Methods of algebraic geometry in matrix theory

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Abstract

The purpose of these lectures to report on the recent solution of a 50 years old problem of describing the set of the eigenvalues of a sum of two hermitian matrices with prescribed eigenvalues

1 Statement of the problem

For a field $\mathbb{F}$ denote by $\mathbb{F}^n$ the vector space of column vectors $f = (f_1, ..., f_n)^T$ with entries in $\mathbb{F}$. We will mostly assume that $\mathbb{F}$ is either the field of reals $\mathbb{R}$ or complexes $\mathbb{C}$. We view $\mathbb{R}^n$ and $\mathbb{C}^n$ as inner product spaces with the inner product (x, y) equal to either $y^T x$ or $y^* x$ respectively. Set

$$\mathbb{R}^n_{\geq} := \{x = (x_1, ..., x_n)^T \in \mathbb{R}^n : \quad x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

Let $\mathcal{S}_n \subset \mathcal{H}_n$ be the real vector spaces of $n \times n$ real symmetric and hermitain matrices respectively. Note that $\mathcal{S}_n$ and $\mathcal{H}_n$ describe the space of selfadjoint operators in $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively, with respect to the standard inner product $(\cdot, \cdot)$. Let $A \in \mathcal{H}_n$. It is well known that $\mathbb{C}^n$ has an orthonormal basis consisting entirely of the eigenvectors of A:

$$\begin{aligned}
&Au_i = \lambda_i u_i, \quad \lambda_i \in \mathbb{R}, \quad u_i \in \mathbb{C}^n, \quad i = 1, ..., n, \\
&(u_i, u_j) = \delta_{ij}, \quad i, j = 1, ..., n, \\
&\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad (\lambda := (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n_{\geq}).
\end{aligned}$$

If $A \in \mathcal{S}_n$ we assume that $u_1, ..., u_n \in \mathbb{R}^n$. Sometimes we will emphasize the dependence on A:

$$\begin{aligned}
&\lambda(A) = (\lambda_1(A), ..., \lambda_n(A)) := \lambda, \\
&u_i(A) := u_i, \quad i = 1, ..., n.
\end{aligned}$$

For $\alpha, \beta \in \mathbb{R}^n_{\geq}$ let

$$K(\alpha,\beta) := \{\gamma \in \mathbb{R}^n_{\geq} : \quad \gamma = \lambda(C),\ \ C = A+B,\ \text{ for all } A, B \in \mathcal{H}_n \text{ with } \lambda(A) = \alpha,\ \lambda(B) = \beta\}.$$

The trace equality

$$\sum_{i=1}^{n} \lambda_i(A+B) = \sum_{i=1}^{n} \lambda_i(A) + \sum_{i=1}^{n} \lambda_i(B), \quad A, B \in \mathcal{H}_n \tag{1.1}$$

implies that $K(\alpha, \beta)$ lies in the hyperplane

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i, \quad \gamma = (\gamma_1, ..., \gamma_n)^T \in \mathbb{R}^n_{\geq}. \tag{1.2}$$

The problem of describing $K(\alpha, \beta)$ was raised in the late 40's in Gelfand's seminar in Moscow [BG]. The aim of these lectures to report the solution of this problem, primary by A. Klyachko [Kly] and A. Knutson and T. Tao [KT]. Consult with [Fu1]. We will also describe the characterization of the set

$$K_{\leq}(\alpha,\beta) := \{\gamma \in \mathbb{R}^n_{\geq} : \quad \gamma = \lambda(C),\ \text{ for all } A, B, C \in \mathcal{H}_n \text{ with } C \leq A+B,\ \lambda(A) = \alpha,\ \lambda(B) = \beta\},$$

due to Friedland [Fr2] and Fulton [Fu2], which enables to generalize these results to non-negative selfadjoint compact operators.

2 Minimax characterizations of eigenvalues

The maximal and minimal characterizations of the first and the last eigenvalue of $A \in \mathcal{H}_n$, which go back to J.W. Rayleigh in 19th century, are

$$\begin{aligned} \lambda_1(A) &= \max_{0 \neq x \in \mathbb{C}^n} \frac{(Ax,x)}{(x,x)} = \max_{(x,x)=1} (Ax,x), \\ \lambda_n(A) &= \min_{0 \neq x \in \mathbb{C}^n} \frac{(Ax,x)}{(x,x)} = \min_{(x,x)=1} (Ax,x). \end{aligned} \tag{2.1}$$

They follow easily if we choose to present the Rayleigh quotient $\frac{(Ax,x)}{(x,x)}$ in the o.n. eigenbasis of A. Since the Rayleigh quotient (or (Ax, x)) is a linear function(al) on $\mathcal{H}_n$ for a fixed x, (2.1) yields that the function $\lambda_1(\cdot) : \mathcal{H}_n \to \mathbb{R}$ ($\lambda_n(\cdot) : \mathcal{H}_n \to \mathbb{R}$) is a convex (concave) function. Clearly, each $\lambda_i(A)$ is a homogeneous function of degree 1:

$$\lambda_i(tA) = t\lambda_i(A), \quad t \in \mathbb{R}_+,\ A \in \mathcal{H}_n,\ i = 1, ..., .$$

Hence

$$\lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B), \quad A, B \in \mathcal{H}_n. \tag{2.2}$$

The characterization of any other eigenvalue of $A \in \mathcal{H}_n$ is either minmax or maxmin characterization. Let

$$< n >:= \{1, 2, ..., n\}.$$

The following characterization is widely known as Courant-Fischer characterization [Gan]. Let $\mathrm{Gr}(k, \mathbb{F}^n)$ be the collection of all k-dimensional subspaces of $\mathbb{C}^n$. For $L \in \mathrm{Gr}(\mathrm{k}, \mathbb{F}^n)$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, let $L^\perp$ be the orthogonal complement of L in $\mathbb{F}^n$ with respect to $(\cdot, \cdot)$. Then

$$\lambda_i(A) = \min_{L \in \mathrm{G}(i-1, \mathbb{C}^n)} \max_{x \in L^\perp, (x,x)=1} (Ax, x), \quad i \in < n > . \tag{2.3}$$

The following inequalities are due to Weyl [Wey]:

Corollary 2.1 *Let $A, B \in \mathcal{H}_n$ and assume that $i, j, i + j - 1 \in < n >$. Then*

$$\lambda_{i+j-1}(A + B) \le \lambda_i(A) + \lambda_j(B).$$

Proof. Let

$$\lambda_i(A) = \max_{x \in L_{i-1}(A)^\perp} (Ax, x), \quad L_{i-1}(A) = \mathrm{span}\ (u_1(A), ..., u_{i-1}(A)),$$
$$\lambda_i(B) = \max_{x \in L_{i-1}(B)^\perp} (Bx, x), \quad L_{i-1}(B) = \mathrm{span}\ (u_1(B), ..., u_{i-1}(B)).$$

Let $L = L_{i-1}(A) + L_{j-1}(B) \in \mathrm{Gr}(k, \mathbb{C}^n)$ where $k \le i + j - 2$. Clearly

$$\lambda_{i+j-1}(A + B) \le \lambda_{k+1}(A + B) \le \max_{x \in L^\perp,\ (x,x)=1} ((A + B)x, x)) \le \lambda_i(A) + \lambda_j(B).$$

□

Remark 2.2 To prove (2.3) one notes that for any $L \in \mathrm{Gr}(i - 1, \mathbb{C}^n)$ $L^\perp \cap L_i(A) \in \mathrm{G}(m, \mathbb{C}^n)$ for some $m \ge 1$. Hence $\lambda_i(A) \le \max_{x \in L^\perp, (x,x)=1}(Ax, x)$. Clearly $\lambda_i(A) = \max_{x \in L_{i-1}(A)^\perp, (x,x)=1}(Ax, x)$.

Let $L \in \mathrm{Gr}(m, \mathbb{C}^n)$. Fix an o.n. basis $x_1, ..., x_m$ in L. Let $A \in \mathcal{H}_n$ and denote by $A(x) = A(x_1, ..., x_m) := ((Ax_i, x_j))_1^m \in \mathcal{H}_m$. Choose another o.n. basis $y_1, ..., y_m$ in L. Then $A(y) = A(y_1, ..., y_m)$ is unitary similar to $A(x)$. Let $\lambda(A|L) = (\lambda_1(A|L), ..., \lambda_m(A|L))^T \in \mathbb{R}^m_\ge$ be the eigenvalues of $A(x)$. The following result was called by Polya and Schiffer [PS] the convoy principle and is attributed to Poincaré. See [Fr1] for its uses for matrices and selfadjoint compact nonnegative operators.

$$\lambda_i(A) = \max_{L \in \mathrm{Gr}(m, \mathbb{C}^n)} \lambda_i(A|L), \quad i = 1, ..., m, \quad m = 1, ..., n. \tag{2.4}$$

Corollary 2.3 (Ky Fan 1949 [Fan]) *Let $A \in \mathcal{H}_n$ and $m \in < n >$. Then*

$$\sum_{i=1}^m \lambda_i(A) = \max_{x_1,...,x_m \in \mathbb{C}^n,\ (x_i,x_j)=\delta_{ij}} \sum_{i=1}^m (Ax_i, x_i).$$

In particular for any $A, B \in \mathcal{H}_n$ and $m \in < n >$

$$\sum_{i=1}^m \lambda_i(A + B) \le \sum_{i=1}^m \lambda_i(A) + \sum_{i=1}^m \lambda_i(B). \tag{2.5}$$

Proof. Clearly, for any o.n. basis $x_1, ..., x_m$ of L we have the equality

$$\mathrm{trace}(A|L) := \sum_{i=1}^{m} (Ax_i, x_i) = \sum_{i=1}^{m} \lambda_i(A|L).$$

Use the convoy principle to deduce the maximum characterization of $\sum_{i=1}^{m} \lambda_i(A)$. $\square$

3 Results of Lidskii and Wielandt

In [L1] V.B. Lidskii announced the following result. Let Π_n be the group of all $n \times n$ permutation matrices. For $x \in \mathbb{R}^n$ let $\Gamma(x)$ be the convex hull spanned by the vectors $Px,\ P \in \Pi_n$. Then

$$K(\alpha, \beta) \subset \alpha + \Gamma(\beta), \quad \text{for all } \alpha, \beta \in \mathbb{R}^n_{\geq}. \tag{3.1}$$

Wielandt was not able to reconstruct the outline of Lidskii's proof in [L1]. To prove (3.1) Wielandt gave a characterization of any sum of the eigenvalues of $A \in \mathcal{H}_n$, which generalizes all the above characterizations. A (complete) *flag* F_* on $\mathbb{C}^n$ is a strictly increasing sequence of subspaces

$$[0] = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n.$$

That is dim $F_i = i,\ i = 0, ..., n$. Let $I \subset < n >$ of cardinality $k = |I|$. Then

$$I = \{i_1, ..., i_k\}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n.$$

A partial flag F_I (associated with I) is a a strictly increasing sequence of subspaces

$$F_{i_1} \subset F_{i_2} \subset \cdots \subset F_{i_k}, \quad \dim F_{i_j} = i_j, \quad j = 1, ..., k.$$

Any partial flag F_I can be completed to a complete flag F_* in many ways unless $I =< n >$. We shall view F_I as a partial flag of some F_*. Let

$$x[I] := \sum_{i \in I} x_i \quad \text{for any } x = (x_1, ..., x_n)^T \in \mathbb{R}^n.$$

Theorem 3.1 (Wielandt [Wie]) *Let $I \subset < n >$, $|I| = m \in < n >$. Then for any $A \in \mathcal{H}_n$*

$$\lambda(A)[I] = \max_{F_I} \min_{x_i \in F_i,\ (x_i, x_j) = \delta_{ij},\ i,j \in I} \sum_{i \in I} (Ax_i, x_i). \tag{3.2}$$

Proof. The proof is by the induction on n. Assume that the Theorem holds for $n \leq N$. Let $n = N + 1$. One needs to show first that for $I = \{1 \leq i_1 < \cdots < i_m \leq N + 1\}$

$$\lambda(A)[I] \geq \min_{x_i \in F_i,\ (x_i, x_j) = \delta_{ij},\ i,j \in I} \sum_{i \in I} (Ax_i, x_i). \tag{3.3}$$

Suppose first that $i_m < N + 1$. Idenitify F_{i_m} with $\mathbb{C}^{i_m}$. Then the induction hypothesis implies that

$$\lambda(A|F_{i_m})[I] \geq \min_{x_i \in F_i,\ (x_i, x_j) = \delta_{ij},\ i,j \in I} \sum_{i \in I} (Ax_i, x_i).$$

Use the convoy principle $\lambda(A)[I] \geq \lambda(A|F_{i_m})[I]$ to deduce (3.3). Assume now that $i_m = N+1$. If $I =< N+1 >$ then (3.3) holds, since for any full flag F_* equality holds in (3.3). Assume that $|I| < N+1$. Then there exits a unique $g \in< N >$ such that $g \notin I$ and $\{g+1, ..., N+1\} \subset I$. Let f be the biggest element in $I \backslash \{g+1, ..., N+1\}$. (If $I = \{g+1, ..., N\}$ then $f = 0$.) Let

$$L = F_f + \text{span}\,(u_{g+1}(A), ..., u_{N+1}(A)), \quad \dim L \leq f + N + 1 - g \leq N.$$

Hence there exists $\tilde{L} \in \text{Gr}(N, \mathbb{C}^{N+1})$ such that $L \subset \tilde{L}$. Note that

$$\begin{aligned}
&F_f \subset F_{g+1} \cap \tilde{L} \subset \cdots \subset F_{N+1} \cap \tilde{L}, \\
&g + i - 1 \leq \dim F_{g+i} \cap \tilde{L} \leq g + i, \quad i = 1, ...N - g, \\
&\dim F_{N+1} \cap \tilde{L} = N.
\end{aligned}$$

