Tensors: theory and applications

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Overview

Ranks of 3-tensors

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List of applications

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

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COR $\mathrm{rank}\ \mathcal{T} \leq \min(mn, ml, nl)$

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PRF: 3-sat with n variables m clauses
satisfiable iff $\mathrm{rank}\ \mathcal{T} = 4n + 2m, \mathcal{T} \in \mathbb{F}^{(2n+3m)\times(3n)\times(3n+m)})$
otherwise rank is larger

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Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \ne (3, 2p+1, 2p+1)$

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Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \neq (3,2p+1,2p+1)$
Fact: $\mathrm{grank}_{\mathbb{C}}(3,2p+1,2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

Bilinear maps and product of matrices

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Product of two 2×2 matrices is done by 7 multiplications

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I checked the conjecture up to $m, n, l \leq 14$

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For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

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Examples [1]
$m = n \ge 2,\ l = (m-1)(n-1) + 1.$
$m = n = 4,\ l = 11, 12$

Rank one approximations

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$

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How many distinct singular values are for a generic tensor?

ℓ_p maximal problem and Perron-Frobenius

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$\|(x_1, \ldots, x_n)^\top\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

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Assume that $\mathcal{T} \geq 0$. Then $\mathbf{x}, \mathbf{y}, \mathbf{z} \geq 0$

For which values of p we have an analog of Perron-Frobenius theorem?

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No, for $p = 2$, and probably for $p < 3$

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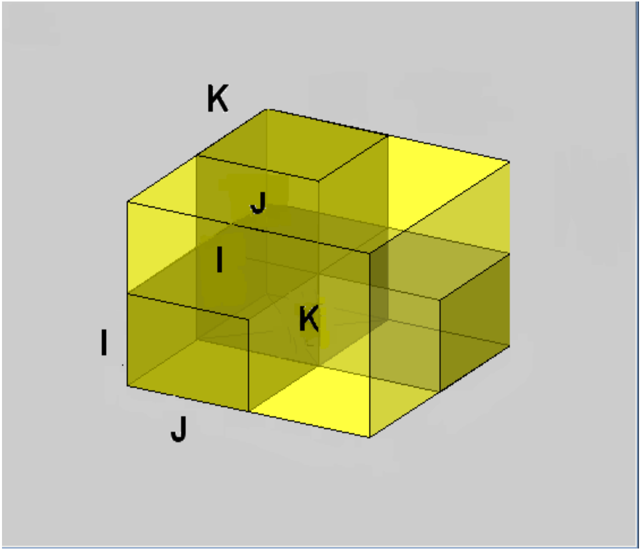
$\max_{\mathbb{U}_1} \|P_{\mathbb{V}}(\mathcal{T})\|$ is an approximation in 2-tensors=matrices

Fast low rank approximation I:



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Approximate $A \in \mathbb{R}^{m \times n}$ by CUR where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of A.

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For given $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$, $F \in \mathbb{R}^{m\times p}$, $E \in \mathbb{R}^{n\times q}$, $G \in \mathbb{R}^{l\times r}$,

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(corresponds to best CUR approximation on the entries read)

For given $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$, $F \in \mathbb{R}^{m\times p}$, $E \in \mathbb{R}^{n\times q}$, $G \in \mathbb{R}^{l\times r}$, where $\langle p\rangle \subset \langle n\rangle \times \langle l\rangle, \langle q\rangle \subset \langle m\rangle \times \langle l\rangle, \langle r\rangle \subset \langle m\rangle \times \langle l\rangle$

$\min_{\mathcal{U}\in\mathbb{C}^{p\times q\times r}} \|\mathcal{A} - \mathcal{U} \times F \times E \times G\|_F$ achieved for $\mathcal{U} = \mathcal{A} \times E^\dagger \times F^\dagger \times G^\dagger$

Fast low rank approximations II:

Approximate $A \in \mathbb{R}^{m\times n}$ by CUR where $C \in \mathbb{R}^{m\times p}$, $R \in \mathbb{R}^{q\times n}$ for some submatrices of A.

$\min_{U\in\mathbb{C}^{p\times q}} \|A - CUR\|_F$ achieved for $U = C^\dagger A R^\dagger$

Faster choice: $U = A[I, J]^\dagger$
(corresponds to best CUR approximation on the entries read)

For given $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$, $F \in \mathbb{R}^{m\times p}$, $E \in \mathbb{R}^{n\times q}$, $G \in \mathbb{R}^{l\times r}$, where $\langle p\rangle \subset \langle n\rangle \times \langle l\rangle, \langle q\rangle \subset \langle m\rangle \times \langle l\rangle, \langle r\rangle \subset \langle m\rangle \times \langle l\rangle$

$\min_{\mathcal{U}\in\mathbb{C}^{p\times q\times r}} \|\mathcal{A} - \mathcal{U} \times F \times E \times G\|_F$ achieved for $\mathcal{U} = \mathcal{A} \times E^\dagger \times F^\dagger \times G^\dagger$

CUR approximation of $\mathcal{A}$ obtained by choosing E, F, G submatrices of unfolded $\mathcal{A}$ in the mode $1, 2, 3$.

List of applications

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Face recognition

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Video tracking

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Factor analysis

References I

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