MULTI-DIMENSIONAL ENTROPY

and the monomer-dimer problem

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1 UNRESTRICTED CAPACITY

$$< n >:= \{1, 2, 3, ..., n\}$$

IS AN ALPHABET ON n LETTERS.

$$\underbrace{\bullet\ \bullet\ \bullet \ldots \bullet\ \bullet\ \bullet}_{m \text{ times}}$$

LINEAR STORAGE OF LENGTH m.

STORAGE MESSAGES n^m.

(UNRESTRICTED) CAPACITY:

$$\log_2 n \quad = \frac{\log_2 n^m}{m}.$$

2 $(0-1)$ LIMITED CHANNEL

(HARD CORE LATTICE or NEAR NEIGHBOR EXCLUSION)

$n = 2$, $< 2 >= \{1, 2\} = \{1, 0\}$ ($2 \equiv 0$).

NO TWO $1's$ ARE NEIGHBORS.

OF m-MESSAGES IS u_m;

$$u_{m+1} = u_m + u_{m-1},\ m = 1, 2, ...$$

FIBONACCI SEQUENCE $2, 3, 5, 8, \ldots$

SHANNON CAPACITY:

$$\lim_{m\to\infty} \frac{\log_2 u_m}{m} =$$

$$\log_2 \frac{1+\sqrt{5}}{2} = 0.694241914...$$

3 SUBSHIFT OF FINITE TYPE

$$a = a_1 ... a_m = (a_i)_1^m : < m > \rightarrow < n >$$

WORD OF LENGTH m.

$< n >^{<m>}$ all messages of length m.

$$< n >^{\mathbb{Z}} = \{a = (\dots a_{-1} a_0 a_1 \dots) =$$

$$(a_i)_{i \in \mathbb{Z}} : \mathbb{Z} \rightarrow < n >\}$$

$$< n >^{\mathbb{N}} = \{a = (a_1 \dots) =$$

$$(a_i)_{i \in \mathbb{N}} : \mathbb{N} \rightarrow < n >\}$$

PROJECTION: $\pi_m((a_i)) = (a_i)_{i=1}^m$.

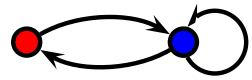
$\mathcal{S} \subset < n >^{\mathbb{N}}$ ($< n >^{\mathbb{Z}}$) is SOFT if

$\exists P \subset < n >^{<r>}$ such that $a \in \mathcal{S} \iff$

any consec. string of r letters in a in P.

EXAMPLE: $r = 2 \Rightarrow P = \Gamma \subset <n> \times <n>$.

$n = 2$, $P = \{$●●, ●●, ●●$\}$



ALLOWABLE WORD OF LENGTH $m+1$

A WALK OF LENGTH m ON Γ

$$\Gamma^m = \{(a_i)_1^{m+1} \in <n>^{m+1}:$$
$$(a_i, a_{i+1}) \in \Gamma\}$$
$$\Gamma^{\mathbb{N}} = \{(a_i) \in <n>^{\mathbb{N}}: \ (a_i, a_{i+1}) \in \Gamma\}$$
$$\Gamma^{\mathbb{Z}} = \{(a_i) \in <n>^{\mathbb{Z}}: \ (a_i, a_{i+1}) \in \Gamma\}$$

any SOFT can be coded as a walk on

$$\Gamma \subset <N> \times <N> \equiv \pi_{r-1}(P) \times \pi_{r-1}(P)$$

$$N = \#\pi_{r-1}(P)$$

$$((a_i)_1^{r-1}, (b_i)_1^{r-1}) \in \Gamma \iff$$
$$b_1 = a_2, \ldots, b_{r-2} = a_{r-1},$$

ASSUMPTION: $\mathcal{S}$ -SOFT

$(a_i)_{i\in\mathbb{N}}$ is m-periodic

$$a_{i+m} = a_i,\ i \in \mathbb{N}.$$

δ_m=$\log$# P-allow. words of length m.

$$\tilde{\delta}_m = \log \#\pi_m(\mathcal{S})$$

$\delta_{m,per}$=$\log$# m-periodic words in $\mathcal{S}$.