Let $\tilde{I} = I \backslash \{N+1\} \cup \{g\}$. Hence there exists a flag $\tilde{F}_{\tilde{I}}$ such that

$$\begin{aligned}
&F_i = \tilde{F}_i, \quad i \in I \backslash \{g+1, ..., N+1\} = \tilde{I} \backslash \{g, ..., N\}, \\
&F_{g+i} \supset \tilde{F}_{g+i-1} \quad i = 1, ..., N+1, \\
&\tilde{F}_N = \tilde{L}.
\end{aligned}$$

By construction

$$\min_{x_i \in F_i,\ (x_i, x_j) = \delta_{ij},\ i,j \in I} \sum_{i \in I} (Ax_i, x_i) \leq \min_{x_i \in \tilde{F}_i,\ (x_i, x_j) = \delta_{ij},\ i,j \in \tilde{I}} \sum_{i \in \tilde{I}} (Ax_i, x_i).$$

Use the induction hypothesis to obtain that

$$\lambda(A|\tilde{F}_N)[\tilde{I}] \geq \min_{x_i \in \tilde{F}_i,\ (x_i, x_j) = \delta_{ij},\ i,j \in \tilde{I}} \sum_{i \in \tilde{I}} (Ax_i, x_i).$$

Since $\tilde{F}_N \supset \text{span}\,(u_{g+1}(A), ..., u_{N+1}(A))$ it follows that the eigenvalues of $A|\tilde{F}_N$ are the N coordinates of the vectors $\lambda(A|L')$ and $(\lambda_{g+1}(A), ..., \lambda_{N+1}(A))$, where $L' \subset \tilde{F}_N$ is the orthogonal complement of span $(u_{g+1}(A), ..., u_{N+1}(A))$ in $\tilde{F}_N$. Use the convoy principle for $\lambda(A|L')$ to deduce $\lambda(A)[I] \geq \lambda(A|\tilde{F}_N)[\tilde{I}]$. Hence (3.3) holds. Let $L_I(A)$ be the partial flag corresponding to the complete flag $L_*(A)$, where $L_i(A) = \text{span}\,(u_1(A), ..., u_i(A))$, $i = 1, ..., n$. It is straightforward to show that

$$\lambda(A)[I] = \min_{x_i \in L_i(A),\ (x_i, x_j) = \delta_{ij},\ i,j \in I} \sum_{i \in I} (Ax_i, x_i).$$

□

Corollary 3.2 *Let $A, B \in \mathcal{H}_n$ and $I \subset< n >$. Then*

$$\lambda(A+B)[I] \leq \lambda(A)[I] + \lambda(B)[< |I| >].$$

Proof. Consider Wielandt's characterization for $\lambda(A+B)[I]$. Ky Fan characterization yields $\sum_{i\in I}(Bx_i,x_i) \le \lambda(B)[<|I|>]$ for any orthonormal set $x_i, i\in I$. Use Wielandt's chracterization for $\lambda(A)[I]$ to deduce the above inequality. $\Box$

Proof of Lidskii's theorem It is well known [HLP] that $x=(x_1,...,x_n)^T \in \Gamma(\beta)$ iff $x[I] \le \beta[<|I|>]$ for all $I \subset <n>$. Corollary 3.2 shows that $\lambda(A+B)-\lambda(A) \in \Gamma(\lambda(B))$. $\Box$

See Bhatia [Bha] for a detailed proof of Wielandt's and Lidskii's inequalities.

4 Horn's results and conjectures

In [Hor] Horn studied in detail the structure of $K(\alpha,\beta)$. Let $\mathcal{U}_n$ be the unitary group $n\times n$ complex valued matrices. Then

$$K(\lambda(A),\lambda(B)) = \{\lambda(A+UBU^*): \quad U\in\mathcal{U}_n\}, \quad \text{for any } A,B\in\mathcal{H}_n. \tag{4.1}$$

Horn showed that a boundary point $\eta\in K(\alpha,\beta)$ corresponds to $C=A+B$, where A,B (and hence C) have a nontrivial common invariant subspace $L\in \mathrm{Gr}(m,\mathbb{C}^n)$, $1\le m<n$. Clearly $L^\perp$ is also a nontrivial subspace of A,B,C. Hence

$$\mathrm{trace}(C|L)=\mathrm{trace}(A|L)+\mathrm{trace}(B|L), \quad \mathrm{trace}(C|L^\perp)=\mathrm{trace}(A|L^\perp)+\mathrm{trace}(B|L^\perp). \tag{4.2}$$

One of these equalities induces the inequality of the type

$$\lambda(A+B)[K] \le \lambda(A)[I]+\lambda(B)[J], \quad I,J,K\subset<n>, \quad 1\le|I|=|J|=|K|<n. \tag{4.3}$$

Horn conjectured the form of the sets (I,J,K) which satisfy (4.3). They are defined recursively as follows. Let

$$U_r^n := \{(I,J,K):\ I,J,K\subset<n>,\ |I|=|J|=|K|=r<n,\ \sum_{i\in I} i+\sum_{j\in J} j = \frac{r(r+1)}{2}+\sum_{k\in K} k\}. \tag{4.4}$$

Horn showed that if η is a boundary point certain quadratic form has to be nonnegative definite. Hence any (I,J,K) coming from (4.2) has to be in U_r^n for some $r\in<n-1>$. Define $T_1^n := U_1^n$. The inequalities (4.3) corresponding to $(I,J,K)\in T_1^n$ are Weyl's inequalities. For $1<r\le n-1$ let

$$T_r^n := \{(I,J,K)\in U_r^n: \quad \text{for all } (U,V,W)\in T_p^r,\ p\in<1,r-1> \quad \textstyle\sum_{u\in U} i_u+\sum_{v\in V} j_v \le \frac{p(p+1)}{2}+\sum_{w\in W} k_w\}. \tag{4.5}$$

Conjecture 4.1 (Horn [Hor]). *$\gamma\in K(\alpha,\beta)$ iff (1.2) holds and*

$$\gamma[K] \le \alpha[I]+\beta[J], \quad \text{for all } (I,J,K)\in T_r^n,\ r\in<1,n-1>. \tag{4.6}$$

Horn proved the validity (4.6) for triples (I, J, K) belonging to the sets T_1^n, T_2^n, T_3^n. He showed that his conjecture holds for $n = 2, 3, 4$. For $n = 2$ it is straightforward to show that

$$K((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \{(\gamma_1, \gamma_2) \in \mathbb{R}^2_{\geq}$$
$$\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2,$$
$$\gamma_1 \leq \alpha_1 + \beta_1,$$
$$\gamma_2 \leq \min(\alpha_1 + \beta_2, \alpha_2 + \beta_1).$$

The above 3 inequalities are Weyl's inequalities. For $n = 3$ Horn's result claims

$$K((\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)) = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3_{\geq}$$
$$\gamma_1 + \gamma_2 + \gamma_3 = \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3,$$
$$\gamma_1 \leq \alpha_1 + \beta_1,$$
$$\gamma_2 \leq \min(\alpha_1 + \beta_2, \alpha_2 + \beta_1),$$
$$\gamma_3 \leq \min(\alpha_1 + \beta_3, \alpha_2 + \beta_2, \alpha_3 + \beta_1),$$
$$\gamma_1 + \gamma_2 \leq \alpha_1 + \alpha_2 + \beta_1 + \beta_2,$$
$$\gamma_1 + \gamma_3 \leq \min(\alpha_1 + \alpha_3 + \beta_1 + \beta_2, \alpha_1 + \alpha_2 + \beta_1 + \beta_3),$$
$$\gamma_2 + \gamma_3 \leq \min(\alpha_2 + \alpha_3 + \beta_1 + \beta_2, \alpha_1 + \alpha_2 + \beta_2 + \beta_3, \alpha_1 + \alpha_3 + \beta_1 + \beta_3).\}$$

Note that out of 12 inequalities (the first) 6 inequalites are due to Weyl, 1 is due to Ky Fan, 4 due to Wielandt and 1 is due to Horn:

$$\gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3. \tag{4.7}$$

Indeed, note that the inequalities (4.6) for $(I, J, K) \in T_1^n$ is the set of Weyl's inequalities. Next

$$T_2^n := \{(I, J, K) \subset < n >: \quad I = (1 \leq i_1 < i_2 \leq n),\ J = (1 \leq j_1 < j_2 \leq n),$$
$$K = (1 \leq k_1 < k_2 \leq n),$$
$$i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 3,\ i_1 + j_1 \leq k_1 + 1,\ \max(i_1 + j_2, i_2 + j_1) \leq k_2 + 1.\} \tag{4.8}$$

Hence (4.7) are the inequalities for $I = J = \{1, 3\},\ K = \{2, 3\}$ which are in T_2^3. The cardinalities of $|T_r^n|$ grows very fast. For example:

$$|T_1^7| = |T_6^7| = 28,\ |T_2^7| = |T_5^7| = 252,\ |T_3^7| = |T_4^7| = 751.$$

See [DST]. It is now known that Horn's inequalities are not minimal for $n \geq 6$. For example

$$(I, J, K) = (\{1, 3, 5\}, \{1, 3, 5\}, \{2, 4, 6\} \in T_3^n, \quad n \geq 6.$$

Hence for any $\gamma \in K(\alpha, \beta) \subset \mathbb{R}^n_{\geq},\ n \geq 6$ we have

$$\gamma_2 + \gamma_4 + \gamma_6 \leq \alpha_1 + \alpha_3 + \alpha_5 + \beta_1 + \beta_3 + \beta_5. \tag{4.9}$$

For $n = 6$ the above inequality follows from the trace equality. Indeed, for $n = 2m$ and $\alpha \in \mathbb{R}^{2m}$ let $\alpha_{\text{odd}}, \alpha_{\text{even}}$ be the sum of odd and even coordinates of $\alpha = (\alpha_1, ..., \alpha_{2m})$. Then

$$2\gamma_{\text{even}} \leq \gamma_{\text{odd}} + \gamma_{\text{even}} = \alpha_{\text{odd}} + \alpha_{\text{even}} + \beta_{\text{odd}} + \beta_{\text{even}} \leq 2(\alpha_{\text{odd}} + \beta_{\text{odd}}).$$

In [L2] the son Lidskii claimed to prove Horn's conjecture by listing 5 lemmas (without proofs), which imply Horn's conjecture. Day, So and Thompson [DST] were able to prove the first 3 lemmas of B.V. Lidskii.

5 Flags and Schubert varieties

Let $V (= \mathbb{F}^n)$ be an n-dimensional vector space over $\mathbb{F}$. Let F_* be a complete flag on V (see §3). Assume that $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and V is an inner product space with the inner product $(\cdot, \cdot)$. Then F_* induces an orthonormal basis in V:

$$F_i = \text{span}(f_1, .., f_i),\ i = 1, ..., n, \quad (f_i, f_j) = \delta_{ij},\ i, j = 1, ..., n. \tag{5.1}$$

In what follows we restrict ourselves to the complex case $\mathbb{F} = \mathbb{C}$. The orthonormal basis $\{f_1, ..., f_n\}$ induced by F_* is defined up to the action of $\mathcal{U}_1$ (the group of complex numbers of modulus 1). That is, $\zeta_1 f_1, ...\zeta_n f_n,\ \zeta_1, ..., \zeta_n \in \mathcal{U}_1$ is the set of all possible o.n. bases in $\mathbb{C}^n$ induced by F_*. Let $\mathcal{D}_n < \mathcal{U}_n$ be the subgroup of all unitary diagonal matrices.

Lemma 5.1 *Let $\mathcal{F}_n$ be the space of all flags in $\mathbb{C}^n$. Then $\mathcal{F}_n$ is isomorphic to the homogeneous space $\mathcal{U}_n/\mathcal{D}_n$ of real dimension $n(n-1)$. $\mathcal{F}_n$ is a fibre bundle over $\mathbb{P}^{n-1}$ with the fiber $\mathcal{F}_{n-1}$. Furthermore $\mathcal{F}_n$ is a smooth complex projective variety of complex dimension $\frac{n(n-1)}{2}$.*

Proof. Let $U = (u_1, ..., u_n) \in \mathcal{U}_n$. Then the n columns of U give an o.n. basis of $\mathbb{C}^n$. A flag F_* induces a unique left coset $U\mathcal{D}_n$. Hence $\mathcal{F}_n \sim \mathcal{U}_n/\mathcal{D}_n$. Clearly

$$\dim_{\mathbb{R}} \mathcal{U}_n/\mathcal{D}_n = n^2 - n = n(n-1).$$

Observe next that a choice of one dimensional subspace F_1 is the definition of a point $z \in \mathbb{P}^{n-1}$. Fix $z \in \mathbb{P}^{n-1}$. By choosing an o.n. basis in $\mathbb{C}^n$ we may assume that z is presented by $u_1 = e_n = (0, ..., 0, 1)^T$. That is, $u_2, .., u_n \in \mathbb{C}^{n-1}$. Hence $\mathcal{F}_n$ is a fibre bundle with a basis $\mathbb{P}^{n-1}$ and a fibre $\mathcal{F}_{n-1}$. For a set $\mathcal{T} \subset \mathbb{F}$ let

$$\begin{aligned}
&\mathbf{M}_{nm}(\mathcal{T}) := \{A : \quad A = (a_{ij})_{i=j=1}^{i=n,j=m},\ a_{ij} \in \mathcal{T},\ i = 1, ..., n,\ j = 1, ..., m\},\\
&\mathbf{M}^o_{nm}(\mathcal{T}) := \{A \in \mathbf{M}_{nm}(\mathcal{T}) : \quad \text{rank}\, A = \min(m, n)\},\\
&\mathbf{M}_n(\mathcal{T}) := \mathbf{M}_{nn}(\mathcal{T}), \quad \mathbf{M}^o_n(\mathcal{T}) := \mathbf{M}^o_{nn}(\mathcal{T}),\\
&\mathbf{GL}(n, \mathbb{F}) := \mathbf{M}^o_n(\mathbb{F}),\\
&\mathbf{UT}(n, \mathbb{F}) := \{A = (a_{ij})_1^n \in \mathbf{GL}(n, \mathbb{F}) : \quad a_{ij} = 0, \text{ for } 1 \leq j < i \leq n\}.
\end{aligned} \tag{5.2}$$

Let $A = (a_1, ..., a_n) \in \mathbf{GL}(n, \mathbb{C})$ be the n columns of A. Then A induces the complete flag

$$F_i = \text{span } (a_1, ..., a_i), \quad i = 1, ..., n. \tag{5.3}$$

Vice versa, a complete flag F_* induces a unique left coset $A\mathbf{UT}(n, \mathbb{C})$ in $\mathbf{GL}(n, \mathbb{C})$. Hence $\mathcal{F}_n \sim \mathbf{GL}(n, \mathbb{C})/\mathbf{UT}(n, \mathbb{C})$. As $\mathbf{GL}(n, \mathbb{C})$ and $\mathbf{UT}(n, \mathbb{C})$ are algebraic groups it follows that $\mathcal{F}_n$ is a smooth projective variety of complex dimension $\frac{n(n-1)}{2}$. $\Box$

Let $I = \{1 \le i_1 < i_2 < \cdots < i_m \le n\} \subset < n >$. Then $F_*(I)$ is the partial flag

$$F_{i_1} \subset \cdots \subset F_{i_m} \subset \mathbb{C}^n, \quad \dim F_i = i, \ i \in I.$$

We view $F_*(I)$ as a partial flag of some complete flag F_*.

