# Counting matchings in graphs with applications to the monomer-dimer models

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- Summary and open problems

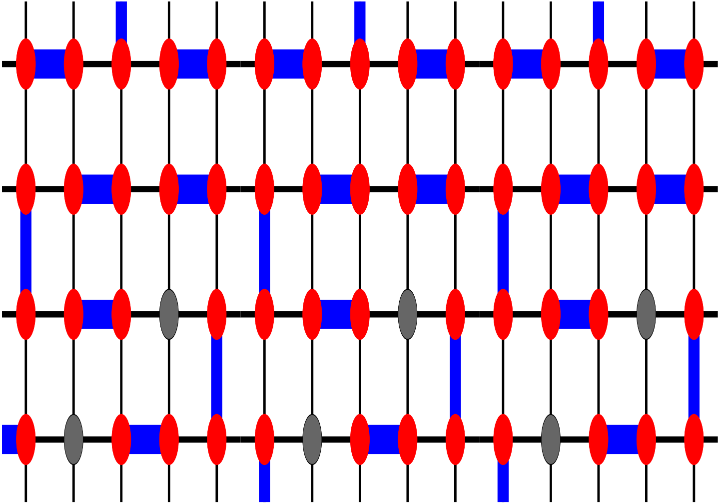

Figure: Matching on the two dimensional grid: Bipartite graph on 60 vertices, 101 edges, 24 dimers, 12 monomers

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- $M$ is $k$-matching $\iff \#M = k$.

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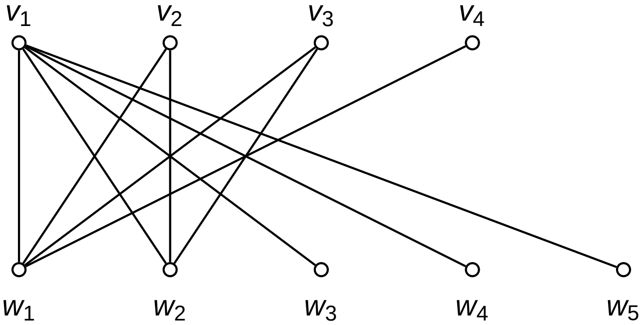
Birkhoff-Egerváry-König theorem (1946-1931-1916)

# Bipartite graphs

Figure: An example of a bipartite graph




Incidence matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

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$r^k \min_{C\in\Omega_n} \mathrm{perm}_k\, C \le \phi(k, G)$ for any $G \in \mathcal{G}(r, 2n)$

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Tverberg permanent conjecture 1963:

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- There are new simple proofs using nonnegative hyperbolic polynomials e.g. Friedland-Gurvits 2008

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$$\phi(n, G) \ge \left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n \quad \textit{for} \quad G \in \mathcal{G}(r, 2n)$$

Gurvits 2006: $A \in \Omega_n$, each column has at most $r$ nonzero entries:

$$\operatorname{perm} A \ge \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{r-1}{r}\right)^{(r-1)n}.$$

$$\textit{Cor} : \ \phi(n, G) \ge \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n$$

$$\textit{Con FKM } 2006 : \ \phi(k, G) \ge \binom{n}{k}^2 \left(\frac{nr-k}{nr}\right)^{nr-k}\left(\frac{kr}{n}\right)^k, \ G \in \mathcal{G}(r, 2n)$$

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F-G 2008 showed weaker inequalities

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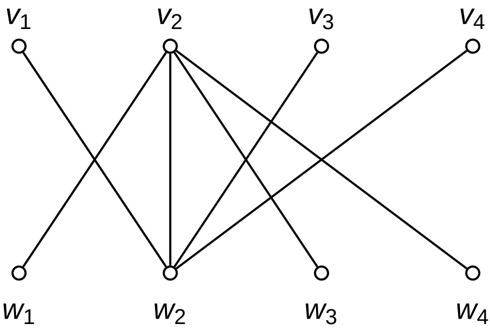
- Prf: Any edge in $e \in E$ can be in at most $(r-1)^2$ different 4-cycles.