$$\delta_{m,per} \le \tilde{\delta}_m \le \delta_m.$$

$\{t_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ IS SUBADDITIVE (SA):

$$t_{p+q} \le t_p + t_q \quad \text{for all } p, q \in \mathbb{N}.$$

$$\{t_i\}_{i\in\mathbb{N}} \text{ SA} \Rightarrow \lim_{i\to\infty} \frac{t_i}{i} = \tau \le \frac{t_p}{p}.$$

CLAIM: $\{\delta_m\}, \{\tilde{\delta}_m\}$- ARE SA.

$$h_{com} := \lim_{m\to\infty} \frac{\delta_m}{m} \quad \text{CAPACITY}$$

$$h := \lim_{m\to\infty} \frac{\tilde{\delta}_m}{m} \quad \text{ENTROPY}$$

$$h_{per} := \limsup_{m\to\infty} \frac{\delta_{m,per}}{m} \quad \text{periodic entropy}$$

$$-\infty \le h_{per} \le h \le h_{com}$$

MAIN THM

for 1- dimensional SOFT $\mathcal{S}$

$$h_{per} = h = h_{com} = \log \rho(\Gamma)$$

Γ -GRAPH INDUCED BY SOFT.

COR. $\mathcal{S}$ is decidable:

EITHER $\mathcal{S} = \emptyset \iff$:

$\exists m$ with no allow. word of length m

OR $\mathcal{S}$ contains an m periodic state.

4 MULTI-DIMENSIONAL CAPACITY

$2 \leq d$ dimension

$$\mathbf{m} := (m_1, ..., m_d) \in \mathbb{Z}^d$$
$$|\mathbf{m}| = |m_1| + ... + |m_d|$$
$$|\mathbf{m}|_{pr} := |m_1| \times \ldots \times |m_d|$$
$$< \mathbf{m} > := < m_1 > \times \ldots \times < m_d >,$$
$$\text{for } \mathbf{m} \in \mathbb{N}^d$$

$a :< \mathbf{m} > \rightarrow < n >$ is $(a_{\mathbf{i}})_{\mathbf{i} \in < \mathbf{m} >}$

$< n >^{< \mathbf{m} >}$ set of all maps a.

$< n >^{\mathbb{N}^d}$ & $< n >^{\mathbb{Z}^d}$ all maps

from $\mathbb{N}^d$ & $\mathbb{Z}^d$ to $< n >$.

$\pi_{\mathbf{m}}((a_{\mathbf{i}})) = (a_{\mathbf{i}})_{\mathbf{i} \in < \mathbf{m} >}$ proj. on $< \mathbf{m} >$.

$\mathcal{S} \subset < n >^{\mathbb{N}^d}$ ($< n >^{\mathbb{Z}^d}$) SOFT if

$\exists P \subset < n >^{< \mathbf{r} >}$ such that $a \in \mathcal{S} \iff$

any consec. box of $\mathbf{r}$ letters in a in P.

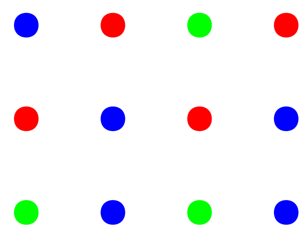
EXAMPLE: $\Gamma = (\Gamma_1, ..., \Gamma_d)$

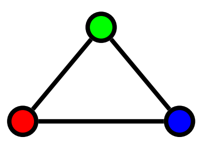
$(a_{\mathbf{i}})_{\mathbf{i} \in <\mathbf{m}>}$ IS ALLOWABLE WORD

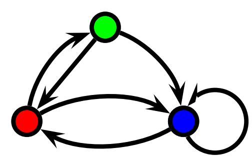
if any line through $\mathbf{i} \in < \mathbf{m} >$

in direction $\mathbf{e}_k = (\delta_{k1}, ..., \delta_{kd})$ is in $\Gamma_k^{m_k - 1}$ for each k

$$< (4, 3) > :=$$







any SOFT can be coded as $\Gamma^{\mathbb{N}^d}$ $(\Gamma^{\mathbb{Z}^d})$

$\exists\ \mathcal{S}$ SOFT **NOT DECIDABLE**:

$\exists \Gamma^{\mathbb{N}^2} \neq \emptyset$ with no periodic state

Berger 1966

$$\Gamma^{\mathbb{N}^d} = \emptyset \iff \exists \mathbf{m}\ \Gamma^{\mathbf{m}} = \emptyset$$

ASSUMPTION: $\mathcal{S}$ -SOFT

$\delta_{\mathbf{m}}$-log# P-allow. words of dim. $\mathbf{m}$.

$$\tilde{\delta}_{\mathbf{m}} = \log \#\pi_{\mathbf{m}}(\mathcal{S})$$

$\delta_{\mathbf{m},per}$-log# $\mathbf{m}$-periodic words in $\mathcal{S}$.

$$\delta_{\mathbf{m},per} \le \tilde{\delta}_m \le \delta_m.$$

$\{\delta_{\mathbf{m}}\}, \{\tilde{\delta}_{\mathbf{m}}\}$- are SA in each coordinate

(split box $< \mathbf{m} >$ to 2 boxes by $x_k = i_k$)

$$h_{com} := \lim_{\mathbf{m}\to\infty} \frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}} \quad \textbf{CAPACITY}$$

$$h := \lim_{m\to\infty} \frac{\tilde{\delta}_{\mathbf{m}}}{|\mathbf{m}|_{pr}} \quad \textbf{ENTROPY}$$

$$h_{per} := \limsup_{\mathbf{m}\to\infty} \frac{\delta_{\mathbf{m},per}}{|\mathbf{m}|_{per}} \quad \textbf{PERIODIC ENTROPY}$$

$$-\infty \le h_{per} \le h \le h_{com} \le \frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}}$$

Berger: $-\infty = h_{per} < 0 \le h \le h_{com}$

Friedland 97: for all $\mathcal{S}$ $h = h_{com}$

5 UPPER ESTIMATES OF MDC

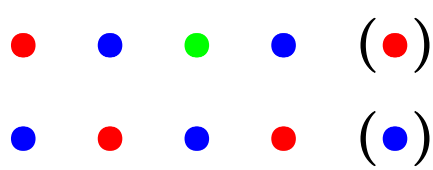
$d = 2$ and $\mathbf{m} = (m_1, m_2)$.

$\Gamma(2, m_1) \subset \Gamma_1^{m_1-1} \times \Gamma_1^{m_1-1}$

$((b_i)_1^{m_1}, (c_i)_1^{m_1}) \in \Gamma(2, m_1) \iff (b_i, c_i) \in \Gamma_2$

for each $i = 1, ..., m_1$.

EXAMPLE: $\Gamma(2, 4)$



$\Gamma_{per}(2, m_1) \subset \Gamma(2, m_1)$:
$(b_{m_1}, b_1), (c_{m_1}, c_1) \in \Gamma_1$
$\rho(\Gamma(2, m_1)) \geq \rho(\Gamma_{per}(2, m_1))$

statistical mechanics - *transfer* matrices:

$A(\Gamma(2, m_1))$ & $A(\Gamma_{per}(2, m_1))$

$$\lim_{m_2 \to \infty} \frac{\delta_{\mathrm{m}}}{m_2} = \log \rho(\Gamma(2, m_1)) \geq m_1 h$$

$$\limsup_{m_2 \to \infty} \frac{\delta_{\mathrm{m},per}}{m_2} = \log \rho(\Gamma_{per}(2, m_1))$$

$$\lim_{m_1 \to \infty} \frac{\log \rho(\Gamma(2, m_1))}{m_1} = h$$

$$\limsup_{m_1 \to \infty} \frac{\log \rho(\Gamma_{per}(2, m_1))}{m_1} = h_{per}$$

6 Upper-Lower Bounds with Symmetricity Con.

Γ_2 - **SYMMETRIC**

$$\max\left(\frac{\log\rho(\Gamma(1,p+2q+1)) - \log\rho(\Gamma(1,2q+1))}{p},\right.$$

$$\left.\frac{\log\rho(\Gamma_{per}(1,p+2q) - \log\rho(\Gamma_{per}(1,2q))}{p}\right) \leq$$

$$h \leq \frac{\log\rho(\Gamma_{per}(1,2m))}{2m} \left(\leq \frac{\log\rho(\Gamma(1,2m)}{2m}\right)$$

for any $m, p \geq 1$ and $q \geq 0$.