Lemma 5.2 *Let $I = \{1 \le i_1 < i_2 < \cdots < i_m \le n\} \subset < n >$. Denote by $\mathcal{F}(I)$ the set of all partial flags $F_*(I)$ in $\mathbb{C}^n$. Then $\mathcal{F}(I)$ is a smooth projective variety of dimension*

$$\dim \mathcal{F}(I) = \sum_{k=1}^{m} (i_k - i_{k-1})(n - i_k), \quad i_0 = 0. \tag{5.4}$$

Proof. Let $I = \{l\}$. Then $\mathcal{F}(\{l\}) = \text{Gr}(l, \mathbb{C}^n)$. Any $F_*(\{l\})$ is spanned by the columns of $A \in \mathbf{M}^o_{nl}(\mathbb{C})$. Hence $F_*(\{l\})$ determines a unique coset $A\mathbf{GL}(l, \mathbb{C})$ in the quotient space $\mathbf{M}^o_{nl}(\mathbb{C})/\mathbf{GL}(l, \mathbb{C})$. Hence $\mathcal{F}(\{l\})$ is a smooth projective variety of dimension

$$\dim \mathcal{F}(\{l\}) = \dim \text{Gr}(l, \mathbb{C}^n) = \dim \mathbf{M}^o_{nl}(\mathbb{C})/\mathbf{GL}(l, \mathbb{C}) = l(n - l).$$

To prove (5.4) for $m > 1$, let $\tilde{n} = n - i_1$ and $\tilde{I} = \{i_2 - i_1, i_2 - i_1, ..., i_m - i_1\} \subset < \tilde{n} >$. Then the above arguments show that $\mathcal{F}(I)$ is a fibre bundle with a basis $\text{Gr}(i_1, \mathbb{C}^n)$ and the fibre $\mathcal{F}(\tilde{I})$. Hence

$$\dim \mathcal{F}(I) = \dim \text{Gr}(i_1, \mathbb{C}^n) + \dim \mathcal{F}(\tilde{I}).$$

Use induction to show (5.4). A straightforward argument shows that $\mathcal{F}(I)$ is given as a quotient of $\mathbf{M}^o_{mi_m}$ by a corresponding subgroup of block upper triangular matrices $GL(I) < \mathbf{GL}(i_m, \mathbb{C})$. Hence $\mathcal{F}(I)$ is a smooth projective variety. $\Box$

Fix a flag F_* in $\mathbb{C}^n$. Let $L \in \text{Gr}(m, \mathbb{C}^n)$. Then

$$\begin{aligned} &[0] = L \cap F_0 \subset L \cap F_1 \subset \cdots \subset L \cap F_n = L, \\ &\dim L \cap F_i \le \dim L \cap F_{i-1} + 1, \quad i = 1, ..., n. \end{aligned} \tag{5.5}$$

Let

$$\begin{aligned} &I(L, F_*) := \{I = \{1 \le i_1 < \cdots < i_m \le n\} : \quad \dim L \cap F_{i_j} = j, \ j = 1, ..., m, \\ &\dim L \cap F_k < j, \text{ for all } k < i_j\}, \quad L \in \text{Gr}(m, \mathbb{C}^n). \end{aligned} \tag{5.6}$$

For $I = \{1 \le i_1 < \cdots i_m \le n\}$ let

$$\begin{aligned}\Omega_I^o(F_*) &:= \{L \in \mathrm{Gr}(m,\mathbb{C}^n): \quad I(L,F_*) = I\},\\ \Omega_I(F_*) &:= \{L \in \mathrm{Gr}(m,\mathbb{C}^n): \quad \dim L \cap F_{i_j} \ge j,\ j = 1,...,m\},\end{aligned} \tag{5.7}$$

the Schubert cell and the Schubert variety corresponding to I.

Lemma 5.3 *Let $I = \{1 \le i_1 < \cdots i_m \le n\}$. Then $\Omega_I^o(F_*) \subset \mathrm{Gr}(m,\mathbb{C}^n)$ is a quasiprojective variety. $\Omega_I(F_*) \subset \mathrm{Gr}(m,\mathbb{C}^n)$ is a projective variety, which is the closure of $\Omega_I^o(F_*)$ in $\mathrm{Gr}(m,\mathbb{C}^n)$. Furthermore*

$$\dim \Omega_I(F_*) = \dim \Omega_I^o(F_*) = \sum_{j=1}^m i_j - j. \tag{5.8}$$

Proof. Without loss of generality we may assume that F_* is the standard flag

$$F_i = \mathrm{span}\ (e_1,...,e_i), \quad e_i = (\delta_{1i},...,\delta_{ni})^T, \quad i = 1,...,n. \tag{5.9}$$

Then L is spanned by the columns of a matrix $A = (a_1,...,a_m) \in \mathbf{M}_{nm}^o(\mathbb{C})$ such that

$$a_j = (a_{1j},...,a_{nj})^T, \quad a_{i_j j} \ne 0,\ a_{ij} = 0,\ i = i_j + 1,...,n,\ j = 1,...,m.$$

Clearly the set of all such A is a quasivariety in $QV(I) \subset M_{nm}^o(\mathbb{C})$. Each $L \in \Omega_I^o(F_*)$ induces a unique coset $A\mathbf{UT}(m,\mathbb{C})$, where $A \in QV(I)$. Hence $\Omega_I^o(F) \sim QV(I)/\mathbf{UT}(m,\mathbb{C})$. This shows that $\Omega_I^o(F_*)$ is a quasivariety in $\mathrm{Gr}(m,\mathbb{C}^n)$ of dimension $\sum_{j=1}^m i_j - \frac{m(m+1)}{2}$. Hence $\Omega_I(F_*)$ is a closed variety in $\mathrm{Gr}(m,\mathbb{C}^n)$, which is the topological (Zariski) closure of $\Omega_I^o(F_*)$. In particular (5.8) holds. □

Lemma 5.4 *There is one to one correspondance between the Schubert cells in $\mathrm{Gr}(m,\mathbb{C}^n)$ and the set of all $m \times n$ matrices of rank m in its reduced row echelon form: Each $L \in \mathrm{Gr}(m,\mathbb{C}^n) \sim \mathbf{M}_{nm}^o(\mathbb{C})/\mathbf{GL}(m,\mathbb{C})$ induces a unique matrix $A(L)$ in the left coset of $\mathbf{M}_{nm}^o/\mathbf{GL}(m,\mathbb{C})$, whose columns span L, such $A(L)^T$ is in its reduced row echelon form. Assume that the first nonzero entry of $A(L)^T$ in the row j, which is equal to 1, is in the column $\tilde{i}_j$ for $j = 1,...,m$. Let $i_j = n - \tilde{i}_{m-j+1} + 1,\ j = 1,...,m$ and set $I = \{1 \le i_1 < \cdots < i_m \le n\}$. Let F_* be the reversed standard flag*

$$F_i = \mathrm{span}\ (e_n,...,e_{n-i+1}), \quad i = 1,...,n.$$

Then $I(L,F_*) = I$.

The proof of the lemma is straightforward and is left to the reader. One can use Lemma 5.4 to find the dimension of the Schubert cell $\Omega_I^o(F_*)$.

6 Hersch-Zwahlen characterization

Lemma 6.1 ([HZ]) *Let $A \in \mathcal{H}_n$ and denote by $F_*(A)$ the flag induced by the eigenvectors of A: $F_i(A) = \text{span }(u_1(A), ..., u_i(A)),\ i = 1, ..., n$. Let $I = \{1 \le i_1 < i_2 < \cdots < i_m \le n\}$. Then*

$$\lambda(A)[I] = \min_{L \in \Omega_I(F_*(A))} \text{trace}(A|L). \tag{6.1}$$

Proof. Let $L \in \Omega_I(F_*(A))$. Then L has an orthonormal basis $x_1, ..., x_m$ such that $x_j \in F_{i_j}(A),\ j = 1, ..., m$. Hence $(Ax_j, x_j) \ge \lambda_{i_j}(A)$ and

$$\lambda(A)[I] \le \text{trace}(A|L).$$

For $L = \text{span }(u_{i_1}(A), u_{i_2}(A), ..., u_{i_m}(A)) \in \Omega_I(F_*(A))$ equality holds in the above inequality. $\Box$

Corollary 6.2 ([HZ]) *Let $A, B, C \in \mathcal{H}_n,\ C = A + B$. Let*

$$I = \{1 \le i_1 < i_2 < \cdots < i_m \le n\},$$
$$J = \{1 \le j_1 < j_2 < \cdots < j_m \le n\},$$
$$K = \{1 \le k_1 < k_2 < \cdots < k_m \le n\}.$$

Set

$$I' = \{n - i_m + 1 < \cdots < n - i_1 + 1\}, \quad J' = \{n - j_m + 1 < \cdots < n - j_1 + 1\}.$$

Suppose that

$$\Omega_{I'}(F_*(-A))) \cap \Omega_{J'}(F_*(-B))) \cap \Omega_K(F_*(C)) \ne \emptyset.$$

Then (4.3) holds.

Proof. Let $L \in \Omega_{I'}(F_*(-A))) \cap \Omega_{J'}(F_*(-B))) \cap \Omega_K(F_*(C))$. Apply Lemma 6.1 to $-A, -B, C$ respectively and use the equality $-A - B + C = 0$ to deduce

$$\lambda(-A)[I'] + \lambda(-B)[J'] + \lambda(C)[K] \le 0.$$

$\Box$

Corollary 6.3 ([HZ]) *Let $I, J, K, I', J' \subset < n >$ be defined as in Corollary 6.2. Suppose that for any three complete flags $F_*(1), F_*(2), F_*(3)$ in $\mathbb{C}^n$ the following condition holds*

$$\Omega_{I'}(F_*(1))) \cap \Omega_{J'}(F_*(2))) \cap \Omega_K(F_*(3)) \ne \emptyset. \tag{6.2}$$

Then for any $A, B \in \mathcal{H}_n$ (4.3) holds.

Proof of (4.7). Let

$$I = J = \{1, 3\}, \quad K = \{2, 3\}.$$

Assume first that $n = 3$. Then $I' = J' = I = J$. We claim that for any three flags in $\mathbb{C}^3$ (6.2) holds. Indeed, choose $L \in \mathrm{Gr}(2, \mathbb{C}^3)$ such that $L \supset F_1(1) + F_1(2)$. As any two dimensional subspaces in $\mathbb{C}^3$ have a common one dimensional subspace $L \in \Omega_{I'}(F_*(1))) \cap \Omega_{J'}(F_*(2))) \cap \Omega_K(F_*(3))$. Hence (4.7) holds for any $A, B \in \mathcal{H}_3$. Let $n > 3$ and $A, B, C \in \mathcal{H}_n,\ C = A + B$. Let $L = F_3(C)$. Then

$$\lambda_2(C) + \lambda_3(C) = \lambda_2(C|L) + \lambda_3(C|L) \le \lambda_1(A|L) + \lambda_3(A|L) + \lambda_1(B|L) + \lambda_3(B|L) \le \lambda_1(A) + \lambda_3(A) + \lambda_1(B) + \lambda_3(B).$$

7 Schubert calculus

Let $I \subset < n >$ be defined as in Corolary 6.2. Set

$$\begin{aligned} &\omega_j := i_{m-j+1} - (m - j + 1), \quad \alpha_j = n - i_j - m + j, \quad j = 1, ..., n, \\ &\omega(I) := \omega = (\omega_1, ..., \omega_m),\ \alpha(I) := \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m_{\geq} \cap \mathbb{Z}^m_+, \\ &||\omega||_1 = \sum_{i=1}^m \omega_i, \quad ||\alpha||_1 = \sum_{i=1}^m \alpha_i. \end{aligned} \tag{7.1}$$

Note that $\omega(I') = \alpha(I)$, and $\alpha(I)$ $(\omega(I))$ is with $1-1$ correspondence with $I \subset < n >$. Moreover $||\alpha(I)||_1$ $(||\omega(I)||_1)$ gives the dimension of $\Omega_{I'}(F_*)$ $(\Omega_I(F_*))$ in $\mathrm{Gr}(m, \mathbb{C}^n)$, which is equal to the codimension of $\Omega_I(F_*)$ $(\Omega_{I'}(F_*))$.

