), x\big) < C. \tag{5.6}$$

In the following, we denote by $\bar{G}$ and $\bar{H}$ the corresponding images of G and H in $\mathrm{QI}(X)$; the commutativity of the diagram above implies that $\bar{H} < \bar{G}$. For a general metric space the QI-group does not have a natural topological group structure. In the arguments below we will take advantage of the large-scale geometry of X to obtain information on $\bar{G}$ to be able to place it in the image of $\mathrm{Isom}(X)$ in $\mathrm{QI}(X)$.

Definition of the subgroup $\bar{G}' < \bar{G}$
We now apply the following splitting theorem by Kleiner and Leeb.

THEOREM 5.12 ([45, Theorem 1.1.2])
For every $C, L > 0$ there are $L', C', D > 0$ such that every (L, C)-quasi-isometry $X \to X$ is within distance D from a product of (L', C')-quasi-isometries $X_i \to X_{\pi(i)}$ between the factors for some permutation $\pi \in \mathrm{Sym}_n$.

At the expense of increasing the constants C and L, we may hence assume that, for each $g \in G$, the quasi-isometry $f_g : X \to X$ as in (5.6) is a product of (L, C)-quasi-isometries $f_g^{(i)} : X_i \to X_{\pi_g(i)}$ for some permutation $\pi_g \in \mathrm{Sym}_n$. Another consequence of Theorem 5.12 is that the product $\mathrm{QI}(X_1) \times \cdots \times \mathrm{QI}(X_n)$ embeds into $\mathrm{QI}(X)$ as a subgroup of index at most $n! = |\mathrm{Sym}_n|$. We define the finite-index subgroup $\bar{G}' < \bar{G}$ by

$$\bar{G}' = \bar{G} \cap \mathrm{QI}(X_1) \times \cdots \times \mathrm{QI}(X_n).$$

Let $\bar{H}' = \bar{H} \cap \bar{G}'$, and let $G' < G$ and $H' < H$ be the preimages of $\bar{G}'$ and $\bar{H}'$.

It follows that there are (L, C)-quasi-isometries $f_g^{(i)} : X_i \to X_i$ for every $i \in \{1, \ldots, n\}$ and $g \in G'$ such that $f_g^{(i)}$ represents the ith component of the quasi-isometry f_g. In view of (5.6) and the quantitative statement in Theorem 5.12 and at the expense of increasing $C, L > 0$ once more, the following holds true: for every $i \in \{1, \ldots, n\}$ and for every bounded set $B_i \subset X_i$ there is a neighborhood of the iden-

tity $V \subset G$ so that

$$\forall_{g \in V \cap G'} \forall_{x \in B_i} \quad d_X\big(f_g^{(i)}(x), x\big) < C. \tag{5.7}$$

Openness of $G' < G$
The group G' is defined as the kernel of a homomorphism of G to a finite group. But we do not know at this point that the homomorphism is continuous, which would imply that $G' < G$ is open and, thus, an lcsc group itself. Next we provide a direct argument that shows the openness of $G' < G$.

Let d_{X_i} denote the metric X_i, and let the metric d_X on X be the ℓ^2-sum of d_{X_i}'s. Let $B_i \subset X_i$ be a bounded subset whose diameter exceeds $3C$, and let $B = B_1 \times \cdots \times B_n$. Let $V \subset G$ be a neighborhood of the identity that satisfies (5.6). Next we show that V is contained in G' and so $G' < G$ is an open subgroup. Suppose it is not. Then there is $g \in V$ such that π_g is nontrivial; that is, there is $i \in \{1, \dots, n\}$ such that $j := \pi_g(i) \neq i$. Let $x_j, x_j' \in B_j$ be points whose distance is at least $3C$. Pick points $x_l \in B_l$ for $l \in \{1, \dots, j-1, j+1, \dots, n\}$. Then either $d_{X_j}(f_g^{(i)}(x_i), x_j) > C$ or $d_{X_j}(f_g^{(i)}(x_i), x_j') > C$. Without loss of generality assume the first case. Let $x = (x_1, \dots, x_n) \in B$. Then

$$d_X\big(f_g(x), x\big) \geq d_{X_j}\big(f_g^{(i)}(x_i), x_j\big) > C,$$

contradicting (5.6). Therefore, $V \subset G'$.

Mapping G' to $\mathrm{Isom}(X_i)$ *for the higher-rank factors X_i*
The rigidity theorem for higher-rank irreducible buildings (such as $X_1, \dots, X_k$) by Kleiner and Leeb [45] is the next important ingredient.

THEOREM 5.13 ([45, Theorem 1.1.3])
Let $i \in \{1, \dots, k\}$. For every $C > 0$ and $L > 0$ there is a constant $D > 0$ such that every (L, C)-quasi-isometry $X_i \to X_i$ is within distance D from a unique isometry $X_i \to X_i$.

Moreover, no two distinct isometries $X_i \to X_i$ for $i \in \{1, \dots, k\}$ are within bounded distance from each other. Hence, the natural homomorphism $\mathrm{Isom}(X_i) \to \mathrm{QI}(X_i)$ is an isomorphism for $i \in \{1, \dots, k\}$, and we obtain homomorphisms

$$\psi_i \colon G' \longrightarrow \mathrm{Isom}(X_i)$$

for which the restriction to $H' < G'$ is the homomorphism $H' \to \mathrm{Isom}(X_i)$. The statement (5.7) and Theorem 5.13 applied to $f_g^{(i)}$ show that there is a constant $E > 0$

such that for every bounded subset $B \subset X_i$ there is a neighborhood of the identity $V \subset G$ such that

$$\forall_{g \in V} \forall_{x \in B} \quad d_{X_i}\big(\psi_i(g)(x), x\big) < E. \tag{5.8}$$

LEMMA 5.14
The map ψ_i is continuous for $i \in \{1, \dots, k\}$.

Proof
Let $U \subset \mathrm{Isom}(X_i)$ be an open neighborhood of the identity. We have to show that there is an open neighborhood of the identity of G that is contained in $\psi_i^{-1}(U)$.

We rely on the following geometric fact about buildings that follows, for example, from [63]: for every constant $D > 0$ and any open neighborhood W of the identity in $\mathrm{Isom}(X_i)$, there is a bounded set $B \subset X_i$, depending on D and W so that

$$\big\{\theta \in \mathrm{Isom}(X_i) \,\big|\, \sup_{x \in B} d_{X_i}\big(\theta(x), x\big) < D\big\} \subset W. \tag{5.9}$$

We apply this general fact to the constant E from (5.8) and the identity neighborhood U. Let us fix a bounded subset $B = B(E, U) \subset X_i$ such that

$$\big\{\theta \in \mathrm{Isom}(X_i) \,\big|\, \sup_{x \in B} d_{X_i}\big(\theta(x), x\big) < E\big\} \subset U. \tag{5.10}$$

Applying the statement (5.8) to this specific subset B provides us with a neighborhood $V = V(E, B) \subset G$ of the identity such that

$$\forall_{g \in V} \forall_{x \in B} \quad d_{X_i}\big(\psi_i(g)(x), x\big) < E. \tag{5.11}$$

Since (5.10) and (5.11) mean that $V \subset \psi_i^{-1}(U)$, the continuity of ψ_i follows. □

Mapping G' to $\mathrm{Homeo}(\partial X_j)$ *for the rank-*1 *factors X_j*
Next we turn to the tree factors X_j with $j \in \{k+1, \dots, n\}$. For every tree of bounded degree T, such as X_j with $k < j \leq n$, the space of ends ∂T is compact, and one has an embedding of groups

$$\mathrm{QI}(T) \hookrightarrow \mathrm{Homeo}(\partial T).$$

For $j \in \{k+1, \dots, n\}$ let ψ_j be the composition

$$\psi_j \colon G' \xrightarrow{\phi_j} \mathrm{QI}(X_j) \hookrightarrow \mathrm{Homeo}(\partial X_j),$$

and denote by $\bar{G}'_j = \psi_j(G')$ the image in $\mathrm{Homeo}(\partial X_j)$.