$\mathcal{S}$ **DECIDABLE**:

$$\mathcal{S} \neq \emptyset \iff \Gamma(1,2) \text{ has cycle}$$

AND COMPUTABLE

$$h_{per} = h$$

$$d \geq 3 \; \Gamma_1, ..., \Gamma_{d-1} \text{ symmetric} \Rightarrow$$

$$h_{per} = h \text{ is computable}$$

$$\frac{\log \rho(\Gamma_{per}((2m_1, ..., 2m_{d-1}))}{2^{d-1}|m_1|...|m_{d-1}|} \geq h \geq$$

$$\frac{\overline{h}(p + 2q) - \overline{h}(2q)}{p}$$

$$\overline{h}(q) := \lim_{\mathrm{m}^{\{1\}} \to \infty} \log \frac{\#W_{\{1\},per}(\mathrm{m}^{\{1\}}, q)}{|m^{\{1\}}|_{pr}}$$

TRUE IF Γ ISOTROPIC & SYMMETRIC:

$$\Gamma_1 = ... = \Gamma_d = \Delta \; - \text{SYMMETRIC}$$

7 Automorphism Subgroups and Computations

$A = (a_{ij})_1^N$ nonnegative matrix

$\mathcal{A}(A) := \{\pi \in S_N : a_{\pi(i)\pi(j)} = a_{ij},\ i,j \in < N >\}$

$G \leq \mathcal{A}(A), \mathcal{O}(G) := < N > /G, M = \#\mathcal{O}(G)$

$$\hat{A} = (\hat{a}_{\alpha\beta})_{\alpha,\beta \in \mathcal{O}(G)},$$
$$\hat{a}_{\alpha\beta} =: \sum_{j \in \beta} a_{ij},\ i \in \alpha,$$
$$\rho(A) = \rho(\hat{A}),$$
$$A = A^T \Rightarrow \hat{A} \text{ is symmetric for}$$
$$< \mathrm{x}, \mathrm{y} >= \sum_{\alpha \in \mathcal{O}(G)} (\#\alpha) x_\alpha y_\alpha.$$
$$M \geq N/\#G,$$

In our computations $M \sim N/\#G$.

$$T((m_1, ..., m_d)) := (\mathbb{Z}/m_1\mathbb{Z}) \times ... \times (\mathbb{Z}/m_d\mathbb{Z})$$

$$\mathcal{A}(\mathrm{m}) = \mathcal{A}(\text{adjacency graph of } T(\mathrm{m}))$$

Γ **is isotropic symmetric graph** $\Rightarrow$

$$\mathcal{A}(\mathrm{m}^{\{d\}}) \leq \mathcal{A}(A(\Gamma_{per}(d, \mathrm{m}^{\{d\}})))$$

$$\#\mathcal{A}(2) = 2,\ \#\mathcal{A}(m) = 2m,\ m \geq 3 \Rightarrow$$
$$\mathcal{A}(m_1) \times ... \times \mathcal{A}(m_{d-1}) \leq \mathcal{A}(\mathrm{m}^d)$$
$$m_i = m_j\ \Rightarrow i \leftrightarrow j \Rightarrow$$
$$\#\mathcal{A}((m, ..., m)) \geq (2m)^{d-1}(d-1)! \text{ for } m \geq 3$$
$$T(4) \sim <(2,2)> \Rightarrow$$
$$\#\mathcal{A}(4, ..., 4) \geq 2^{2(d-1)}(2(d-1))!$$

Friedland-Peled 03

8 d-Dimensional Monomer-Dimers

Dimer: $(\mathbf{i},\mathbf{j}),\ \mathbf{j}=\mathbf{i}+\mathbf{e}_k \in \mathbb{Z}^d$.

any partition of $\mathbb{Z}^d$ to dimers (1-factor).

Monomer: occupies $\mathbf{i} \in \mathbb{Z}^d$.

any partition of $\mathbb{Z}^d$ to monomer-dimers

is 1-factor of a subset of $\mathbb{Z}^d$.