Lemma 7.1 *Let $I, J, K \in U^n_m$. Suppose that for any three flags $F_*(1), F_*(2), F_*(3)$ the condition (6.2) holds. Then $\Omega_{I'}(F_*(1)))\cap\Omega_{J'}(F_*(2)))\cap\Omega_K(F_*(3))$ consists of a finite number of points if the flags $F_*(1), F_*(2), F_*(3)$ are in general position.*

Proof. Observe that

$$I, J, K \in U^n_m \iff I, J, K \subset < n >,\ |I| = |J| = |K| = m,\ ||\omega(I)||_1 + ||\omega(J)||_1 = ||\omega(K)||_1. \tag{7.2}$$

As the codimension of $\Omega_{I'}(F_*(1))$ $(\Omega_{J'}(F_*(2)))$ is $||\omega(I)||_1$ $(||\omega(J)||_1)$ we view the variety $\Omega_{I'}(F_*(1))$ $(\Omega_{J'}(F_*(2)))$ given by $||\omega(I)||_1$ $(||\omega(J)||_1)$ algebraically independent conditions. Hence $\Omega_{I'}(F_*(1))) \cap \Omega_{J'}(F_*(2))) \cap \Omega_K(F_*(3))$ is the solution of $||\omega(I)||_1 + ||\omega(J)||_1$ algebraic conditions restricted to $\Omega_K(F_*(3))$, which is of dimension $||\omega(I)||_1 + ||\omega(J)||_1$. If $F_*(1), F_*(2), F_*(3)$ are in general positions, these algebraic conditions restricted to $\Omega_K(F_*(3))$ can give only a finite number of solutions. $\Box$

$$S^n_m := \{(I, J, K) \in U^n_m : \quad \text{such that for any three flags (6.2) holds}\}. \tag{7.3}$$

For $I, J \subset < n >$ satisfying the condition of Corollary 6.2 define

$$I \le J \iff i_p \le j_p, \quad p = 1, ..., m.$$

The following result was known for sometime [Fu1]:

Lemma 7.2 *Let $I, J, K \subset < n >$, $|I| = |J| = |K| = m < n$. Assume that the condition* (6.2) *is satisfied for any three flags $F_*(1), F_*(2), F_*(3)$. Then there exists $I_1, J_1, K_1 \in S_m^n$ satisfying $I_1 \geq I,\ J_1 \geq J,\ K_1 \leq K$.*

The basis of the integer homology of $\mathrm{Gr}(m, \mathbb{C}^n)$ is determined by the cycles σ_I, representing the Schubert varieties $\Omega_I(F_*),\ I \subset < n >$. For $I \subset < n >$ let $\sigma_\alpha \in H_{||\alpha||_1}(\mathrm{Gr}(m, \mathbb{C}^n), \mathbb{Z})$ be the dual cycle to σ_I. That is, the cup product of σ_I and σ_α is the generator of the top homology $H_{m(n-m)}(\mathrm{Gr}(m, \mathbb{C}^n), \mathbb{Z})$. Equivalently

$$\sigma_I \cdot \sigma_\alpha = \sigma_\alpha \cdot \sigma_I = \sigma_{point},$$

where σ_{point} represents the homology element of the point in $H_0(\mathrm{Gr}(m, \mathbb{C}^n), \mathbb{Z})$. We view σ_α as an element in cohomology $H^{||\alpha||_1}(\mathrm{Gr}(m, \mathbb{C}^n), \mathbb{Z})$ given by a corresponding differential form of degree $||\alpha||_1$. Then for any $\alpha, \beta \in \mathbb{R}^m_{\geq} \cap \mathbb{Z}^m_+$ with $||\alpha||_1 + ||\beta||_1 \leq m(n-m)$ we have the formula

$$\sigma_\alpha \cdot \sigma_\beta = \sum_{\gamma \in \mathbb{R}^m_{\geq} \cap \mathbb{Z}^m_+,\ ||\gamma||_1 = ||\alpha||_1 + ||\beta||_1} c^\gamma_{\alpha\beta} \sigma_\gamma. \tag{7.4}$$

Here $c^\gamma_{\alpha\beta}$ are nonnegative integers. These integers give the precise version of Lemma 7.1:

Lemma 7.3 *Let $I, J, K \in U_m^n$. Then*

$$\Omega_{I'}(F_*(1))) \cap \Omega_{J'}(F_*(2))) \cap \Omega_K(F_*(3)) = c^{\omega(K)}_{\omega(I),\omega(J)} \sigma_{point}.$$

That is, if $c^{\omega(K)}_{\omega(I),\omega(J)} = 0$ then the condition (6.2) *does not hold for "most" of three flags $F_*(1), F_*(2), F_*(3)$. If $c^{\omega(K)}_{\omega(I),\omega(J)} \neq 0$ then the condition* (6.2) *does holds for any three flags $F_*(1), F_*(2), F_*(3)$. Furthermore for "most" of three flags $F_*(1), F_*(2), F_*(3)$, i.e. three flags in general position, $\Omega_{I'}(F_*(1))) \cap \Omega_{J'}(F_*(2))) \cap \Omega_K(F_*(3))$ consits of $c^{\omega(K)}_{\omega(I),\omega(J)}$ distinct points.*

The coeffients $c^\gamma_{\alpha\beta}$ appear naturally in representation theory, as well as in invariant factors [Fu1]. With each vector $\alpha \in \mathbb{R}^m_{\geq} \cap \mathbb{Z}^m_+$ one associates the Young diagram, whose row i has length α_i. (We allow here trivial rows with 0 length.) Then V_α corresponds to the irreducible representation of $\mathbf{GL}(m, \mathbb{C})$ or the symmetric group $\mathbf{S}_m$. The *weight* of V_α is $||\alpha||_1$. Consider the tensor product of such two irreducible presentation $V_\alpha \otimes V_\beta$. It is known that such a product is a direct sum of irreducible representations V_γ of the weight $||\gamma||_1 = ||\alpha||_1 + ||\beta||_1$ of multiplicity $c^\gamma_{\alpha\beta}$. That is

$$V_\alpha \otimes V_\beta = \sum_{\gamma \in \mathbb{R}^m_{\geq} \cap \mathbb{Z}^m_+,\ ||\gamma||_1 = ||\alpha||_1 + ||\beta||_1} \oplus c^\gamma_{\alpha\beta} V_\gamma. \tag{7.5}$$

Theorem 7.4 *(*[KT]-The saturation conjecture) *Let $\alpha, \beta, \gamma \in \mathbb{R}^m_{\geq} \cap \mathbb{Z}^m_+,\ ||\gamma||_1 = ||\alpha||_1 + ||\beta||_1$. Then for any integer $N > 1$*

$$c^\gamma_{\alpha\beta} \neq 0 \iff c^{N\gamma}_{(N\alpha)(N\beta)} \neq 0.$$

Theorem 7.4 is instrumental in proving $T_m^n = S_m^n,\ m = 1, ..., n-1$ [Fu1]. In what follows we need the following lemma.

Lemma 7.5 *Let $I, J, K \subset < n >,\ |I| = |J| = |K| = m < n$ and assume that there exists $(I_1, J_1, K_1) \in S_m^n$ such that $I \leq I_1,\ J \leq J_1,\ K \geq K_1$ Then for any triple*

$$A_1, A_3, A_3 \in \mathcal{H}_n, \quad A_1 + A_2 + A_3 = rE_n, \tag{7.6}$$

where E_n is the $n \times n$ identity matrix, the following inequalities hold:

$$\lambda(A_1)[I'] + \lambda(A_2)[J'] + \lambda(A_3)[K] \leq \lambda(A_1)[I_1'] + \lambda(A_2)[J_1'] + \lambda(A_3)[K_1] \leq mr. \tag{7.7}$$

Proof. As $I' \geq I_1',\ J' \geq J_1',\ K \geq K_1$ and the eigenvalues of hermitian matrices are arranged in a decreasing order we deduce the left hand side of (7.7). To prove the right hand side of (7.7) choose $L \in \Omega_{I_1'}(F_*(A_1)) \cap \Omega_{J_1'}(F_*(A_2)) \cap \Omega_K(F_*(A_3))$ and apply (6.1) to (7.6). $\square$

Corollary 7.6 *Let $A, B \in \mathcal{H}_n$. Then any inequality induced by the triples $I, J, K \subset < n >$ given by Corollary 6.3 follows from the inequality corresponding to some $(I_1, J_1, K_1) \in S_m^n$.*

8 Stable filtrations

A filtration U_* of subspaces in $\mathbb{C}^n$ is an infinite sequence of decreasing subspaces where only a finite number of subspaces are different from the trivial subspace [0]:

$$C^n = U_0 \supset U_1 \supset \cdots U_k \supset \cdots, \quad \dim U_i = 0 \text{ for } i > N. \tag{8.1}$$

Each filtration of subspaces defines a unique partial flag $F_*(I)$, where $U_k = F_{i_{j(k)}}$ for some $i_{j(k)} \in I$ for each $k \geq 1$ such that $\dim U_k \geq 1$. Furthermore, for each $i \in I$ F_i appears in the above filtration. Let

$$\alpha_i := \#\{U_j : \quad \dim U_j \geq i\},\ i = 1, ..., n, \quad \alpha := (\alpha_1, ..., \alpha_n) \in \mathbf{R}^n_{\geq} \cap \mathbb{Z}^n_+. \tag{8.2}$$

Then $F_*(I)$ is a complete flag iff $\alpha > 0$ and the coordinates of α are pairwise distinct. Vice versa:

Lemma 8.1 *Let F_* be a given complete flag in $\mathbb{C}^n$. Assume that $\alpha \in \mathbf{R}^n_{\geq} \cap \mathbb{Z}^n_+$. Then there exists a unique filtration (8.1) such that (8.2) holds and U_* induces a partial flag $F_*(I)$.*

Proof. First $U_i = [0]$ for $i > \alpha_1$. If $\alpha_1 = 0$ then U_* is a trivial fibration. Assume that $\alpha_1 > 0$ and $\alpha_1 = \cdots = \alpha_{k-1} > \alpha_k,\ 1 < k \leq n+1$. (Here $\alpha_{n+1} = 0$.) Then $U_{\alpha_1} = \cdots = U_{\alpha_k+1} = F_{k-1}$. Other U_i are determined similarly. $\square$

Lemma 8.2 *Let (8.1) be a given filtration in $\mathbb{C}^n$ with the corresponding α given by (8.2). Let $< \cdot, \cdot >$ be any inner product on $\mathbb{C}^n$. Denote by $P(U_k)$ the orthonormal projection on U_k for $k = 1, ...$ Then the operator $A = \sum_{k=1}^{\infty} P(U_k)$ is a selfadjoint operator with respect to $< \cdot, \cdot >$ with the eigenvalue vector α.*

Proof. Let $F_*(I)$ be the partial flag induced by the filtration (8.1). Complete $F_*(I)$ to a full flag. Then F_* together with $< \cdot, \cdot >$ induces an orthonormal basis $f_1, ..., f_n$ in $\mathbb{C}^n$ such that $F_i = \text{span}\,(f_1, ..., f_i),\ i = 1, ..., n$. In this o.n. basis each $P(U_i)$ is represented by a diagonal matrix, whose first dim U_i diagonal entries are equal to 1 and all other diagonal entries are equal to zero. In this basis A is represented by a diagonal matrix whose $i - th$ diagonal entry is equal to α_i for $i = 1, ..., n$. $\square$

Lemma 8.3 *Assume that the filtration* (8.1) *induces a complete flag* F_*. *Let* α *be given by* (8.2). *Let* $L \in \text{Gr}(m, \mathbb{C}^n)$ *and assume that* $I(L, F_*)$ *is given by* (5.6). *Then*

$$\sum_{k=1}^{\infty} \dim\,(L \cap U_k) = \sum_{i \in I(L,F_*)} \alpha_i. \tag{8.3}$$

Proof. Let $I(L, F_*) = \{1 \leq i_1 < i_2 < \cdots < i_m \leq n\}$. Let a and b be the values of the left hand side and the right hand side of (8.3) respectively. If dim $L \cap U_k = j \geq 1$ then the contribution of U_k to a is j. U_k contributes 1 to α_{i_l} for $l = 1, ..., j$. That is U_k contributes j to b. $\square$

Definition 8.4 *l-filtration* $U_*(1), ..., U_*(l)$ *of* $\mathbb{C}^n$ *is called stable if for any subspace* $[0] \neq L \neq \mathbb{C}^n$

$$\mu(L) := \frac{1}{\dim L} \sum_{i=1}^{l} \sum_{j=1}^{\infty} \dim L \cap U_j(i) < \mu(\mathbb{C}^n) := \frac{1}{n} \sum_{i=1}^{l} \sum_{j=1}^{\infty} \dim U_j(i).$$

The characterization of $K(\alpha, \beta)$ is deduced from the following theorem.

Theorem 8.5 ([Tot],[Kly]) *Let* $U_*(1), ..., U_*(l)$ *be an l-filtration of* $\mathbb{C}^n$ *which induces* l *complete flags* $F_*(1), ..., F_*(l)$ *in general position. Then* $U_*(1), ..., U_*(l)$ *is stable iff there exists an inner product* $< \cdot, \cdot >$ *on* $\mathbb{C}^n$ *such that*

$$\sum_{i=1}^{l} \sum_{j=1}^{\infty} P(U_j(i)) = \mu(\mathbb{C}^n)\text{Id}. \tag{8.4}$$

To prove this theorem Totaro uses geometric invariant theory. Klyachko uses Donaldson's theory for bundles over $\mathbb{P}^2$.