LEMMA 5.15
The map ψ_j is continuous and has a closed image for every $j \in \{k+1, \dots, n\}$.

Proof
The continuity statement is proved in [35, Theorem 3.5]. The argument is analogous to the one for Lemma 5.14. However, instead of the geometric fact about higher-rank buildings (5.9) one uses the following fact about trees, which is a consequence of the Mostow–Morse lemma for Gromov-hyperbolic spaces: for a tree T of bounded degree and constants $L, C, D > 0$ and for any identity neighborhood $U \subset \mathrm{Homeo}(\partial T)$ there is a compact subset $B \subset T$ so that the image in $\mathrm{Homeo}(\partial T)$ of any (L, C)-quasi-isometry $f : T \to T$ with

$$\sup\{d_T(f(x), x) \mid x \in B\} < D$$

belongs to U. Since the image of H' is cocompact in G' and the image of H' in $\mathrm{Homeo}(\partial X_j)$ is closed, the image $\bar{G}'_j$ of G' in $\mathrm{Homeo}(\partial X_j)$ is also closed. □

Identifying the image of ψ_j
Fix an index $j \in \{k+1, \dots, n\}$. Let H'_j denote the image of H' in $\mathrm{Isom}(X_j)$, and let $\bar{H}'_j$ denote the corresponding subgroup of $\mathrm{Homeo}(\partial X_j)$. Consider the action of $\bar{G}'_j$ on the Cantor set ∂X_j that comes from the quasiaction of $\bar{G}'_j$ on the tree X_j. The latter is given by the family $\{f_g^{(j)}\}$ of (L, C)-quasi-isometries of the locally finite tree X_j such that $f_{g_1}^{(j)} \circ f_{g_2}^{(j)}$ is within uniformly bounded distance from $f_{g_1 g_2}^{(j)}$ for all $g_1, g_2 \in G'$. Note that the maps $f_g^{(j)}$ may be associated to the images $\psi_j(g) \in \bar{G}'_j$ rather than $g \in G'$. We proceed with two claims:

(1) There exist a locally finite tree T_j, an action $\rho : \bar{G}'_j \to \mathrm{Isom}(T_j)$, and a quasi-isometry $q : T_j \to X_j$ so that the $f_g^{(j)}$'s are within bounded distance from the map $q \circ \rho(g) \circ q'$, where $q' : X_j \to T_j$ is a coarse inverse of q.
(2) One can take tree T_j to be X_j.

Claim (1) follows from the deep work of Mosher, Sageev, and Whyte on quasiactions on trees (see [54, Theorem 1]).

There is also an alternative argument based on the result of Carette–Dressen [24, Theorem C]. Indeed, the fact that the action of H'_j on the tree X_j is strongly transitive implies that the action of $\bar{H}'_j$ on the Cantor set ∂X_j is 3-*proper* and 3-*cocompact* (meaning the action on the space of distinct triples is proper and cocompact), and these properties pass to the action of $\bar{G}'_j$ that contains $\bar{H}'_j$ as a cocompact subgroup. By [24, Theorem C] such an action of a locally compact group on a Cantor set is sufficient to construct an isometric action on a locally finite tree T_j with ∂T_j being identified with the given Cantor set.

We now offer a few general remarks on trees and groups acting on trees. An action of a group on a tree is called *minimal* if there are no proper subtrees invariant under the group action. If an action on a locally finite tree is minimal, then the induced action on the boundary of the tree is minimal in the sense of dynamical systems: there are no proper closed invariant subsets. This is a consequence of the limit set being the unique minimal closed invariant subset of the boundary (see [28, Théorème 5.1] for a more general result, which is attributed to Gromov, in the context of Gromov-hyperbolic spaces).

Conversely, if a group G is acting by automorphisms on a locally finite tree T and $L \subset \partial T$ is a minimal G-invariant set consisting of more than two points, then the union of all geodesics in T with endpoints in L, called the *convex hull* $\mathrm{co}(L)$ of L, is a minimal G-invariant subtree of T. Any G-invariant subtree T' has $\partial T' = L$ and, therefore, contains $\mathrm{co}(L)$. Moreover, there exists a retraction $r\colon T' \to \mathrm{co}(L)$ given by nearest point projection to $\mathrm{co}(L)$. This retraction is equivariant with respect to $\mathrm{Aut}(T)$.

Returning to our situation, we may assume without loss of generality that the tree T_j has no vertices of degree 1 and so $T_j = \mathrm{co}(\partial T_j)$. We identify its boundary ∂T_j with ∂X_j in a $\bar{G}'_j$-equivariant way. Since the action of H_j on the tree X_j is minimal, the topological action of $\bar{H}'_j$ on the Cantor set $\partial X_j = \partial T_j$ is minimal, and therefore, the action of $\bar{H}'_j$ on the tree T_j is minimal as well—that is, T_j does not contain any proper $\bar{H}'_j$-invariant subtree.

LEMMA 5.16

The action homomorphism $\bar{G}'_j \to \mathrm{Isom}(T_j)$ is injective and continuous and has a closed image.

Proof

Since both the isometry group and the quasi-isometry group of either X_j or T_j inject into the homeomorphism group of $\partial X_j = \partial T_j$, the map $\bar{G}'_j \to \mathrm{Isom}(T_j)$ is injective. Since $\mathrm{Isom}(T_j)$ embeds as a closed subgroup into $\mathrm{Homeo}(\partial T_j)$, continuity and closedness follow from the continuity and closedness of the composition

$$\bar{G}'_j \to \mathrm{Isom}(T_j) \to \mathrm{Homeo}(\partial T_j) = \mathrm{Homeo}(\partial X_j). \qquad \square$$

The argument for claim (2) relies on the fact that the structure of the tree X_j can be read from the intersection patterns of vertex stabilizers, which are maximal open compact subgroups of $\bar{H}'_j$ (see, e.g., [1, Corollary 11.35, Theorem 11.38] for the more general discussion of groups with BN-pairs acting on Euclidean buildings). In our case, $\bar{H}'_j$ acts on the tree X_j, transitively on edges and with two orbits of vertices. Let $e = (v_1, v_2)$ be an edge in X_j, and denote

$$K_i = \mathrm{Stab}_{\bar{H}'_j}\big(\{v_i\}\big) \quad (i = 1, 2).$$

Then K_1 and K_2 are maximal compact (and open) nonconjugate subgroups of $\bar{H}'_j$. The group $\bar{H}'_j$ is generated by $K_1 \cup K_2$, and it acts transitively on the edges of X_j, with $K_1 \cap K_2$ being the stabilizer of edge $e = (v_1, v_2)$. For each $i = 1, 2$, the subgroup $K_1 \cap K_2$ is a maximal subgroup in K_i, and all maximal subgroups on K_i are conjugate in K_i (because K_i acts transitively on the edges emerging from v_i).