# An example

Figure: Edge neighborhood of $\overline{V_2 W_2}$ of 4- regular graph on 8 vertices



# Upper perfect matching bounds for general graphs

$G = (V, E)$ Non-bipartite graph on $2n$ vertices

$$\phi(n, G) \leq \prod_{v \in V} ((\deg v)!)^{\frac{1}{2 \deg v}}$$

If $\deg v > 0, \forall v \in V$ equality holds iff $G$ is a disjoint union of complete balanced bipartite graphs

Kahn-Lóvasz unpublished, Friedland 2008-arXiv, Alon-Friedland 2008-arXiv.

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   $p_1(n, r) =$
   $\frac{n^4r^4}{24} + \frac{n^3r^3}{4}(1-2r) + \frac{n^2r^2}{24}\left(19 - 60r + 52r^2\right) + nr\left(\frac{5}{4} - 5r + 7r^2 - \frac{7r^3}{2}\right)$

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Notation:

$$f(x) = \sum_{i=0}^{N} a_i x^i \preceq g(x) = \sum_{i=0}^{N} b_i x^i \iff$$

$$a_i \le b_i \text{ for } i = 1, \ldots, N.$$

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  If $n$ even $G$ multi-bipartite 2-regular graph then $\Phi_G(x) \succeq \Phi_{C_n}(x)$.

# Relations between matching polynomials

- For $0 \le i \le j$
  $\Phi_{C_i}(x)\Phi_{C_j}(x) - \Phi_{C_{i+j}}(x) = (-1)^i x^i \Phi_{C_{j-i}}(x)$
- $P_n$ path $1 \to 2 \to \ldots \to n$.
- $p_n(x) := \Phi_{P_n}(x), q_n(x) := \Phi_{C_n}(x)$
- $p_k(x) = p_{k-1}(x) + xp_{k-2}(x)$
- $q_k(x) = p_k(x) + xp_{k-2}(x)$
- If $n = 0, 1 \mod 4$
  $p_{n-1} = p_1 p_{n-1} \prec p_3 p_{n-3} \prec \cdots \prec p_{2\lfloor \frac{n}{4} \rfloor - 1} p_{n-2\lfloor \frac{n}{4} \rfloor + 1} \prec$
  $p_{2\lfloor \frac{n}{4} \rfloor} p_{n-2\lfloor \frac{n}{4} \rfloor} \prec p_{2\lfloor \frac{n}{4} \rfloor - 2} p_{n-2\lfloor \frac{n}{4} \rfloor + 2} \prec \cdots \prec p_2 p_{n-2} \prec p_0 p_n = p_n$

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- Characterization of maximal and minimal matching polynomial graphs in family of graphs with given number of vertices of degrees one and two

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  $\psi(x,\ G_1) := 1 + 15x + 75x^2 + 145x^3 + 96x^4 + 12x^5$
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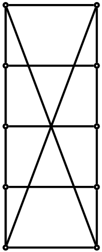
- For $2n$ from 12 to 24 the extremal graphs, with the maximal $\phi(l, G)$:

$$
\begin{array}{ll}
\frac{2n}{6}K_{3,3} & \text{if } 6|2n \\
\frac{2n-8}{6}K_{3,3}\bigcup Q_3 & \text{if } 6|(2n-2) \\
\frac{2n-10}{6}K_{3,3}\bigcup(G_1 \text{ or } M_{10}) & \text{if } 6|(2n-4)
\end{array}
$$

# Two bipartite 3-regular graphs on 10 vertices



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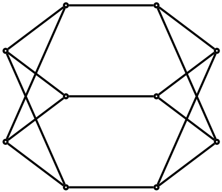