Theorem 8.6 ([Kly]) *Let* $\alpha, \beta \in \mathbf{R}^n_{\geq}$. *Then* $K(\alpha, \beta)$ *is a polyhedron in* $\mathbb{R}^n_{\geq}$ *which is given by the trace equality* (1.2) *and the inequalites*

$$\gamma[K] \leq \alpha[I] + \beta[J], \quad \text{for all } (I, J, K) \in S^n_r,\ r \in < 1, n-1 > . \tag{8.5}$$

Proof. Since $K(\alpha, \beta)$ is a continuous in the parameters α, β it is enough to prove the theorem for $\alpha, \beta \in \mathbb{Q}^n$ such that the coordinates of α, β are pairwise distinct. Fix such a pair α, β. As $K(\alpha, \beta)$ is a closed set, it is enough to show that if $\gamma \in \mathbb{Q}^n$, all the coordinates of γ are pairwise distinct, (1.2) holds, and

$$\gamma[K] < \alpha[I] + \beta[J], \quad \text{for all } (I, J, K) \in S^n_r,\ r \in < 1, n-1 >, \tag{8.6}$$

then $\gamma \in K(\alpha, \beta)$. Since for any $t > 0$ $K(t\alpha, t\beta) = tK(\alpha, \beta)$ it is enough to show that $t\gamma \in K(t\alpha, t\beta)$. Hence we can choose t to be a big positive integer so that

$$\hat{\alpha} = (\hat{\alpha}_1, ..., \hat{\alpha}_n) := t\alpha, \ \hat{\beta} = (\hat{\beta}_1, ..., \hat{\beta}_n) := t\beta, \ \hat{\gamma} = (\hat{\gamma}_1, ..., \hat{\gamma}_n) := t\gamma \in \mathbb{Z}^n.$$

Choose N a big enough positive integer so that the coordinates of $\alpha(i), \ i = 1, 2, 3$ are positive distinct integers:

$$\begin{aligned} &\alpha(i) := (\alpha_1(i), ..., \alpha_n(i)), \quad i = 1, 2, 3, \\ &\alpha_j(1) = N - \hat{\alpha}_{n-j+1}, \ \alpha_j(2) = N - \hat{\beta}_{n-j+1}, \ \alpha_j(3) = N + \hat{\gamma}_j, \ j = 1, ..., n. \end{aligned}$$

Then $\hat{\gamma} \in K(\hat{\alpha}, \hat{\beta})$ iff there exists $A_1, A_2, A_3 \in \mathcal{H}_n$ satisfying (7.6) with $r = 3N$ such that $\lambda(A_i) = \alpha(i), \ i = 1, 2, 3$. The definition of $\alpha(i), \ i = 1, 2, 3$ and the assumption (8.6) yields

$$\begin{aligned} &\sum_{i=1} \sum_{j=1}^{n} \alpha_j(i) = 3Nn, \\ &\alpha(1)[I'] + \alpha(2)[J'] + \alpha(3)[K] < 3Nm, \quad (I, J, K) \in S_m^n, \ m \in < 1, n-1 > . \end{aligned} \tag{8.7}$$

Let $F_*(i), \ i = 1, 2, 3$ be three complete flags in general position. Let $U_*(i)$ be the filtration defined by $\alpha(i)$ and $F_*(i)$ for i=1,2,3. We claim that the 3 filtration $U_*(i), \ i = 1, 2, 3$ is stable. Let $L \in \mathrm{Gr}(m, \mathbb{C}^n), \ m \in < 1, n-1 >$. Let

$$I_1' = I(L, F_*(1)), \quad J_1' = I(L, F_*(2)), \quad K_1 = I(L, F_*(3).$$

Then

$$L \in \Omega_{I_1'}(F_*(1)) \cap \Omega_{J_1'}(F_*(2)) \cap \Omega_{K_1}(F_*(3)).$$

Since the three flags $F_*(i), \ i = 1, 2, 3$ are in general position the Schubert calculus implies the existence of $(I, J, K) \in S_m^n$ such that $I_1' \geq I', \ J_1' \geq J', \ K_1 \geq K$. (8.7) yields

$$\frac{1}{m}(\alpha(1)[I_1'] + \alpha(2)[J_1'] + \alpha(3)[K_1]) \leq \frac{1}{m}(\alpha(1)[I'] + \alpha(2)[J'] + \alpha(3)[K]) < 3N \tag{8.8}$$

Lemma 8.3 yields that the left hand side the right hand side of (8.8) is $\mu(L)$ and $\mu(\mathbb{C}^n)$ respectively. Hence 3 filtration $U_*(1), U_*(2), U_*(3)$. is stable. Theorem 8.5 yields the existence of a hermitian inner product $< \cdot, \cdot >$ on $\mathbb{C}^n$ such that (8.4) holds. Let

$$B_i := \sum_{j=1}^{\infty} P(U_i(j)), \quad i = 1, 2, 3.$$

Pick an orthonormal basis $e_1, ..., e_n$ in $\mathbb{C}^n$ with respect to the $< \cdot, \cdot >$. Let $A_i \in \mathcal{H}_n$ represent B_i for $i = 1, 2, 3$. Then $A_1 + A_2 + A_3 = 3NE_n$. Lemma 8.2 implies that $\alpha(i) = \lambda(A_i), \ i = 1, 2, 3$. $\square$

9 Majorizing sums

For $\alpha, \beta \in \mathbb{R}^n_{\geq}$ let

$$a_K(\alpha,\beta) := \min_{I,J,\ (I,J,K)\in S^n_{|K|}} \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j, \quad K \subset <n>,\ 1 \leq |K| < n,$$
$$a_{<n>}(\alpha,\beta) := \sum_{i=1}^{n} \alpha_i + \beta_i. \tag{9.1}$$

Then $K(\alpha,\beta)$ is characterized by the following set of inequalities:

$$-x_i + x_{i+1} \leq 0, \quad i = 1, ..., n-1,$$
$$x[K] \leq a_K, \quad K \subset <n>, \tag{9.2}$$

$$-x[<n>] \leq -a_{<n>}, \tag{9.3}$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and

$$a_K = a_K(\alpha,\beta), \quad K \subset <n>. \tag{9.4}$$

Proposition 9.1 *Let $\alpha, \beta \in \mathbb{R}^n_{\geq}$. Then $y \in K_{\leq}(\alpha,\beta)$ if and only if the system* (**??**), (**??**), (**??**) *and*

$$-x_i \leq -y_i, \quad i = 1, ..., n, \tag{9.5}$$

is solvable.

Proof. Let

$$\text{diag}\ (x) := \text{diag}\ (x_1, ..., x_n), \quad x = (x_1, ..., x_n) \in \mathbb{F}^n. \tag{9.6}$$

Assume first that the system of equations (**??**), (**??**), (**??**) and (**??**) is solvable. Then

$$\text{diag}\ (y) \leq \text{diag}(x) = A + B, \quad \text{for some } A, B \in \mathcal{H}_n,\ \lambda(A) = \alpha,\ \lambda(B) = \beta.$$

Hence $y \in K_{\leq}(\alpha,\beta)$. Vise versa, if $y \in K_{\leq}(\alpha,\beta)$ then $y = \lambda(F)$, and $F \leq C = A + B$, $\lambda(A) = \alpha$, $\lambda(B) = \beta$. Then $x = \lambda(C)$ satisfies (**??**) and (**??**), where (**??**) holds. As $F \leq C$ and (**??**) holds. $\square$

Definition 9.2 *Let $a := (a_I)_{\emptyset \neq I \subset <n>}$ be a given real vector with $2^n - 1$ coordinates. Let*

$$K(a) := \{x \in \mathbb{R}^n_{\geq} : \quad x[I] \leq a_I,\ I \subset <n>,\ |I| < n, \text{ and } x[<n>] = a_{<n>}\},$$
$$\hat{K}(a) := \{x \in \mathbb{R}^n_{\geq} : \quad x[I] \leq a_I,\ I \subset <n>\},$$
$$K'(a) := \{y \in \mathbb{R}^n_{\geq} : \quad \exists x \in K(a),\ y \leq x\}. \tag{9.7}$$

Clearly

$$K(a) \subset K'(a) \subset \hat{K}(a). \tag{9.8}$$

Lemma 9.3 *Let $n > 1$ and assume that $a = (a_I)_{I\subset<n>}$ is a given vector. Suppose that $K(a)$ is a nonempty set. Then $K'(a)$ is a polyhedral set in $\mathbb{R}^n$ given by (**??**) and the inequalities*

$$\sum_{i=1}^{n} w_i^l(n)x_i \le -a_{<n>} + \sum_{I\subset<n>,\ 0<|I|<n} u_I^l(n)a_I, \quad l = 1, ..., M(n), \tag{9.9}$$

for some fixed vectors $(w_i^l(n))_{i=1}^n,\ (u_I^l(n))_{I\subset<n>,\ 0<|I|<n},\ l = 1, ..., M(n)$ independent of a.

Proof. The system (**??**) can be stated as $Ux \le b$, where $b^T = (0^T, a^T),\ 0 \in \mathbb{R}^{n-1}$. The system of equations (**??**), (**??**) and (**??**) can be written in matrix form as

$$\begin{aligned} &Vx \le c, \\ &V^T = (U^T, -e, -E_n), \quad c^T = (b^T, -a_{<n>}, -y^T),\ e := (1, ..., 1)^T \in \mathbb{R}^n. \end{aligned} \tag{9.10}$$

Proof. A variant of Farkas lemma [?] (§7.3) yields that the solvability of (**??**) is equivlalent to the implication

$$z \ge 0,\ z^T V = 0 \Rightarrow z^T c \ge 0. \tag{9.11}$$

Here $z^T = (t^T, u^T, v, w^T)$ is a row vector partitioned as c^T:

$$t = (t_1, ..., t_{n-1})^T,\ u = (u_I)_{I\subset<n>},\ v \in \mathbb{R},\ w = (w_1, ..., w_n)^T \in \mathbb{R}^n.$$

It is straightforward to show that any solution z of $z^T V = 0$ is equivalent to the validity of the following identity in n variables in $x \in \mathbb{R}^n$:

$$\sum_{I\subset<n>} u_I x[I] = \sum_{i=1}^{n-1} t_i(x_i - x_{i+1}) + \sum_{i=1}^{n} (w_i + v)x_i. \tag{9.12}$$

The Farkas-Minkowski-Weyl theorem [?] (§7.2) yields that the cone $zV = 0, z \ge 0$ is finitely generated. First we divide the extremal vectors $z = (t, u, v, w)$ to two sets: $v = 0$ and $v \ne 0$. The subset with $v = 0$ corresponds to the set

$$z^{l,1}(n) := (t^{l,1}(n), u^{l,1}(n), 0, w^{l,1}(n)), \quad l = 1, ..., M_1(n).$$

We normalize the second set of extremal vectors by letting $v = 1$. We divide the second set to the subsets determined by $w = 0$:

$$z^{l,2}(n) := (t^{l,2}(n), u^{l,2}(n), 1, 0), \quad l = 1, ..., M_2(n),$$

and $w \ne 0$:

$$z^{l,3}(n) := (t^l(n), u^l(n), 1, w^l(n)), \quad u^l_{<n>}(n) = 0, \quad w^l(n) \ne 0, \quad l = 1, ..., M(n).$$

Note that the set $z^{l,2}(n), l = 1, ..., M_2(n)$ contains an extremal vector $\zeta = (0, u, 1, 0)$, where $u_{<n>} = 1$ and all other coordinates of u are equal to zero. Hence the extremal vector $z^{l,3}(n)$ satisfies the condition $u^l_{<n>} = 0$ for $l = 1, ..., M(n)$.

We claim that the number of nonzero coordinates in any extremal vector z is at most $n + 1$. Let z be an extremal ray of the cone $zV = 0, z \geq 0$. Assume that z has exactly p nonvanishing coordinates. Let $\hat{V}$ be a $p \times n$ submatrix of V corresponding to the nonzero elements of z. Let $wV = 0$ and assume that $w_i = 0$ if $z_i = 0$. Then the nonzero coordinates of w satisfy n equations. As z is an extremal ray it follows that $w = \alpha z$ for some $\alpha \in \mathbf{R}$. Hence the n columns of $\hat{V}$ span $p - 1$ dimensional subspace, i.e. rank $\hat{V} = p - 1 \leq n$.

We claim that the set $z^{l,3}(2)$ is empty. Consider an extremal vector $z^{l,3}(n)$. By the definition $v^l(n) = 1$, $w^l(n) \neq 0$ and $u^l_{<n>}(n) = 0$. Use (**??**) to deduce that $u^l(n) \neq 0$. Assume now that $n = 2$. Since $z^{l,3}(2)$ has at most 3 nonzero coordinates, we deduce that each vector $u^l(2), v^l(2) = 1, w^l$ has exactly one nonzero coordinate and $t^l(2) = 0$. As $u^l_{<2>}(2) = 0$ (**??**) can not hold.

The system $zV = 0$ is equivalent to $(t^T, u^T)U = ve^T + w$, where U is the matrix representing the system (**??**). Hence

$$\begin{aligned}
&z^T c = (t^T, u^T)b - va_{<n>} - w^T y = \\
&(t^T, u^T)b - va_{<n>} - ((t^T, u^T)U - ve^T)y = (t^T, u^T)(b - Uy) + v(e^T y - a_{<n>}), \\
&zc^T = ua^T - va_{<n>} - wy^T.
\end{aligned} \tag{9.13}$$

The inequality (**??**) and the definition of $z^{l,1}(n)$ yield that $z^{l,1}(n)^T c \geq 0$. The inequality (**??**), (**??**) and the definition of $z^{l,2}(n)$ yield that $z^{l,2}(n)^T c \geq 0$ if $y \in K(a)$. The last part of (**??**) yields the validity of $z^{l,2}(n)^T c \geq 0$ in general. Hence $y \in K'(a)$ iff $z^{l,3}(n)^T c \geq 0,\ l = 1, ..., M(n)$, which are equivalent to (**??**). □

As the set of vectors of the form $z^{l,3}(2)$ is empty we deduce:

Corollary 9.4 *For $n = 2$ $K'(a) = \hat{K}(a)$.*

In [Fr2] we showed that for $n = 3$ $K'(a(\alpha, \beta)) = \hat{K}(a(\alpha, \beta))$. That is, for $n = 2, 3$ any $y \in \hat{K}(a(\alpha, \beta))$ satisfies (**??**). In [Fr2] we posed the problem if this statement holds for any $n > 3$. This problem was answered positively by Fulton in [Fu2].