LEMMA 5.17
There are $k \in \mathbb{N}$ and an $\bar{H}'_j$-equivariant cellular homeomorphism $f : X_j \to T_j$ such that f is a homothety with stretch factor k.

Proof
Upon subdividing, we may assume that $\bar{H}'_j$ acts on T_j without inversion. By the continuity part of Lemma 5.16, the orbits of the compact subgroups K_1 and K_2 in T_j are bounded. By the Bruhat–Tits fixed-point theorem [16, Corollary 2.8, p. 179], there is a vertex in T_j that is fixed under K_1 or K_2, respectively. Choose a pair (w_1, w_2) of vertices in T_j with $K_i w_i = w_i$ that has minimal distance.

Since $\bar{H}'_j$ is generated by K_1 and K_2, the $\bar{H}'_j$-orbit $T' = \bigcup_{h \in \bar{H}'_j} [hw_1, hw_2]$ of the geodesic segment $[w_1, w_2]$ between w_1 and w_2 is connected. Thus, T' is a $\bar{H}'_j$-invariant subtree of T_j. By the proceeding discussion this implies that $T' = T_j$.

The segment $[w_1, w_2]$ is fixed by $K_1 \cap K_2$. The map that sends the vertex v_i to w_i for $i = 1, 2$ and the edge $[v_1, v_2]$ onto $[w_1, w_2]$ by an affine homeomorphism can be extended to an $\bar{H}'_j$-equivariant cellular surjective map $f : X_j \to T_j$.

Next we show that f is locally injective. It is enough to show local injectivity at the K_i-fixed vertex v_i of X_j. By symmetry let us consider v_1 only. Consider its neighbors v_2 and $v'_2 = hv_2$ for some $h \in \bar{H}'_j$. Since K_1 and K_2 are not conjugated, the f-images w_1, w_2, w'_2 of v_1, v_2, and v'_2 are pairwise distinct. Let w be the center of the tripod given by w_1, w_2, w'_2. We want to show that $w = w_1$, which implies the local injectivity of f at v_1. Both $K_1 \cap K_2$ and $K_1 \cap hK_2h^{-1}$ are contained in the stabilizer S of w. Since $K_1 \cap K_2$ is maximal in K_1 and $K_1 \cap hK_2h^{-1} \neq K_1 \cap K_2$ it follows that $S = K_1$. So $w = w_1$ since we chose (w_1, w_2) to have minimal distance among the vertices fixed by K_1 or K_2, respectively.

As a surjective, locally injective map between trees, f is a homeomorphism. Let k be the distance between w_1 and w_2. Obviously, f is locally a homothety with stretch factor k. Thus, it is globally so. □

Conjugating with a homothety as in Lemma 5.17 we obtain a topological isomorphism

$$\mathrm{Isom}(T_j) \cong \mathrm{Isom}(X_j)$$

compatible with the two embeddings of $\bar{H}'_j$. Hence, $\bar{G}'_j$ can be embedded as a closed intermediate subgroup $\bar{H}'_j < \bar{G}'_j < \mathrm{Isom}(X_j)$, and the homomorphism ψ_j can be regarded as a homomorphism $G' \to \bar{G}'_j \hookrightarrow \mathrm{Isom}(X_j)$ composed with the natural embedding $\mathrm{Isom}(X_j) \to \mathrm{Homeo}(\partial X_j)$. Thus, we also denote the map $G' \to \bar{G}'_j \to \mathrm{Isom}(X_j)$ by ψ_j. Let $\Delta\colon G' \to \prod_{i=1}^n G'_i$ be the diagonal embedding. We obtain a continuous homomorphism with closed image

$$\psi'\colon G' \xrightarrow{\prod \psi_i \circ \Delta} \prod_{i=1}^{n} \mathrm{Isom}(X_i)$$

whose restriction to H' corresponds to the embedding $\bar{H}' \hookrightarrow \prod_{i=1}^n \mathrm{Isom}(X_i)$.

Finally we argue that this homomorphism $\psi' : G' \to \prod \bar{G}'_i \hookrightarrow \prod \mathrm{Isom}(X_i)$ extends to a homomorphism $\psi\colon G \to \mathrm{Isom}(\prod X_i)$. Recall that, by Theorem 5.12, the image $\bar{G}$ of G in $\mathrm{QI}(X)$, where $X = X_1 \times \cdots \times X_n$, is a finite extension of the product $\bar{G}' = \bar{G}'_1 \times \cdots \times \bar{G}'_n$ with $\bar{G}'_i < \mathrm{QI}(X_i)$, by the finite group $\pi(G) < \mathrm{Sym}_n$. The conjugation action of G on $G' = G'_1 \times \cdots \times G'_n$ that descends to the conjugation action of $\bar{G}$ on $\bar{G}' = \bar{G}'_1 \times \cdots \times \bar{G}'_n$ and for each $g \in G$ induces isomorphisms $p_{g,i} : \bar{G}'_i \to \bar{G}'_{\pi_g(i)}$. As a consequence of the proof so far we identify the groups $\bar{G}'_i$ as closed intermediate subgroups

$$\bar{H}'_i < \bar{G}'_i < \mathrm{Isom}(X_i) \quad \big(i \in \{1, \ldots, n\}\big)$$

and observe that the isomorphisms $p_{g,i} : \bar{G}'_i \to \bar{G}'_{\pi_g(i)}$ are continuous (as in Lemmas 5.14 and 5.16). Any continuous group isomorphism $\bar{G}'_i \cong \bar{G}'_j$ is induced by an isometry $X_i \cong X_j$—for higher-rank buildings (corresponding to $i, j \le k$) this follows from [45, Theorem 5.13], and for rank-1 cases (corresponding to $k \le i, j \le n$) it can be deduced from Lemma 5.17. Therefore, for every $g \in G$, $\pi_g(i) = j$ only if X_i is isometric to X_j. Thus, each $g \in G$ defines an isometry $\psi(g)$ of $X = X_1 \times \cdots \times X_n$, thereby defining the claimed continuous homomorphism $\psi : G \to \mathrm{Isom}(\prod_{i=1}^n X_i)$ that extends $\psi' : G' \to \prod \mathrm{Isom}(X_i)$. This completes the proof of Theorem 5.11 and, therefore, also the proof of the main classification result—Theorem A.