Dimer and Monomer-Dimer are SOFT

$$0=\tilde{h}_1 \le \tilde{h}_2 \le \ldots \le \tilde{h}_d \le \ldots (\text{dimers})$$

$$\log \frac{1+\sqrt{5}}{2} = h_1 \le h_2 \le \ldots \le h_d \le \ldots$$
$$(\text{monomer} - \text{dimer})$$

Fisher, Kasteleyn and Tempreley 61

$$\tilde{h}_2 = \frac{1}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)^2} = 0.29156090\ldots$$

9 Upper and Lower Bounds

$A = (a_{ST}), S, T \subset T(\mathrm{m}^{\{d\}})$ is transfer matrix

represents multigraph and is "Hankel": a_{ST} is

the number of monomer-dimer (dimer) covers of

$T(\mathrm{m}^{\{d\}})\backslash(S \cup T)$ and $a_{ST} = 0$ if $S \cap T \neq \emptyset$

S (T) the locations of dimers going down (up)

$$\widetilde{h}_d \leqslant \frac{\log \widetilde{\beta}(2\mathrm{m}^{\{d\}})}{2^{d-1}|\mathrm{m}^{\{d\}}|_{pr}} \quad \text{Ciucu 98}$$

$$h_d \leqslant \frac{\log \beta(2\mathrm{m}^{\{d\}})}{2^{d-1}|\mathrm{m}^{\{d\}}|_{pr}} \quad \text{FP 03}$$

$$h_2 \geqslant \frac{\log \beta(p + 2q) - \log \beta(2q)}{p} \quad \text{FP 03}$$

Similar lower bounds for exist for $h_3, \widetilde{h}_3$ FP 03.

$\widetilde{h}_3 \leq \frac{\log \widetilde{\beta}(6,4)}{6 \cdot 4} = 0.4575469308$

Lundow 2001 using matrix of order 184854

which splits to a direct sum of 3 matrices.

10 The Value of h_2

Friedland-Peled 03

(confirming Baxter's heuristic computations 1968):

$$h_2 = 0.66279897 \text{ using } m = 14, 15, 16$$

$$.66279897190 \leqslant h_2 \leqslant .662798972844913$$

$$m = 16,\ N = 2^{16} = 65536,$$

$$M = 2250,\ \frac{N}{2m} = 2,048.$$

M. Jerrum-87:

Two-dimensional monomer-dimer systems

are computationally intractible.

m_1	$\#\mathcal{O}(m_1)$	$\log\beta(m_1)$	
4	**6**	**2.6532941163**	
5	**8**	**3.3135066910**	
6	**13**	**3.9769139475**	
7	**18**	**4.6395628723**	
8	**30**	**5.3023993987**	
9	**46**	**5.9651887945**	
10	**78**	**6.6279902386**	
11	**126**	**7.2907885674**	
12	**224**	**7.9535877093**	
13	**380**	**8.6163866375**	
14	**687**	**9.2791856222**	
15	**1224**	**9.9419845918**	
16	**2250**	**10.60478356551861**	
17	**4112**	$\in$ **(11.267582535, 11.267582554)**	$\in$

Table 1: Spectral radii for h_2

11 Matchings and permanents

$\mathbf{perm}_{\ s}\mathbf{A} =$ sum perm. all $s \times s$ submatrices of A.

Tverberg's conjecture (Friedland 1982)

$\mathbf{perm}_{\ s}\mathbf{A} \geq \mathbf{perm}_{\ s}\mathbf{J_n}$ for any $n \times n$ d.s. matrix.

G is r-regular bipartite with n in each class.

$W(G,s)$ set of all matchings of size s in G.

$$\#W(G,s) \geqslant \binom{n}{s}^2 s! \left(\frac{r}{n}\right)^s$$

$\lambda_d(p)$-monomer dimer entropy

with dimer density $p \in [0,1]$.

$$\lambda_d(p) \geqslant \frac{1}{2}(-p\log p - 2(1-p)\log(1-p) + p\log 2d - p)$$

$h_d = \max_{p\in[0,1]} \lambda_d(p) \geqslant$
$\frac{1}{2}(-p(d)\log p(d) - 2(1-p(d))\log(1-p(d))$
$+p(d)\log 2d - p(d))$

$p(d) = \frac{4d+1-\sqrt{8d+1}}{4d}$.