10 Characterization of $K_{\leq}(\alpha, \beta)$

Theorem 10.1 ([Fu2]) *Let $\alpha, \beta \in \mathbb{R}^n_{\geq}$. Then the set $K_{\leq}(\alpha, \beta)$ is given by the inequalities* (**??**)*, where $a = a(\alpha, \beta)$ is given by* (**??**)*. That is, $\gamma \in K_{\leq}(\alpha, \beta)$ iff γ satisfies Horn's inequalities* (4.6) *and the trace inequality*

$$\sum_{i=1}^{n} \gamma_i \leq \sum_{i=1}^{n} \alpha_i + \beta_i. \tag{10.1}$$

To prove the above theorem we need a few lemmas [Fu2].

Lemma 10.2 *Let F_* be a complete flag in $V = \mathbb{C}^n$. Let $U \in \mathrm{Gr}(r, \mathbb{C}^n)$. Set $I = I(U, F_*)$ and let I^c be the complement of I in $< n >$. Then F_* induces the flags $\tilde{F}_*$, $\hat{F}_*$ in U and V/U repectively:*

$$\begin{aligned} &\tilde{F}_* : \ \tilde{F}_j = F_{i_j} \cap U, \ j = 1, ..., r, \quad I = \{1 \le i_1 < i_2 < \cdots < i_r \le n\}, \\ &\hat{F}_* : \ \hat{F}_j = (F_{i_j^c} + U)/U, \ j = 1, ..., n-r, \quad I^c = \{1 \le i_1^c < i_2^c < \cdots < i_{n-r}^c \le n\}. \end{aligned} \tag{10.2}$$

Proof. Clearly $F_i \cap U$, $(F_i + U)/U$, $i = 1, ..., n$ induce filtrations in U and V/U respectively. The definition of $I(U, F_*)$ (5.6) yields that dim $\tilde{F}_j = j$, $j = 1, ..., r$. Furthermore, as

$$\dim\,(F_i + U)/U = \dim\, F_i/F_i \cap U = i - \dim\, F_i \cap U, \quad i = 1, ..., n,$$

we easily deduce that dim $\hat{F}_j = j$ for $j \in I^c$. □

The proof of the following lemma is straightforward and is left to the reader:

Lemma 10.3 *Let Z, U, T be three subspaces in $V = \mathbb{C}^n$. Assume that $Z \supset U$. Then*

$$\dim\,(Z \cap T) = \dim U \cap T + \dim Z/U \cap (T + U/U).$$

Let I and I^c be two complementary sets in $< n >$ of cardinality r and $n - r$ respectively given as in (**??**). Let $P = \{1 \le p_1 < p_2 < \cdots < p_l\}$. Let

$$\begin{aligned} &I_P := \{i_{p_1}, ..., i_{p_l}\}, \quad for\ P \subset < r >, \\ &I_P^+ := I \cup \{i_{p_1}^c, ..., i_{p_l}^c\}, \quad for\ P \subset < n - r > . \end{aligned}$$

Lemma 10.4 *Let F_* be a complete flag in $V = \mathbb{C}^n$. Let $I \subset < n >$, $|I| = r$ and assume that $U \in \Omega_I(F_*) \subset \mathrm{Gr}(r, V)$. Let $\tilde{F}_*$ and $\hat{F}_*$ be the induced flags in U and V/U respectively.*
(i) *If X is a subspace of U of dimension x, with $X \in \Omega_P(\tilde{F}_*)$ for some $P \subset < r >$, $|P| = x$ then $X \in \Omega_{I_P}(F_*)$.*
(ii) *If $Y = Z/U$ is a subspace of V/U of dimension y, with $Y \in \Omega_P(\hat{F}_*)$ for some $P \subset < n - r >$, $|P| = y$ then $Z \in \Omega_{I_P^+}(F_*)$.*

Proof. Let X satisfy the assumptions of (i). Then

$$s \le \dim X \cap \tilde{F}_{p_s} = \dim X \cap (U \cap F_{i_{p_s}}) = \dim X \cap F_{i_{p_s}}, \quad s = 1, ..., x.$$

Hence $X \in \Omega_{I_P}(F_*)$.

Assume that Y satisfies the assumptions of (ii). Observe that the function dim $Z \cap F_i$ on the interval $[0, n] \cap \mathbb{Z}_+$ strictly increases (by 1) exactly at the integers in the set $I(Z, F_*)$. Lemma **??** yields that

$$\dim Z \cap F_i = \dim U \cap F_i + \dim Y \cap (F_i + U)/U, \quad i = 1, ..., n.$$

As dim $Z \cap F_i$ can jump only by one, we deduce that the jumps of dim $Z \cap F_i$ are at the jumps of dim $U \cap F_i$ and at the the jumps of dim $Y \cap (F_i + U)/U$, which are at $I(U, F_*)$ and $(I(U, F^*)^c)_{I(Y, \hat{F}_*)}$. Hence

$$I(Z, F_*) = I(U, F_*) \cup (I(U, F^*)^c)_{I(Y, \hat{F}_*)} \le I_P^+ \Rightarrow Z \in \Omega_{I(Z, F_*)}(F_*) \subset \Omega_{I_P^+}(F_*).$$

□

Proof of Theorem ??. We prove the theorem by induction on n. Let

$$\tilde{\alpha} = (\tilde{\alpha}_1, ..., \tilde{\alpha}_n) = (-\alpha_n, ..., -\alpha_1), \quad \tilde{\beta} = (\tilde{\beta}_1, ..., \tilde{\beta}_n) = (-\beta_n, ..., -\beta_1).$$

As in §7, it is equivalent to show the existence of $A_1, A_2, A_3 \in \mathcal{H}_n$ with eigenvalue vectors $\tilde{\alpha}, \tilde{\beta}, \gamma$ such that $A_1 + A_2 + A_3 \leq 0$. For $n = 1$ the theorem clearly holds. Assume that the theorem holds for any $n < N$. Let $n = N$. Assume that γ satisfies all the inequalities (4.6) and the trace inequality holds (**??**). Suppose that at least one inequality is an equality. Assume first the trace equality (1.2) holds. Then $\gamma \in K(\alpha, \beta) \Rightarrow \gamma \in K_{\leq}(\alpha, \beta)$. Assume now that we have an equality

$$\tilde{\alpha}[I] + \tilde{\beta}[J] + \gamma[K] = 0 \quad \text{for some } (I', J', K) \in S_r^n,\ r \in < 1, n-1 > . \tag{10.3}$$

(We assumed here that $S_r^n = T_r^n,\ r \in < 1, n-1 >$.) Let

$$\alpha' := (\tilde{\alpha}_i)_{i \in I},\ \beta' := (\tilde{\beta}_j)_{j \in J},\ \gamma' := (\gamma_k)_{k \in K} \in \mathbb{R}^r_{\geq},$$
$$\alpha'' := (\tilde{\alpha}_i)_{i \in I^c},\ \beta'' := (\tilde{\beta}_j)_{j \in J^c},\ \gamma'' := (\gamma_k)_{k \in K^c} \in \mathbb{R}^{n-r}_{\geq}.$$

We claim:
(a) there exist $B_1, B_2, B_3 \in \mathcal{H}_r$ with the eigenvalue vector α', β', γ' respectively such that $B_1 + B_2 + B_3 = 0$;
(b) there exist $C_1, C_2, C_3 \in \mathcal{H}_{n-r}$ with the eigenvalue vector $\alpha'', \beta'', \gamma''$ respectively such that $C_1 + C_2 + C_3 \leq 0$.

Assume that (a) and (b) holds. Then $A_i := B_i \oplus C_i, i = 1, 2, 3$ yield the theorem in this case. Theorem 8.6 yields that (a) is equivalent to the inequalities

$$\alpha'[P] + \beta'[Q] + \gamma'[R] \leq 0, \quad \text{for all } (P', Q', R) \in S_l^r,\ l \in < 1, r-1 > . \tag{10.4}$$

Note that

$$\alpha'[P] = \tilde{\alpha}[I_P],\ \beta'[Q] = \tilde{\beta}[J_Q],\ \gamma'[R] = \gamma[K_R].$$

We claim that for any three flags $F_*(1), F_*(2), F_*(3)$

$$\Omega_{I_P}(F_*(1)) \cap \Omega_{J_Q}(F_*(2)) \cap \Omega_{K_R}(F_*(3)) \neq \emptyset. \tag{10.5}$$

As $(I', J', K) \in S_r^n$ pick $U \in \Omega_I(F_*(1)) \cap \Omega_J(F_*(2)) \cap \Omega_K(F_*(3)) \subset \mathrm{Gr}(r, \mathbb{C}^n)$. Let $\tilde{F}_*(i)$ be the induced complete flag in U for $i = 1, 2, 3$. Let $(P', Q', R) \in S_l^r$. Pick $X \in \Omega_P(\tilde{F}_*(1)) \cap \Omega_Q(\tilde{F}_*(2)) \cap \Omega_R(\tilde{F}_*(3)) \subset \mathrm{Gr}(l, U)$. Part (i) of Lemma **??** yields that X is in the intersection of the three sets given in (**??**). Combine (**??**) with Lemma **??**, the left hand side of (7.7) and (4.6) to deduce (**??**).

To prove (b) we the induction hypothesis that it is ehough to show

$$\alpha''[P] + \beta''[Q] + \gamma''[R] \leq 0,$$
$$\text{for all } (P', Q', R) \in S_l^{n-r},\ l \in < 1, n-r-1 >, \quad \text{and } P = Q = R =< n-r > . \tag{10.6}$$

In view of (**??**) each of the above inequalities is equivalent to

$$\tilde{\alpha}[I_P^+] + \tilde{\beta}[J_Q^+] + \gamma[K_R^+] \leq 0.$$

This inequality follows from part (ii) of Lemma **??** and the arguments as above.

Assume finally that γ satisfies all strict inequalities (4.6) and the strict trace inequality holds (**??**). Then there exists $\overline{\gamma} \in \mathbb{R}^n_{\geq}$ such that $\gamma \leq \overline{\gamma}$, $\overline{\gamma}$ satisfies all the inequalities (4.6) and (**??**) where at least one inequality is an equality. We showed that $\overline{\gamma} \in K_{\leq}(\alpha, \beta)$. Trivially $\gamma \in K_{\leq}(\alpha, \beta)$. □

11 Selfadjoint operators in a separable Hilbert space

Let $\mathbf{H}$ be a separable infinite dimensional Hilbert space with an inner product $(u, v) \in \mathbf{C}$ for $u, v \in \mathbf{H}$. ($\mathbf{H}$ has a countable orthonormal basis $\{e_i\}_1^\infty$.) Denote by $\mathcal{H}$ the set of all linear, bounded, selfadjoint operators $A : \mathbf{H} \to \mathbf{H}$. That is $(Ax, y) = (x, Ay)$ for all $x, y \in \mathbf{H}$ and $||A|| := \sup_{0 \neq x \in \mathbf{H}} \frac{|(Ax,x)|}{(x,x)} < \infty$. Recall the well known spectral properties of $A \in \mathcal{H}$ [**?**] or [**?**]. Denote by spec(A) the spectrum of A, i.e. all $z \in \mathbb{C}$ such that $(zI - A)^{-1}$ does not exist. Then spec(A) is a compact set located in the closed interval $[-||A||, ||A||]$. Recall the spectral decomposition of A:

$$A = \int_{[-||A||-1, ||A||+1]} x dE(x).$$

Here $E(x), x \in \mathbf{R}, 0 \leq E(x) \leq I$ is the resolution of the identity of commuting increasing family of orthogonal projections induced by A, which is continuous from the right. Furthermore $E(-||A|| - 0) = 0$ and $E(||A|| + 0) = I$. Note that

$$I = \int_{[-||A||-1, ||A||+1]} dE(x)$$

For an open or a closed (Borel) set $T \subset \mathbf{R}$ denote by $P(A, T)$ the spectral projection of A on T:

$$P(A, T) := \int_T dE(x).$$

We let dim $P(A, T)$ be the dimension of the subspace $P(A, T)\mathbf{H}$. Note that $0 \leq$ dim $P(A, T) \leq \infty$. Observe that dim $P(A, (a, b))$ is finite and positive iff spec$(A) \cap (a, b)$ consists of a finite number of eigenvalues of A, each one with a finite dimensional eigenspace. We say that $\mu(A)$ is the first accumulation point of the spectrum of A if

$$\text{dim } P(A, (\mu(A) + \epsilon, \infty)) < \infty, \quad \text{dim } P((\mu(A) - \epsilon, \infty)) = \infty$$

for every positive ϵ. $\mu(A)$ must be either a point of the continuous spectrum or a point spectrum with an infinite corresponding eigenspace. (It is a maximal point in spec(A) with this property.) Denote by $\mathcal{CH} \subset \mathcal{H}$ the set of all selfadjoint compact operators in