6. Proofs of Theorems B–F

Proof of Theorem B

Let $\Gamma < H$ be a lattice embedding that is

(i) either an irreducible lattice embedding into a connected, center-free, semisimple real Lie group H without compact factors, which is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$,

(ii) or an S-arithmetic lattice embedding as in Definition 4.4.

Such Γ satisfies all the assumptions of Theorem A. As a lattice, the group Γ is Zariski-dense in the semisimple algebraic group H by Borel's density theorem [50, Theorem II(4.4)]. Thus, Theorem 1.4(2) ensures that Γ satisfies conditions (CAF) and (NbC). As an irreducible lattice, Γ satisfies (Irr). Moreover, lattices in semisimple groups are known to be finitely generated (see [50, Section IX(3.2)]). So any lattice envelope of Γ is compactly generated by Lemma 3.7. We also note that neither Γ nor any finite-index subgroup $\Gamma' < \Gamma$ has nontrivial finite normal subgroups. Indeed, such a subgroup group $N \lhd \Gamma'$ would be centralized by a finite-index subgroup $\Gamma'' = \ker(\Gamma' \to \mathrm{Aut}(N))$, which forms an irreducible lattice in H. So $N \subset \mathcal{Z}_H(\Gamma'')$, while by Borel's density theorem from [50, Section II(6.3)], $\mathcal{Z}_H(\Gamma'') = \mathcal{Z}(H)$ is the center of H, which is trivial.

Let G be some lcsc group, and let $\Gamma < G$ be a lattice embedding. By Theorem A there is an open subgroup of finite index $G' < G$ and a compact normal subgroup $K \lhd G'$ so that, by denoting $G'_0 := G'/K$ and $\Gamma' := \Gamma \cap G'$, the lattice embedding $\Gamma' \hookrightarrow G'_0$ (by the above discussion $\Gamma' \cap K = \{1\}$) induced from $\Gamma < G$ is of one of the following three types:

(1) G'_0 is a connected, center-free, semisimple, real Lie group without compact factors and $\Gamma' \hookrightarrow G'_0$ is an irreducible lattice;

(2) $\Gamma' \hookrightarrow G'_0$ is an S-arithmetic lattice, possibly up to tree extension, in the sense of Definition 4.4;

(3) G'_0 is a tdlc group with trivial amenable radical.

First, note that $\Gamma < H$ as in (i) is incompatible with $\Gamma' < G'_0$ being of type (2), and $\Gamma < H$ as in (ii) is incompatible with $\Gamma' < G'_0$ being of type (1). Indeed, Margulis's superrigidity theorem [50, Theorem VII(7.1)] precludes the same group Γ' from being both an S-arithmetic lattice (with both real and non-Archimedean unisotropic factors present) and being a lattice in a purely real Lie group.

We will show in Theorem 6.2 below that $\Gamma < H$ as in (i) or (ii) is incompatible with nontrivial lattice embeddings $\Gamma' < G'_0$ of type (3). Before delving into this argument, let us now discuss in some detail the situations, where $\Gamma < H$ is as in (i) and $\Gamma' < G'_0$ is of type (1), and then the case of $\Gamma < H$ as in (ii) and $\Gamma' < G'_0$ of type (2). We recall the famous strong rigidity theorem of Mostow, Prasad, and Margulis.

THEOREM 6.1 (Strong rigidity theorem)
Let H and $H^\flat$ be connected, center-free, semisimple real Lie groups without nontrivial compact factors, with $H \ncong \mathrm{PSL}_2(\mathbb{R})$, and let $\Lambda < H$ and $\Lambda^\flat < H^\flat$ be irreducible lattices. Then any isomorphism $\Lambda \cong \Lambda^\flat$ extends to a continuous isomorphism $H \cong H^\flat$.

The strong rigidity theorem was discovered by Mostow in [55] for all the cases where H and $H^\flat$ are semisimple real Lie groups and $\Lambda < H$ and $\Lambda^\flat < H^\flat$ are *uniform* irreducible lattices. Prasad in [60] extended this result to some nonuniform lattices, including all nonuniform lattices in rank-1 real Lie groups. Higher-rank cases, for both uniform and nonuniform lattice embeddings, follow from Margulis's superrigidity theorem ([50, Theorem VII(7.1)]). The latter result is even more general and implies strong rigidity for irreducible lattices in a larger class of groups, including $\Lambda < H$ being an S-arithmetic lattice as in Setup 4.1 and $H^\flat$ being a product of real and non-Archimedean simple Lie groups over local fields.

Let $\Gamma < H \ncong \mathrm{PSL}_2(\mathbb{R})$ be an irreducible lattice in a real semisimple group as in (i), and assume that a finite-index subgroup $\Gamma' < \Gamma$ and the group G_0' are as in (1). Then by the strong rigidity theorem applied to $\Lambda = \Gamma'$ and $H^\flat = G_0'$, we get a continuous isomorphism $\rho' : G_0' \cong H$ that intertwines the lattice embedding $j' \colon \Gamma' \hookrightarrow G' \to G_0'$ and the restriction $i' = i|_{\Gamma'}$ of the lattice embedding $i : \Gamma \hookrightarrow H$.

Peeking into the proof of Theorem A (see Theorem 5.1 and Proposition 5.3), we see that $K \lhd G'$ is the amenable radical of G. So we also have the induced lattice embedding $\Gamma \hookrightarrow G_0 := G/K$. The action of G on G' by conjugation gives a continuous homomorphism $\pi : G \to \mathrm{Aut}(H)$ with $K < \ker(\pi)$ so that the restriction $\pi' = \pi|_{G'}$ coincides with the composition $G' \to G_0' \xrightarrow{\rho'} H \cong \mathrm{inn}(H) \hookrightarrow \mathrm{Aut}(H)$. This shows that the following diagram is commutative; that is, two homomorphisms $\pi' \circ j'$ and $\mathrm{inn} \circ i'$ describing embeddings $\Gamma' \hookrightarrow H \hookrightarrow \mathrm{Aut}(H)$ coincide:

$$\begin{array}{ccccccc} \Gamma' & & \xhookrightarrow{\quad i' \quad} & & & & H \\ \big\downarrow{\scriptstyle j'} & & & & & & \big\downarrow{\scriptstyle \mathrm{inn}} \\ G' & \longrightarrow & G_0' & \xrightarrow{\rho'} & H & \hookrightarrow & \mathrm{Aut}(H) \end{array} \tag{6.1}$$

We claim that the diagram below is also commutative—that is, the homomorphisms $\alpha = \pi \circ j : \Gamma \to \mathrm{Aut}(H)$ and $\beta = \mathrm{inn} \circ i : \Gamma \to \mathrm{Aut}(H)$ also coincide:

$$\begin{array}{ccccc} \Gamma & & \xhookrightarrow{\quad i \quad} & & H \\ \big\downarrow{\scriptstyle j} & & & & \big\downarrow{\scriptstyle \mathrm{inn}} \\ G & \longrightarrow & G_0 & \longrightarrow & \mathrm{Aut}(H) \\ & \searrow & & \nearrow_{\pi} & \end{array} \tag{6.2}$$

Indeed, Γ' is normal in Γ and $\alpha|_{\Gamma'} = \beta|_{\Gamma'}$, so for any $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$ we have $\alpha(\gamma') = \beta(\gamma')$ and $\alpha(\gamma\gamma'\gamma^{-1}) = \beta(\gamma\gamma'\gamma^{-1})$. This implies that the element $\alpha(\gamma)^{-1}\beta(\gamma) \in \mathrm{Aut}(H)$ centralizes the lattice $\alpha(\Gamma') = \beta(\Gamma')$ in H. Borel's density theorem and the fact that $\mathcal{Z}(H) = \{1\}$ imply that $\alpha(\gamma) = \beta(\gamma)$ for every $\gamma \in \Gamma$. This proves the commutativity of the diagram (6.2) and thereby proves Theorem A for $\Gamma < H$ as in (i) and $\Gamma' < G_0'$ satisfying (1).