$G_1$

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- Permutation $\sigma : \langle nr \rangle \to \langle nr \rangle$ induces $G(\sigma) \in \mathcal{G}_{\mathrm{mult}}(r, 2n)$ and vice versa
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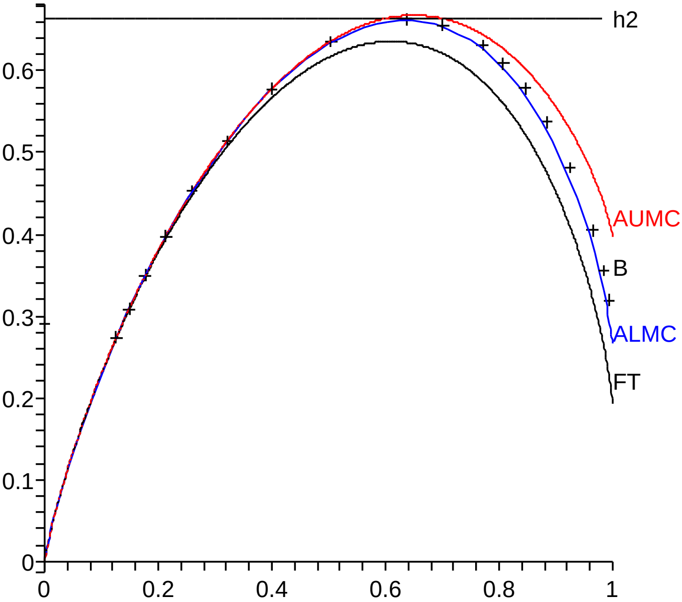
$$p(t) := P_r'(t) \in (0, 1), \quad h_{K(r)}(p(t)) := P_r(t) - tp(t)$$

# $r = 4$



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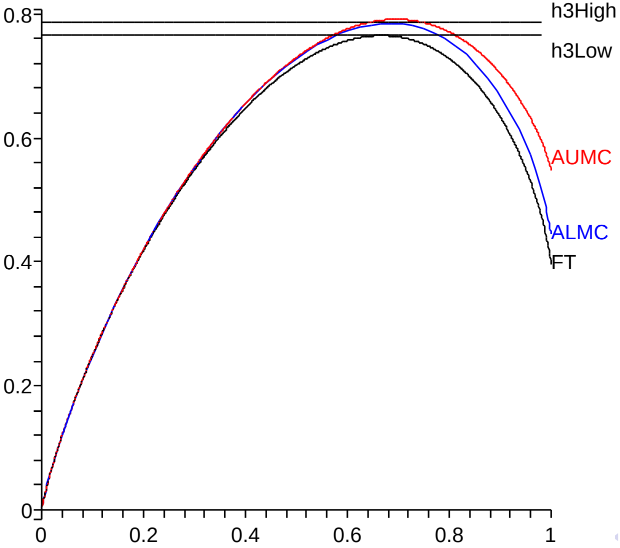

# Lower asymptotic bounds Friedland-Gurvits 2008

Thm: $r \geq 3, s \geq 1$ integers,
$B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries.
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- Cor: $r$-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$

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$$\frac{1}{2}(r+s-1)\log(1-\frac{1}{r+s}) - \frac{1}{2}(s-1+p)\log(1-\frac{1-p}{s})$$

Prf combines properties positive hyperbolic polynomials, capacity and the measure on $\mathcal{G}(r, 2n)$

- Cor: $r$-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$
- Con: under Thm assumptions

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{k_n} B_n}{2n} \geqslant f(r, p) - \frac{p}{2}\log r$$

# Lower asymptotic bounds Friedland-Gurvits 2008

Thm: $r \geq 3, s \geq 1$ integers,
$B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries.
$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$ then

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- For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$ conjecture holds