H. Then $A \in \mathcal{CH}$ iff **H** has an orthonormal basis consisting of the eigenvectors of A and $\mu(A) = \mu(-A) = 0$. Denote by $\mathcal{H}_+$, $\mathcal{H}_+^o$ the cone of nonnegative and positive selfadjoint operators in **H** respectively. That is $A \in \mathcal{H}_+$ $(\mathcal{H}_+^o)$ if $(Ax, x) \geq 0$ (> 0) for any $x \neq 0$. Let $\mathcal{CH}_+$, $\mathcal{CH}_+^o$ the cone of compact nonnegative and compact positive selfadjoint operators in **H** respectively. For $A, B \in \mathcal{H}$ let $A \leq B$ $(A < B)$ iff $B - A \in \mathcal{H}_+$ $(B - A \in \mathcal{H}_+^o)$, i.e. $B - A$ is nonnegative (respectively positive). Then $A \in \mathcal{CH}_+^o$ iff **H** has an orthonormal basis $\{e_i\}_1^\infty$ such that

$$
\begin{aligned}
&Ae_i = \lambda_i e_i, \quad \lambda_i > 0, \quad i = 1, ..., \\
&\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \cdots, \\
&\lim_{n \to \infty} \lambda_i = 0.
\end{aligned}
\tag{11.1}
$$

We say that $\{\lambda_i\}_{i=1}^\infty$ is the eigenvalue sequence of A. If $A \in \mathcal{CH}_+$ then (**??**) holds with the following modifications. First $\lambda_i \geq 0$. Second $\{e_i\}_1^\infty$ is an orthonormal sequence which is a basis for a closed subspace $\mathbf{H}_1$. Third $A\mathbf{H}_1^\perp = 0$, where $\mathbf{H}_1^\perp$ is the orthogonal complement of $\mathbf{H}_1$. In this case $\{\lambda_i\}_{i=1}^\infty$ is called the eigenvalue sequence of A. Note that if $\{\lambda_i\}_{i=1}^\infty$ has only finite number of positive numbers then we can (and will) assume that $\mathbf{H}_1 = \mathbf{H}$. If $\{\lambda_i\}_{i=1}^\infty$ is a sequence of positive numbers then $A \in \mathcal{CH}_+^o$ iff $\{e_i\}_1^\infty$ is an orthonormal basis of **H**. $A \in \mathcal{CH}_+$ is said to be in the trace class if $\sum_{i=1}^\infty \lambda_i < \infty$. Then trace $A := \sum_{i=1}^\infty \lambda_i$.

Let $V \subset \mathbf{H}$ be an n-dimensional subspace. Pick an orthonormal basis $f_1, ..., f_n \in V$. For $A \in \mathcal{H}$ denote by $A(f_1, ..., f_n) = A|V \in \mathcal{H}_n$ the $n \times n$ matrix whose (i, j) entry is (Af_i, f_j). Let

$$\lambda_1(A, V) \geq \lambda_2(A, V) \geq \cdots \geq \lambda_n(A, V)$$

be the n eigenvalues of the Hermitian matrix $A|V$. As in the finite dimensional case the above eigenvalues do not depend on a particular choice of an orthonormal basis $f_1, ..., f_n$ of V. Clearly $|\lambda_i(A, V)| \leq ||A||$, $i = 1, ..., n$.

We now recall the convoy principle [Fr1].

Lemma 11.1 *Let $A \in \mathcal{CH}_+$ have the eigenvalue sequence $\{\lambda_i\}_{i=1}^\infty$. Let $n \geq k \geq 1$ be any integers. Assume that $V \subset \mathcal{H}$ is any n-dimensional subspace. Then $\lambda_k(A, V) \leq \lambda_k$ and this inequality is sharp.*

Proof. For simplicity of exposition assume in addition that $A > 0$. Choose an orthonormal basis $f_1, ..., f_n$ of V so that $A|V$ is the diagonal matrix diag $(\lambda_1(A, V), ..., \lambda_n(A, V))$. Let $f = \sum_{i=1}^k \alpha_i f_i \neq 0$ be such that $(f, e_i) = 0, i = 1, ..., k-1$, where $\{e_i\}_1^\infty$ is an orthonormal basis of $\mathcal{H}$ given in (**??**). Deduce from (**??**) and from the choice of $f_1, ..., f_n$ that

$$\lambda_k(A, V) \leq \frac{(Af, f)}{(f, f)} \leq \lambda_k.$$

For V =span$(e_1, ..., e_n)$ we obtain that $\lambda_k(A, V) = \lambda_k$. $\Box$

For $A \in \mathcal{H}$ let

$$\lambda_k(A, \mathbf{H}) := \sup_{V \subset \mathbf{H}, \dim V = k} \lambda_k(A, V), \quad k = 1, ..., .$$

For $A \in \mathcal{CH}_+$ Lemma **??** yields that

$$\lambda_k(A, \mathbf{H}) = \lambda_k, \quad k = 1, ...,.$$

Lemma 11.2 *Let $A \in \mathcal{H}$. Then the sequence $\{\lambda_i(A, \mathbf{H})\}_1^\infty$ is a nonincreasing sequence which lies in $[-||A||, ||A||]$. Let $\{f_i\}_1^\infty$ be any orthonormal basis in $\mathbf{H}$. Set V_n =span$(f_1, ..., f_n)$ for $n = 1, ...,$. Then the sequence $\{\lambda_k(A, V_n)\}_{n=k}^\infty$ is an increasing sequence which converges to $\lambda_k(A, \mathbf{H})$ for each $k = 1, 2, ...,$.*

Proof. . Fix a complete flag

$$W_1 \subset W_2 \subset \cdots \subset W_i \cdots, \quad \dim W_i = i, \quad i = 1, ...,$$

of subspaces in $\mathcal{H}$. Then the convoy principle for matrices yields that

$$\lambda_i(A, W_{i+1}) \geq \lambda_i(A, W_i) \geq \lambda_{i+1}(A, W_{i+1}), \quad i = 1, 2, ...,.$$

(These inequalities are natural extensions of the Cauchy interlacing inequalites for matrices.) Hence the sequence $\{\lambda_i(A, \mathbf{H})\}_1^\infty$ is a nonincreasing sequence which lies in $[-||A||, ||A||]$. Furthermore we obtain that $\{\lambda_k(A, V_n)\}_{n=k}^\infty$ is a nondecreasing sequence. From the definition of $\lambda_k(A, \mathbf{H})$ we immediately deduce that

$$\lambda_k(A, V_n) \leq \lambda_k(A, \mathbf{H}), \quad n = k, k+1, ...,.$$

Let

$$\tilde{\lambda}_k := \lim_{n\to\infty} \lambda_k(A, V_n), \quad k = 1, ...,.$$

Hence $\tilde{\lambda}_k \leq \lambda_k(A, \mathbf{H}), k = 1, ...,$. We claim that for any k-dimensional subspace $W \subset \mathcal{H}$

$$\tilde{\lambda}_k \geq \lambda_k(A, W).$$

Assume that $g_1, ..., g_k$ is an orthonormal basis in W so that the matrix $((Ag_i, g_j))_1^k$ is the diagonal matrix diag$(\lambda_1(A, W), ..., \lambda_k(A, W))$. Let $P_n : \mathcal{H} \to V_n$ be the orthogonal projection on V_n. That is

$$P_n x = \sum_{i=1}^n (x, f_i) f_i.$$

Then $\lim_{n\to\infty} P_n x = x$ for every $x \in \mathcal{H}$, i.e. P_n converges to I in the strong topology. Hence, for $n > N$, say, $P_n g_1, ..., P_n g_k$ are linearly independent. Let $g_{1,n}, ...g_{k,n} \in V_n$ be the k orthonormal vectors obtained from $P_n g_1, ..., P_n g_n$ using the Gram-Schmidt process. Then

$$\lim_{n\to\infty} g_{i,n} = g_i, \quad i = 1, ..., k.$$

Hence the matrix $((Ag_{i,n}, g_{j,n}))_{i,j=1}^k$ converges to diag $(\lambda_1(A, W), ..., \lambda_k(A, W))$. Let W_n = span $(g_{1,n}, ..., g_{k,n})$. Then

$$\lim_{n\to\infty} \lambda_k(A, W_n) = \lambda_k(A, W).$$

As $W_n \subset V_n$ the convoy principle implies

$$\lambda_k(A, W_n) \le \lambda_k(A, V_n) \le \tilde{\lambda}_k.$$

Hence

$$\lambda_k(A, W) \le \tilde{\lambda}_k \quad \Rightarrow \quad \lambda_k(A, \mathbf{H}) \le \tilde{\lambda}_k \Rightarrow \quad \lambda_k(A, \mathbf{H}) = \tilde{\lambda}_k.$$

□

Lemma 11.3 *Let $A \in \mathcal{H}$. Then the nonincreasing sequence $\{\lambda_i(A, \mathbf{H})\}_1^\infty$ converges to $\mu(A)$.*

Proof. Suppose first that dim $P(A, (a, b)) > 0$. Let

$$A(a, b) := \int_{(a,b)} x dE(x).$$

Then

$$a \le \frac{(Ax, x)}{(x, x)} \le b, \quad 0 \ne x \in P(A, (a, b))\mathbf{H}.$$

Let $\epsilon > 0$. Let $f_1, ..., f_{k-1}$ be an orthonormal basis of $V = P(A, (\mu(A) + \epsilon, \infty))\mathbf{H}$. (If $k = 1$ then $V = 0$.) Hence $V^\perp = P(A, (-\infty, \mu(A) + \epsilon])\mathbf{H}$. Let $W \subset \mathbf{H}$ be any subspace of dimension k. Then $V^\perp \cap W$ contains a nonzero vector $x \in P(A, (-\infty, \mu(A) + \epsilon])\mathbf{H}$. The convoy principle and the above observation yield that

$$\lambda_k(A, W) \le \mu(A) + \epsilon.$$

Hence

$$\lambda_k(A, \mathbf{H}) \le \mu(A) + \epsilon.$$

Recall that $U := P(A, (\mu(A) - \epsilon, \infty))\mathbf{H}$ is infinite dimensional. Let $W \subset U$, dim $W = l$. Then the convoy principle and the above observation yield that

$$\lambda_l(A, W) \ge \mu(A) - \epsilon.$$

Hence $\lambda_l(A, \mathbf{H}) \ge \mu(A) - \epsilon$. This inequality holds for any l. Hence

$$\lim_{l \to \infty} \lambda_l(A, \mathbf{H}) \ge \mu(A) - \epsilon.$$

Since ϵ was an arbitrary positive number we deduce the lemma. □

Corollary 11.4 *Let $A \in \mathcal{H}_+$. The following are equivalent:*
(a) *$A \in \mathcal{CH}_+$ and A is in the trace class.*
(b) *For a given orthonormal basis $\{f_i\}_1^\infty$ of $\mathbf{H}$ the nonnegative series $\sum_{i=1}^\infty (Af_i, f_i)$ converges.*

Furthermore under the assumption (a)

$$\text{trace } A = \sum_{i=1}^\infty (Af_i, f_i) \tag{11.2}$$

for any orthonormal basis $\{f_i\}_1^\infty$.

Proof. (a) $\Rightarrow$ (b): Let $V_n = \text{span}\,(f_1, ..., f_n)$. Then

$$\text{trace}\ A \geq \sum_{i=1}^{n} \lambda_i(A) \geq \text{trace}\ A|V_n = \sum_{i=1}^{n} (Af_i, f_i). \tag{11.3}$$

(b) $\Rightarrow$ (a): Fix $k \geq 1$ and let $n \geq k$. Then

$$\sum_{i=1}^{\infty} (Af_i, f_i) \geq \sum_{i=1}^{n} (Af_i, f_i) = \text{trace}\ (A|V_n) \geq \sum_{i=1}^{k} \lambda_i(A, V_n).$$

Let $n \to \infty$. Use Lemma **??** to deduce that $\sum_{i=1}^{\infty} (Af_i, f_i) \geq \sum_{i=1}^{k} \lambda_i(A, \mathbf{H})$. Clearly each $\lambda_k(A, \mathbf{H}) \geq 0$. Hence

$$\sum_{i=1}^{\infty} (Af_i, f_i) \geq \sum_{i=1}^{\infty} \lambda_i(A, \mathbf{H}). \tag{11.4}$$

Thus $\lim_{k\to\infty} \lambda_k(A, \mathbf{H}) = 0$. Hence $A \in \mathcal{CH}_+$ and $\lambda_k(A, \mathbf{H}) = \lambda_k(A),\ k = 1, ..., n$. Use (**??**) and (**??**) to deduce (**??**). $\square$

12 Characterization of $K_{\leq}(\alpha, \beta)$ for operators in $\mathcal{CH}_+$

Let

$$\Gamma := \{\gamma = \{\gamma_i\}_1^{\infty} : \quad \gamma_i \in \mathbb{R},\ \gamma_i \geq \gamma_{i+1} \geq 0,\ i = 1, ...,\ \lim_{i\to\infty} \gamma_i = 0.\} \tag{12.1}$$

Thus Γ is the set of all eigensequences $\lambda(A) = \{\lambda_i(A)\}_1^{\infty}$ for $A \in \mathcal{CH}_+$. The following theorem follows from [Fr2] and [Fu2]:

Theorem 12.1 *Let* $\mathbf{H}$ *be a separable Hilbert space. Assume that* $\alpha, \beta, \gamma \in \Gamma$. *Then the following are equivalent:*
(a) *There exist* $A, B, C \in \mathcal{CH}_+$ *with* $C \leq A + B$ *and* $\alpha = \lambda(A),\ \beta = \lambda(B),\ \gamma = \gamma(C)$.
(b) *For each* $n = 1, ...,$ *the vectors* $\alpha^n := (\alpha_1, ..., \alpha_n),\ \beta^n := (\beta_1, ..., \beta_n),\ \gamma^n := (\gamma_1, ..., \gamma_n)$ *satisfy the Horn inequalities* (4.6) *and* (**??**), *i.e.* $\gamma^n \in K_{\leq}(\alpha^n, \beta^n)$.