Next, assume that $\Gamma < H$ is an S-arithmetic lattice as in (ii), and assume that $\Gamma' < G'_0$ is of type (2); that is, it is an S-arithmetic lattice, possibly up to a tree extension. We view G'_0 as an intermediate closed group $H^\flat < G'_0 < \mathrm{Isom}(X)$, where $H^\flat = \prod \mathbf{H}_i^\flat(K_{v_i})$ is a product of real and non-Archimedean simple Lie groups and $X = \prod X_i$ is a product of irreducible symmetric spaces and irreducible Bruhat–Tits buildings. In fact, we can write $X = X^{\mathrm{ss}} \times X^{\mathrm{td}}$, where X^{ss} is a (possibly reducible) symmetric space and X^{td} is a (possibly reducible) Euclidean building; the notation is motivated by the inclusions $(H^\flat)^{\mathrm{ss}} < \mathrm{Isom}(X^{\mathrm{ss}})$ and $(H^\flat)^{\mathrm{td}} < \mathrm{Isom}(X^{\mathrm{td}})$.

Note that the image of Γ' in G'_0 is contained in $H^\flat$, so there are two S-arithmetic lattice embeddings $i' \colon \Gamma' \hookrightarrow H$ and $j' \colon \Gamma' \hookrightarrow H^\flat$. The superrigidity theorem ([50, Theorem VII(7.1)]) implies a form of strong rigidity that provides a continuous isomorphism $\pi' \colon H^\flat \to H$ intertwining these embeddings: $\pi' \circ j' = i'$, similarly to the situation in diagram (6.1). In particular, we may think of H and G'_0 acting by isometries on the same space $X = X^{\mathrm{ss}} \times X^{\mathrm{td}}$.

Note that the connected component $(G/K)^0$ of the identity is precisely $\mathrm{Isom}(X^{\mathrm{ss}})^0$, and the action G by automorphisms of $(G/K)^0$ gives a continuous homomorphism $G \to \mathrm{Isom}(X^{\mathrm{ss}})$. Thus, the continuous homomorphism $G' \to G'_0 \hookrightarrow \mathrm{Isom}(X^{\mathrm{ss}})^0 \times \mathrm{Isom}(X^{\mathrm{td}})$ extends to the continuous homomorphism

$$G \to G_0 \hookrightarrow \mathrm{Isom}(X^{\mathrm{ss}}) \times \mathrm{Isom}(X^{\mathrm{td}}) < \mathrm{Isom}(X).$$

Considering two isometric actions of Γ on X that agree on the normal subgroup of finite index Γ' and using the fact that the centralizer of Γ' in $\mathrm{Isom}(X)$ is trivial, we deduce that an analogue of diagram (6.2) is also commutative (i.e., the two Γ-actions on X coincide). This completes the proof of Theorem A in case (ii) and (2).

The following statement will complete the proof of Theorem B.

THEOREM 6.2
Let $\Gamma < H$ be a lattice embedding as in (i) or (ii). Then any lattice embedding of Γ into a tdlc group with trivial amenable radical is trivial.

Let us say that a Polish group L has property *NSS (no small subgroups)* if there is an identity neighborhood $1 \in V \subset L$ that contains no nontrivial compact subgroups. Examples of such groups include discrete countable groups, $\mathrm{Homeo}(S^1)$—the group of homeomorphisms of the circle with compact open topology—and all real Lie groups. In proving Theorem 6.2 we will use the following.

LEMMA 6.3
Let $\Gamma < G$ be a lattice embedding in a tdlc group G with $\mathrm{Rad}_{\mathrm{am}}(G) = \{1\}$. Let L be a Polish group with property NSS, and let $\rho\colon G \to L$ be a continuous homomorphism that is injective on Γ. Then $\Gamma < G$ is a trivial lattice embedding.

Proof
Let $V \subset L$ be a neighborhood that does not contain nontrivial compact subgroups. In a tdlc group, compact open subgroups form a basis of neighborhoods of the identity. Let U be a compact open subgroup of G contained in $\rho^{-1}(V)$. Then $\rho(U)$ is a compact subgroup of V; therefore, it is trivial. Hence, the subgroup $\ker(\rho)$ is open, since it contains an open subgroup U. As $\ker(\rho) \cap \Gamma = \{1\}$ and $\Gamma < G$ is a lattice, $\ker(\rho)$ has finite Haar measure. Thus, $\ker(\rho)$ is compact and is therefore trivial because $\mathrm{Rad}_{\mathrm{am}}(G) = \{1\}$ by assumption. This implies that G is discrete and is a trivial lattice envelope for Γ. □

By Selberg's lemma, Γ as above is virtually torsion-free. Hence, any lattice embedding of Γ into a tdlc group is uniform according to Lemma 3.9. To construct the homomorphism $\rho : G \to L$ as in Lemma 6.3, we will use the homomorphism

$$\rho_0\colon G \to \mathrm{QI}(G) \xrightarrow{\cong} \mathrm{QI}(\Gamma). \tag{6.3}$$

We consider several cases.

Case $\Gamma < H$ is a nonuniform lattice
By a theorem of Struble [69, Theorem, p.127], the group G possesses a continuous, proper, left-invariant metric d, and d has to be quasi-isometric to the word metric. So the kernel can be described as

$$\ker(\rho_0) = \Big\{ g \in G \;\Big|\; \sup_{h \in G} d(gh, h) < \infty \Big\},$$

which implies that $\ker(\rho_0)$ is a Borel subgroup of G.

For a nonuniform, irreducible lattice $\Gamma < H$, the group $\mathrm{QI}(\Gamma) \cong \mathrm{QI}(G)$ coincides with the *commensurator* of Γ in H, which is a countable group. This was first proved by Schwartz in [65] for rank-1 real Lie groups; then by Schwartz in [66], by Farb and Schwarz in [31], and by Eskin in [30] for higher-rank real Lie groups; and by Wortman in [74] for S-arithmetic cases.