# Known lower and upper bounds for $p$-matchings

FKLM accepted JOSS 08:

$$\mathrm{low}_r(p) \geq \max(\mathrm{low}_{r,1}(p), \mathrm{low}_{r,2}(p))$$
$$\mathrm{upp}_r(p) \leq \min(\mathrm{upp}_{r,1}(p), \mathrm{upp}_{r,2}(p))$$

Lower estimates are based on F-G inequalities
and Newton inequalities:
$f(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$ have nonpositive roots
then $\binom{n}{k}^{-1} a_k$ log concave sequence
Upper estimates are based on Bregman inequalities :

$$\phi(k, G) \leq \binom{n}{k} \frac{(r!)^{\frac{k}{n}} (n!)^{\frac{n-k}{n}}}{(n-k)!}$$

and

$$\max_{G \in \mathcal{G}_{\mathrm{mult}}(r,2n)} \phi(k, G) = \binom{n}{k} r^k$$

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Prf: Newton inequalities

# $r = 4$ lower bounds

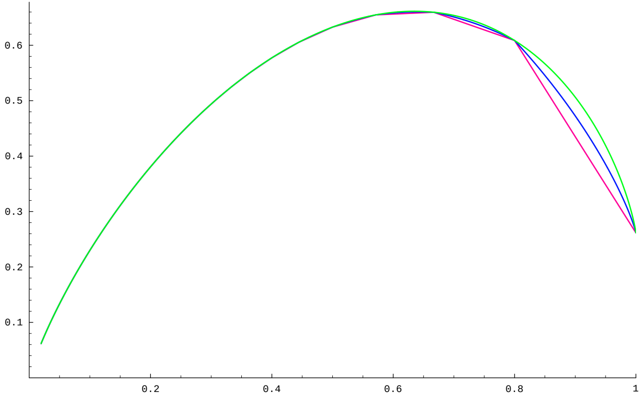


Figure: $fl(p, 4)$-red, $\mathrm{low}_{4,1}(p)$-blue, $f(p, 4)$-green

# $r = 4$ lower bounds differences

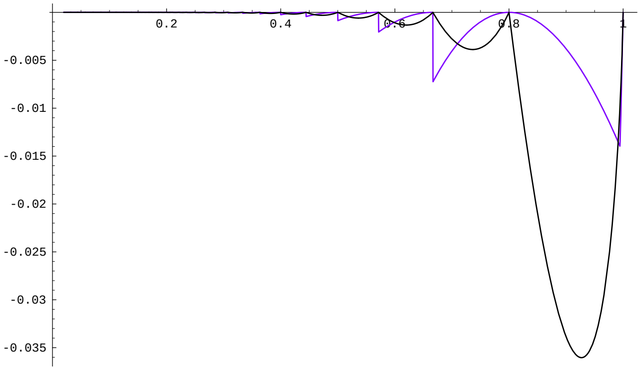


Figure: $\mathrm{low}_{4,1}(p) - f(p,4)$-black, $\mathrm{low}_{4,2}(p) - f(p,4)$-blue

# $r = 4$ upper bounds

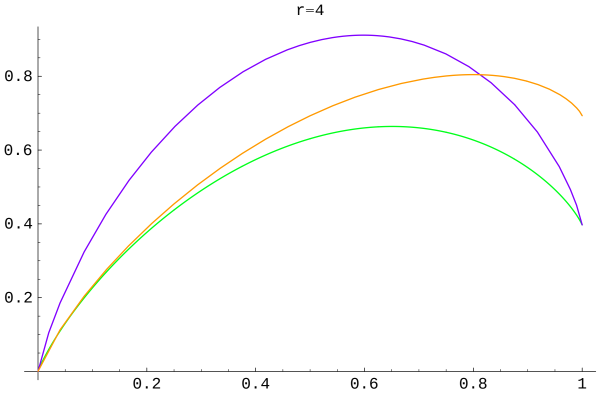


Figure: $h_{K(4)}$-green, $\mathrm{upp}_{4,1}$-blue, $\mathrm{upp}_{4,2}$-orange

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