Proof. (a) $\Rightarrow$ (b): Let $\{f_i\}_1^{\infty}$ be the orthonormal sequence in $\mathbf{H}$ corresponding to $\lambda(C)$: $Cf_i = \gamma_i f_i,\ i = 1, ...$ Let $V_n := \text{span}\,(f_1, ..., f_n)$. Then $C|V_n = \text{diag}\,(\gamma_1, ..., \gamma_n)$ and $\lambda(C|V_n) = \gamma^n$. Let $\alpha(V_n) := \lambda(A|V_n),\ \beta(V_n) := \lambda(B|V_n)$. Clearly $C|V_n \leq A|V_n + B|V_n$. Hence $\gamma^n \in K_{\leq}(\alpha(V_n), \beta(V_n))$. The convoy principle implies that $\alpha^n \geq \alpha(V_n),\ \beta^n \geq \beta(V_n)$. Use Theorem **??** to deduce that $\gamma^n \in K_{\leq}(\alpha^n, \beta^n)$.
(b) $\Rightarrow$ (a): Let $C_n = \text{diag}\,(\gamma_1, ..., \gamma_n) \in \mathcal{H}_n$. Then there exist $A_n = (a_{ij,n})_{i=j=1}^{n},\ B_n = (b_{ij,n})_{i=j=1}^{n} \in \mathcal{H}_n$ such that $\lambda(A_n) = \alpha^n,\ \lambda(B_n) = \beta^n$ and $C_n \leq A_n + B_n$. Clearly

$$|a_{ij,n}| \leq \alpha_1,\ |b_{ij,n}| \leq \beta_1, \quad i, j = 1, ..., n.$$

Hence there exists a subsequence $1 \leq n_1 < n_2 < \ldots$ such that

$$\lim_{l\to\infty} a_{ij,n_l} = a_{ij},\ \lim_{l\to\infty} b_{ij,n_l} = b_{ij}, \quad i, j = 1, ...$$

Let $\tilde{A} := (a_{ij})_{i=j=1}^{\infty}$, $\tilde{B} := (b_{ij})_{i=j=1}^{\infty}$, $\tilde{C} := \mathrm{diag}\,(\gamma_1, \gamma_2, ...) = (c_{ij})_{i=j=1}^{\infty}$ be three infinite hermitian matrices. Fix an orthonormal basis $\mathbf{f} := \{f_i\}_1^{\infty}$ in $\mathbf{H}$. Clearly $\tilde{C}$ represents an operator $C \in \mathcal{CH}_+$ in the basis $\mathbf{f}$. We claim that $\tilde{A}, \tilde{B}$ represent $A, B \in \mathcal{CH}_+$ in the basis $\mathbf{f}$, such that $\lambda(A) \le \alpha,\ \lambda(B) \le \beta$. It is enough to prove this claim for A.

Fix a positive integer k. Then for $n \ge k$ $A_{k,n} := (a_{ij,n})_{i=j=1}^{k}$ is a nonnegative (definite) matrix, whose norm (its first eigenvalue) is bounded by α_1. Let $n = n_l$ and $l \to \infty$. Then $\tilde{A}_k := (a_{ij})_{i=j=1}^{k}$ is a nonnegative matrix, whose norm is bounded by α_1. Since k was arbitrary §26 of [**?**] implies that $\tilde{A}$ represents a linear bounded selfadjoint nonnegative operator in l_2. Hence $A \in \mathcal{H}_+$. The convoy principle yields that

$$\lambda_j(A_{k,n}) \le \lambda_j(A_n) = \alpha_j, \quad j = 1, ..., k.$$

Let $n = n_l$ and $l \to \infty$. Then

$$\lambda_j(\tilde{A}_k) \le \alpha_j, \quad j = 1, ..., k.$$

Let $V_k = \mathrm{span}\ (f_1, ..., f_k)$. Fix m. Hence for $k \ge m$ $\lambda_m(A, V_k) = \lambda_m(\tilde{A}_k) \le \alpha_m$. Use Lemma **??** to deduce $0 \le \lambda_m(A, \mathbf{H}) \le \alpha_m$. As the sequence $\{\alpha_i\}_1^{\infty}$ converges to 0 it follows that the sequence $\{\lambda_i(A, \mathbf{H})\}_1^{\infty}$ converges to 0. Lemma **??** implies that $A \in \mathcal{CH}_+$. Hence $\lambda_m(A, \mathbf{H}) = \lambda_m(A)$. Thus $\lambda(A) \le \alpha$.

We claim that $C \le A + B$. Clearly $C_k \le (a_{ij,n})_{i=j=1}^{k} + (b_{ij,n})_{i=j=1}^{k}$. Let $n = n_l$ and $l \to \infty$. Then $C_k \le \tilde{A}_k + \tilde{B}_k$. As k was arbitrary we deduce that $C \le A + B$. Let $\{e_i\}_1^{\infty}$ and $\{g_i\}_1^{\infty}$ be two orthonormal systems in $\mathbf{H}$ spanning the closed subspaces $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively such that

$$\begin{aligned} &Ae_i = \lambda_i(A)e_i,\ Bg_i = \lambda_i(B)g_i, \quad i = 1, ..., \\ &A\mathbf{H}_1^{\perp} = B\mathbf{H}_2^{\perp} = 0. \end{aligned}$$

Let $\hat{A},\ \hat{B} \in \mathcal{CH}_+$ be given by

$$\begin{aligned} &\hat{A}e_i = \alpha_i e_i,\ \hat{B}g_i = \beta_i g_i, \quad i = 1, ..., \\ &\hat{A}\mathbf{H}_1^{\perp} = \hat{B}\mathbf{H}_2^{\perp} = 0. \end{aligned}$$

Then $A \le \hat{A},\ B \le \hat{B}$ and $\lambda(\hat{A}) = \alpha,\ \lambda(\hat{B}) = \beta$. $\Box$

As in the finite dimensional case for $\alpha, \beta \in \Gamma$ let

$$\begin{aligned} K_{\le}(\alpha, \beta) &:= \{\gamma \in \Gamma : \quad \exists A, B, C \in \mathcal{CH}_+, \text{ where } C \le A + B \text{ and } \lambda(A) = \alpha,\ \lambda(B) = \beta,\ \lambda(C) = \gamma\}, \\ K(\alpha, \beta) &:= \{\gamma \in \Gamma : \quad \exists A, B, C \in \mathcal{CH}_+, \text{ where } C = A + B \text{ and } \lambda(A) = \alpha,\ \lambda(B) = \beta,\ \lambda(C) = \gamma\}. \end{aligned} \tag{12.2}$$

Theorem **??** characterizes the set $K_{\le}(\alpha, \beta)$. It is an infinite polyhedral set given by a countable number of inequalities, where each inequality is in a finite number of variables. Let

$$\Gamma_1 := \{\{\gamma_i\}_1^{\infty} \in \Gamma : \quad \sum_{i=1}^{\infty} \gamma_i < \infty\}.$$

That is $\gamma \in \Gamma_1$ iff there exists $C \in \mathcal{CH}_+$ in the trace class such that $\gamma = \lambda(C)$. The following theorem follows from [Fr2] and [Fu2]:

Theorem 12.2 *Let $\alpha, \beta \in \Gamma_1$. Then*

$$K(\alpha, \beta) = \{\{\gamma_i\}_1^\infty \in K_{\leq}(\alpha, \beta) : \quad \sum_{i=1}^{\infty} \gamma_i = \sum_{i=1}^{\infty} \alpha_i + \beta_i\}.$$

Proof. Suppose that $A, B, C \in \mathcal{CH}_+$ and $C = A + B$. Then $\lambda(C) \in K_{\leq}(\lambda(A), \lambda(B))$. Furthermore for any orthonormal basis $\{f_i\}_1^\infty$

$$\sum_{i=1}^{\infty}(Cf_i, f_i) = \sum_{i=1}^{\infty}(Af_i, f_i) + (Bf_i, f_i). \tag{12.3}$$

Hence if $\lambda(A) = \alpha,\ \lambda(B) = \beta,\ \lambda(C) = \gamma$ then (**??**) and Corollary **??** yield that C is in the trace class and

$$\sum_{i=1}^{\infty} \gamma_i = \text{trace } C = \text{trace } A + \text{trace } B = \sum_{i=1}^{\infty} \alpha_i + \beta_i.$$

Assume now that $\gamma \in K_{\leq}(\alpha, \beta)$. Hence there exists $A, B, C \in \mathcal{CH}_+$, where $C \leq A + B$ and $\lambda(A) = \alpha,\ \lambda(B) = \beta,\ \lambda(C) = \gamma$. Thus

$$\sum_{i=1}^{\infty}(Cf_i, f_i) \leq \sum_{i=1}^{\infty}(Af_i, f_i) + (Bf_i, f_i) = \text{trace } A + \text{trace } B,$$

and C is in the trace class. Suppose that $\sum_{i=1}^{\infty} \gamma_i = \sum_{i=1}^{\infty} \alpha_i + \beta_i$. Let $V_n = \text{span }(f_1, ... f_n),\ n = 1, ...$ As $A + B - C \geq 0$ we deduce that $A + B - C|V_n \geq 0$ for each $n \geq 1$. Hence

$$\text{trace } A + B - C|V_n = \sum_{i=1}^{n}((A + B - C)f_i, f_i) \geq 0$$

and each summand is nonnegative. The equality $\text{trace } C = \text{trace } A + \text{trace } B$ yields that $((A+B-C)f_i, f_i) = 0,\ i = 1, ...$ Therefore $A+B-C|V_n = 0,\ n = 1, ...$ and $A+B-C = 0$. $\square$

It is left to characterize $K(\alpha, \beta)$ where $\alpha, \beta \in \Gamma$ and $\alpha + \beta \notin \Gamma_1$. The arguments of the proof of Theorem **??** shows that $K(\alpha, \beta) \subset K_{\leq}(\alpha, \beta) \backslash \Gamma_1$.

Conjecture 12.3 *Let $\alpha, \beta \in \Gamma$ and assume that $\alpha + \beta \notin \Gamma_1$. Then $K(\alpha, \beta) = K_{\leq}(\alpha, \beta) \backslash \Gamma_1$.*

References

[AG] N.I. Akhiezer and I.M. Glazman, *Theory of Linear Operators in Hilbert Space*, Vol. I-II, Dover, N.Y., 1993.

[BG] F.A. Berezin and I.M. Gelfand, Some remarks on spherical functions on Symmetric Riemannian manifolds, *Amer. Math. Soc. Transl.* 21 (1962), 193-238.

[Bha] R. Bhatia, *Perturbation bounds for matrix eigenvalues*, Pitman Res. Notes 162, 1987.

[DST] J. Day, W. So, R.C. Thompson, The spectrum of Hermitian matrix sum, *Lin. Alg. Appl.* 280 (1998), 289-332.

[Fan] Ky Fan, On a theorem of Weyl concerning eigenvalues of linear transformations, *Proc. Nat. Acad. Sci. USA* 35 (1949), 652-655.

[Fr1] S. Friedland, Extremal eigenvalue problems for convex sets of symmetric matrices and operators, *Israel J. Math.* 15 (1973), 311-331.

[Fr2] S. Friedland, Finite and infinite dimensional generalizations of Klyachko's theorem, *Lin. Alg. Appl.* 322 (2000), 3-22.

[Fu1] W. Fulton, Eigenvalue, invariant factors, highest weights, and Schubert calculus, *Bul. Amer. Math. Soc.* 37 (2000), 209-249.

[Fu2] W. Fulton, Eigenvalues of majorized Hermitian matrices and Littlewood-Richardson coefficients, *Lin. Alg. Appl.* 322 (2000), 23-36.

[Gan] F.R. Gantmacher, *The Theory of Matrices*, vol. I, Chelsea, New York, 1960.

[HLP] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press 1934.

[HZ] J. Hersch and B. Zwahlen, Évaluations par défault pour une summe quelconque de valeurs propres γ_k d'un opérateur $C = A + B$, à l'aide de valeurs propres α_i de A et β_j de B, C, *R. Acad. Sc. Paris* 254 (1962), 1559-1561.

[Hor] A. Horn, Eigenvalues of sums of Hermitian matrices, *Pacific J. Math.* 12 (1962), 225-241.

[Kly] A.A. Klyachko, Stable bundles, representation theory and Hermitian matrices, *Selecta Math. (N.S.)* 4 (1998), 419-445.

[KT] A. Knutson and T. Tao, The honeycomb model of Berenstein-Zelevensky polytope I. Klyachko saturation conjecture, math.RT/9807160, *J. Amer. Math. Soc.* (to appear).

[L1] V.B. Lidskii, The proper values of sum and product of symmetric matrices, *Dokl. Akad. Nauk SSSR* 74 (1950), 769-772.

[L2] B.V. Lidskii, Spectral polyhedron of the sum of two hermitian matrices, *Funct. Anal. Appl.* 16 (1982), 139-140.

[PS] G. Pólya and M. Schiffer, Convexity of functionals by transplantation, *J. d'Anal. Math.* 3 (1953/4), 245-xxx.

[RS] M. Reed and B. Simon, *Functional Analysis*, Academic Press, San Diego, 1998.

[Sch] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Tiptree, 1986.

[Tot] B. Totaro, Tensor products of semistable are semistable, *Geometry and Analysis on complex Manifolds*, World Sci. Publ., 1994, pp. 242-250.

[Wey] H. Weyl, Das asymptotische Verteilungsgesetz der Eigewerte lineare partieller Differentialgleichungen, *Math. Ann.* 71 (1912), 441-479.

[Wie] H. Wielandt, An extremum property of sums of eigenvalues, *Proc. Amer. Math. Soc.* 6 (1955), 106-110.