Since $\ker(\rho_0)$ has countable index in G, the Borel subgroup $\ker(\rho_0)$ has positive Haar measure. Thus, $\ker(\rho_0)$ is an open subgroup (see [13, IX Section 6, No. 8, Lemma 9]). Thus, the quotient homomorphism

$$G \longrightarrow L := G/\ker(\rho_0)$$

is a continuous homomorphism into a countable discrete group. This homomorphism is injective on Γ because Γ embeds in $\mathrm{QI}(\Gamma)$. Thus, the proof of Theorem 6.2 is concluded by applying Lemma 6.3.

For the rest of the proof, we assume $\Gamma < H$ to be a uniform lattice, we write H as a product of simple factors $H = H_1 \times \cdots \times H_n$, and we let $X = X_1 \times \cdots \times X_n$ be the product of the associated symmetric spaces or Bruhat–Tits buildings. At least one of the factors H_i is a simple real Lie group, and we will assume H_1 is such. Since Γ is quasi-isometric to X, we can view the homomorphism (6.3) as

$$\rho_0 : G \longrightarrow \mathrm{QI}(\Gamma) \cong \mathrm{QI}(X).$$

In view of our splitting theorem, Theorem 5.12, from Kleiner and Leeb, there is an open normal subgroup of finite index $G' < G$ whose image $\rho_0(G')$ is contained in the subgroup $\mathrm{QI}(X_1) \times \cdots \times \mathrm{QI}(X_n)$ of finite index in $\mathrm{QI}(X)$. By denoting $\Gamma' = \Gamma \cap G'$, the homomorphism ρ_0 is compatible with the natural embedding $\Gamma' \to \mathrm{Isom}(X_1) \to \mathrm{QI}(X_1)$:

$$\begin{array}{ccccc} \Gamma' & \hookrightarrow & \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_n) & \xrightarrow{\mathrm{pr}_1} & \mathrm{Isom}(X_1) \\ \downarrow & & & & \downarrow \\ G' & \longrightarrow & \mathrm{QI}(X_1) \times \cdots \times \mathrm{QI}(X_n) & \xrightarrow{\mathrm{pr}_1} & \mathrm{QI}(X_1) \end{array}$$

Let us now consider the homomorphism $G' \to \mathrm{QI}(X_1)$ and denote its image by $\bar{G}'_1$.

Case $\Gamma < H$ is uniform and $X_1 = \mathcal{H}^2$
This case occurs if $H_1 \cong \mathrm{PSL}_2(\mathbb{R})$ is one of several factors of H. Note that the projection $\Gamma \to H_1$ is injective. The boundary $\partial\mathcal{H}^2$ is homeomorphic to the circle S^1, and we have a homomorphism

$$\rho\colon G' \to \mathrm{QI}(\mathcal{H}^2) \hookrightarrow \mathrm{Homeo}(S^1).$$

This homomorphism is continuous by [35, Theorem 3.5], and the restriction to Γ is the composition of injective homomorphisms $\Gamma \longrightarrow H_1 \cong \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \mathrm{Homeo}(S^1)$. The Polish group $\mathrm{Homeo}(S^1)$ has the NSS property, so Lemma 6.3 applies, and the proof of Theorem 6.2 is concluded in this case.

Case $\Gamma < H$ is uniform and $X_1 = \mathcal{H}^m$ with $m \geq 3$.
The boundary at infinity $\partial\mathcal{H}^m$ is a sphere S^{m-1} with a natural conformal structure that is preserved by $\mathrm{Isom}(\mathcal{H}^m)$. Moreover, there is an isomorphism $\mathrm{Isom}(\mathcal{H}^m) \cong \mathrm{Conf}(S^{m-1})$ that extends to an isomorphism

$$\mathrm{QI}(\mathcal{H}^m) \cong \mathrm{QConf}(S^{m-1})$$

between the group of quasi-isometries and the group of quasiconformal homeomorphisms of the sphere. The topology of $\mathrm{Isom}(\mathcal{H}^m)$ coincides with that of $\mathrm{Conf}(S^{m-1}) < \mathrm{Homeo}(S^{m-1})$. By a result of Tukia [72, Corollary G], a subgroup $\bar{G}_1'$ of $\mathrm{QConf}(S^{m-1})$ that is uniformly quasiconformal and acts cocompactly on triples of points of S^{m-1} can be conjugated into $\mathrm{Conf}(S^{m-1})$.

Since the homomorphism $G' \to \mathrm{QI}(X) \to \mathrm{QI}(\mathcal{H}^m)$ represents a quasiaction of G on $\mathcal{H}^m$, the image $\bar{G}_1' \subset \mathrm{QI}(\mathcal{H}^m) \cong \mathrm{QConf}(S^{m-1})$ is a uniformly quasiconformal group of homeomorphisms of S^{m-1} due to the Mostow–Morse lemma. The homomorphism

$$G' \longrightarrow \mathrm{QI}(\mathcal{H}^m) \hookrightarrow \mathrm{Homeo}(S^{m-1})$$

is continuous by [35, Theorem 3.5]. Since Γ acts cocompactly on triples of points in S^{m-1}, the same is true for $\bar{G}_1'$. By Tukia's result, $\bar{G}_1'$ can be conjugated into the simple Lie group $\mathrm{Conf}(S^{m-1}) \cong \mathrm{Isom}(\mathcal{H}^m)$. Thus, we obtain a continuous homomorphism

$$\rho\colon G' \longrightarrow \mathrm{Isom}(\mathcal{H}^m).$$

Since $\mathrm{Isom}(\mathcal{H}^m)$ is a Lie group, Lemma 6.3 applies, and Theorem 6.2 is proved in this case.

Case $\Gamma < H$ is uniform and $X_1 = \mathcal{H}^m_{\mathbb{C}}$ with $m \geq 2$.
This symmetric space is a complex hyperbolic space $\mathcal{H}^m_{\mathbb{C}}$; its boundary sphere $\partial\mathcal{H}^m_{\mathbb{C}}$ has a natural conformal class of a sub-Riemannian Carnot–Carathéodory structure. Carnot–Carathéodory analogues of the theory of quasiconformal mappings and a result analogous to Tukia's theorem imply (see [27]) that $\bar{G}_1'$ can be quasiconformally conjugated into $\mathrm{Conf}(\partial\mathcal{H}^m_{\mathbb{C}}) = \mathrm{Isom}(\mathcal{H}^m_{\mathbb{C}})$. This gives us a homomorphism

$$\rho\colon G' \longrightarrow \mathrm{Isom}(\mathcal{H}^m_{\mathbb{C}})$$

that is continuous by [35, Theorem 3.5] and takes values in a Lie group. So by applying Lemma 6.3, we prove Theorem 6.2 in this case.

Case $\Gamma < H$ is uniform and $X_1 = \mathcal{H}^m_{\mathbb{H}}$ or $\mathcal{H}^2_{\mathbb{O}}$.
By the result of Pansu [58, Théorème 1] in the case of X_1 being the quaternionic hyperbolic space $\mathcal{H}^m_{\mathbb{H}}$ and the Cayley plane $\mathcal{H}^2_{\mathbb{O}}$, the natural homomorphism $\mathrm{Isom}(X_1) \to \mathrm{QI}(X_1)$ is actually an isomorphism. The embedding $\mathrm{Isom}(X_1) \hookrightarrow \mathrm{Homeo}(\partial X_1)$ is homeomorphic on its image, so the homomorphism

$$\rho\colon G' \longrightarrow \mathrm{QI}(X_1) \cong \mathrm{Isom}(X_1)$$

is continuous (see [35, Theorem 3.5]). Thus, Lemma 6.3 yields the proof of Theorem 6.2 in this case too.

Case $\Gamma < H$ *is uniform and* $\mathrm{rk}(X_1) \geq 2$
For irreducible symmetric space X_1 of higher rank, the work of Kleiner and Leeb in [45] shows that the natural homomorphism $\mathrm{Isom}(X_1) \to \mathrm{QI}(X_1)$ is an isomorphism. Thus, we obtain a homomorphism

$$\rho\colon G' \longrightarrow \mathrm{QI}(X_1) \cong \mathrm{Isom}(X_1)$$

into a Lie group and would like to apply Lemma 6.3. The proof of Theorem 6.2 will be completed once we verify the continuity of the homomorphism ρ.

According to Lemma 3.30 from which we obtained ρ, there is a constant $C > 0$ such that for every bounded subset $B \subset X_1$ there is a neighborhood of the identity $V \subset G'$ such that

$$\forall_{g\in V}\forall_{x\in B} \quad d_{X_1}\big(\rho(g)(x), x\big) < C. \tag{6.4}$$

Let $U \subset \mathrm{Isom}(X_1)$ be an open neighborhood of the identity. We have to show that there is open neighborhood of the identity of G that is contained in $\rho^{-1}(U)$.

We rely on the following geometric fact about symmetric spaces: for every constant $D > 0$ and any open neighborhood W of the identity in $\mathrm{Isom}(X_1)$, there is a bounded set $B \subset X_1$, depending on D and W so that

$$\big\{\theta \in \mathrm{Isom}(X_1) \;\big|\; \sup_{x\in B} d_{X_1}\big(\theta(x), x\big) < D\big\} \subset W. \tag{6.5}$$

We apply this general fact to the constant C from (6.4) and the identity neighborhood U. Let us fix a bounded subset $B = B(C, U) \subset X_1$ such that

$$\big\{\theta \in \mathrm{Isom}(X_1) \;\big|\; \sup_{x\in B} d_{X_1}\big(\theta(x), x\big) < C\big\} \subset U. \tag{6.6}$$

Applying the statement (6.4) to this specific subset B provides us with a neighborhood $V = V(C, B) \subset G$ of the identity such that

$$\forall_{g\in V}\forall_{x\in B} \quad d_{X_1}\big(\rho(g)(x), x\big) < C. \tag{6.7}$$

Since (6.6) and (6.7) mean that $V \subset \phi^{-1}(U)$, the continuity of ϕ follows. As mentioned above, the now proved Theorem 6.2 also completes the proof of Theorem B. □

Proof of Theorem C
Let Γ be a finite extension of a finitely generated non-Abelian free group, and let $\Gamma < G$ be a lattice envelope. Since Γ satisfies all the assumptions of Theorem A by Theorem 1.4, we only need to discuss three possibilities.

(1) $\Gamma < G$ is virtually isomorphic to an irreducible lattice in a center-free semisimple real Lie group H. The first ℓ^2-Betti number of Γ is positive. By Olbrich's

work [56] (see also Chapter 5 about computations of ℓ^2-invariants of locally symmetric spaces in Lück's book [47]) the only such lattices with positive first ℓ^2-Betti numbers are the ones in $\mathrm{PSL}_2(\mathbb{R})$. In this case, Γ is a nonuniform lattice, since, for example, Γ has virtual cohomological dimension 1.

(2) $\Gamma < G$ is virtually isomorphic to an S-arithmetic lattice as in Setup 4.1. In particular, it is virtually isomorphic to a lattice embedding into a product of two noncompact lcsc groups. This can be ruled by the nonvanishing of the first ℓ^2-Betti number of Γ (see the argument in the proof of Theorem F). Alternatively, this situation can be ruled out by the fact that S-arithmetic lattices as in Section 4 are not Gromov-hyperbolic.

(3) $\Gamma < G$ is virtually isomorphic to a uniform lattice $\Lambda < H$ in a tdlc group H with trivial amenable radical. Since Γ and thus Λ are virtually isomorphic to a non-Abelian free group, we can appeal to the work of Mosher, Sageev, and Whyte in [54]. Theorem 9 in [54] states that there are a tree T and a continuous homomorphism $H \to \mathrm{Isom}(T)$ with cocompact image and compact kernel. Since H has trivial amenable radical, the kernel is trivial. Hence, $\Gamma < G$ is virtually isomorphic to a lattice embedding into a closed cocompact subgroup of the automorphism group of a tree. □

Proof of Theorem D

Let Γ be a uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$, and let $\Gamma < G$ be another lattice embedding. As a Gromov-hyperbolic group, Γ satisfies all the assumptions of Theorem A according to Theorem 1.4. So we only need to discuss three possibilities.

(1) $\Gamma < G$ is virtually isomorphic to an irreducible lattice in a semisimple real Lie group H. As in the proof above, by the positivity of the first ℓ^2-Betti number of Γ, one is reduced to the case of $H \simeq \mathrm{PSL}_2(\mathbb{R})$. In this case, Γ has to be cocompact, but the two embeddings of $\Gamma \to \mathrm{PSL}_2(\mathbb{R})$ need not be conjugate.

(2) $\Gamma < G$ is virtually isomorphic to an S-arithmetic lattices, as in Setup 4.1. This case is ruled out for the same reason as case (2) in the proof of Theorem C.

(3) $\Gamma < G$ is virtually isomorphic to a uniform lattice in a tdlc group. This is covered by [35, Theorem C]: the only possibility is a trivial lattice embedding. □

Proof of Theorem E

Let M be a closed Riemannian manifold of dimension at least 5 with sectional curvatures ranging in

$$\left[-\left(1+\frac{1}{n-1}\right)^2,-1\right].$$

Let $\Gamma = \pi_1(M)$ be the fundamental group, and let $\Gamma < G$ be a lattice envelope of Γ. Since Γ is finitely generated, G is compactly generated by Lemma 3.7. The group Γ is Gromov-hyperbolic, so it has the properties (CAF), (NbC), and (Irr) according to Theorem 1.4. As a fundamental group of a closed aspherical manifold, Γ is torsion-free. In particular, Γ also has property (BT). By Theorem A, we have the following three possibilities for the lattice embedding $\Gamma < G$ up to virtual isomorphism. Since Γ is torsion-free, we may assume, upon replacing Γ by a finite-index subgroup, G by a finite-index subgroup and a quotient by a compact normal subgroup, and M by a finite cover, that we still have $\Gamma = \pi_1(M)$ and $\Gamma < G$ is isomorphic (not just virtually) to one of the following cases.

(1) $\Gamma < G$ is an (irreducible) lattice embedding into a semisimple real Lie group G.
(2) $\Gamma < G$ is an S-arithmetic lattice in the sense of Setup 4.1,
(3) $\Gamma < G$ is a uniform lattice, and G is a tdlc group with trivial amenable radical.

We need to show that the only possibility is case (3) with G being discrete unless M is homeomorphic to a hyperbolic manifold. Since Γ is Gromov-hyperbolic, we can rule out case (2). By the same reason, case (1) is only possible if G has real rank 1 and $\Gamma < G$ is uniform; note here that nonuniform lattices contain a free Abelian subgroup of rank 2. Let us analyze the situation where $\Gamma < G$ is a uniform lattice in a simple Lie group of real rank 1. Let $X = G/K$ be the associated symmetric space which is thus real, complex, quaternionic, or Cayley-hyperbolic. As aspherical spaces with the same fundamental group, the locally symmetric space $\Gamma\backslash X$ and M are homotopy-equivalent.

If X is real hyperbolic, then M is homeomorphic to the closed hyperbolic manifold $\Gamma\backslash X$ by the following striking result of Farrell and Jones [33] in their work on the Borel conjecture. So this possibility is ruled out by assumption.

THEOREM 6.4 (Topological rigidity; see [33])
Let Y be a closed nonpositively curved manifold of dimension not equal to 3, 4. If a closed manifold Z is homotopy-equivalent to Y, then Y and Z are homeomorphic.

The possibilities that X is complex, quaternionic, or Cayley-hyperbolic are ruled out by applying the following result by Mok, Siu, and Yeung [52] to $Y = \Gamma\backslash X$ and $Z = M$, along with the pinching assumption for M.

THEOREM 6.5 (Geometric rigidity; [52, Theorem 1])
Let Y and Z be homotopy-equivalent, closed Riemannian manifolds. Assume that Z is negatively curved, and assume that Y is complex, quaternionic, or Cayley-

hyperbolic. Then Y and Z are isometric up to scaling. In particular, the sectional curvatures of Z cannot range in $[-a,-1]$ for $a<4$.

In summary, case (2) never occurs. Case (1) can only occur if M is homeomorphic to a hyperbolic manifold. As for case (3), we have to show that the tdlc group G is discrete upon dividing out a compact normal subgroup. Consider the natural homomorphism

$$\phi\colon \Gamma \to \mathrm{Isom}(\tilde{M}) \to \mathrm{QI}(\tilde{M}) \to \mathrm{Homeo}(\partial\tilde{M}) = \mathrm{Homeo}(S^{n-1}).$$

Of course, ϕ coincides with the natural action $\Gamma \to \mathrm{Homeo}(\partial\Gamma)$ of Γ on its Gromov boundary. We can refer to Lemma 3.30 or, in this situation, to [35, Theorem 3.5] to conclude that the map ϕ extends to a homomorphism

$$\bar{\phi}\colon G \to \mathrm{Homeo}(S^{n-1}).$$

In the latter reference it is also shown that $\bar{\phi}$ is continuous, the kernel $K := \ker(\bar{\phi})$ is compact, and the image of $\bar{\phi}$ is locally compact in the subspace topology. By the open mapping theorem from [13, IX, Section 5.3], G is topologically isomorphic to $\mathrm{im}(\bar{\phi})$. Since G has trivial amenable radical, K is trivial.

Since the tdlc group G acts continuously and faithfully on the sphere S^{n-1}, a positive solution of the Hilbert–Smith conjecture would imply that G is discrete and thus finish the proof. The general Hilbert–Smith conjecture remains open, but in our situation we can appeal to the work of Mj in [51] that contains the following result.

THEOREM 6.6 (Hilbert–Smith conjecture for boundary actionsp; [51, Corollary 2.2.5])
Let Λ be a Gromov-hyperbolic Poincaré duality group. Let $Q < \mathrm{Homeo}(\partial\Lambda)$ be a subgroup that is finite-dimensional and locally compact in the subspace topology and that lies in the image of the natural homomorphism $\mathrm{QI}(\Lambda) \to \mathrm{Homeo}(\partial\Lambda)$. Let h be the Hausdorff dimension of $\partial\Lambda$ with respect to the visual metric, and let t be the topological dimension of $\partial\Lambda$. We assume that $h < t+2$. Then Q is a Lie group.

We apply this theorem to $Q = \mathrm{im}(\bar{\phi}) \cong G$ and $\Lambda = \Gamma$. The topological dimension of $\partial\Gamma = S^{n-1}$ is $t = n-1$. It remains to verify that we have $h < t+2 = n+1$. The Hausdorff dimension h equals the volume entropy h_{vol} of $\tilde{M}$ (see [75, Theorem C]); that is,

$$h = h_{\mathrm{vol}} := \limsup_{R\to\infty} \frac{1}{R} \log \mathrm{vol}\big(B_{\tilde{M}}(x,R)\big).$$

As a direct consequence of the Bishop–Gunther comparison theorem [38, Theorem 3.101, p. 169] we obtain that, if the n-dimensional manifold $\tilde{M}$ has sectional

curvature in $[-b^2, -a^2]$, then

$$(n-1)a \le h_{\mathrm{vol}} \le (n-1)b.$$

In the present situation, $b = 1 + 1/(n-1)$, so we obtain

$$h = h_{\mathrm{vol}} \le (n-1)\Big(1 + \frac{1}{n-1}\Big) = n < n+1 = t+2.$$

So the assumptions in Theorem 6.6 are satisfied, and we conclude that G is at the same time a Lie group and a tdlc group and, thus, discrete. □

Proof of Theorem F

Let Γ be a group with positive first ℓ^2-Betti number $\beta_1^{(2)}(\Gamma) > 0$ and property (BT). Assume that Γ admits a nonuniform compactly generated lattice envelope G. Since, by Theorem 1.4, Γ satisfies all the conditions of Theorem A and (BT), $\Gamma < G$ is virtually isomorphic to some lattice embedding $\Gamma' < G'$ that is one of the first two types in Theorem A; the totally disconnected case is ruled out by (BT). The nonvanishing of the first ℓ^2-Betti number is preserved under virtual isomorphism (this is easily deduced from basic properties), even under quasi-isometry (see [64, Theorem 1]). So $\beta_1^{(2)}(\Gamma') > 0$. Hence, Γ' cannot be a lattice in a product of two noncompact lcsc groups. That is because $\beta_1^{(2)}(\Gamma') > 0$ implies that the first ℓ^2-Betti number of any lattice envelope of Γ' is positive (see [46, Theorem B]), and the first ℓ^2-Betti number of a product of two noncompact unimodular lcsc groups is zero (see [59, Theorem 6.5]). This excludes that $\Gamma' < G'$ is an S-arithmetic lattice as in Setup 4.1, so type (2) is ruled out. Hence, $\Gamma' < G'$ is a lattice in a connected center-free semisimple Lie group without compact factors. By Olbrich's work [56] (see also Chapter 5 about computations of ℓ^2-invariants of locally symmetric spaces in Lück's book [47]), the only such lattices with positive first ℓ^2-Betti numbers are the ones in $\mathrm{PSL}_2(\mathbb{R})$. Since $\Gamma' < G'$ is also nonuniform, the group Γ' (hence, Γ) is virtually isomorphic to a free group. □

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