12 The Graphs for $\lambda_2(p)$

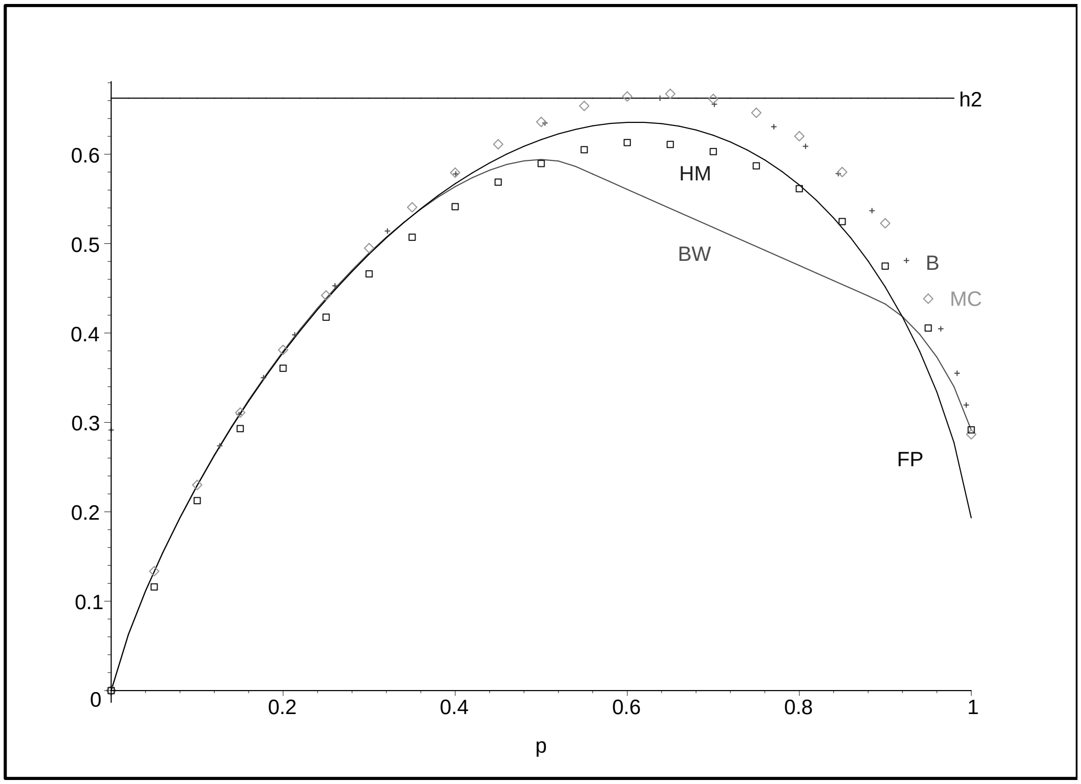


Figure 1: HM is the lower bound of Hammersley-Menon, BW is the lower bound of Bondy-Welsh, FP is the lower bound of Friedland-Peled, MC is the Monte Carlo estimate of Hammersley-Menon, B are Baxter's estimates, and h2 is the true value of $h_2 = \max \lambda_2(p)$.

13 Estimates for $\widetilde{h}_3$ and h_3

Schrijver 1998

$$\#W(G,n) \geqslant \left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n$$

$$\widetilde{h}_d = \lambda_d(1) \geqslant$$
$$\frac{1}{2}((2d-1)\log(2d-1) - (2d-2)\log 2d)$$
$$(S)\ 0.440075842 \leqslant \widetilde{h}_3 \leqslant 0.4575469308\ (L)$$

FP gives

$$0.7652789557 \leqslant h_3 \leqslant .7862023450$$

(m_1, m_2)	$\#\mathcal{O}(m_1, m_2)$	$\log \beta(m_1, m_2)$	$\frac{\log \beta(m_1, m_2)}{m_1 m_2}$
(2, 2)	**6**	**3.224405658**	**0.8061014**
(3, 2)	**13**	**4.768958913**	**0.7948264**
(4, 2)	**34**	**6.367778959**	**0.7959723**
(5, 2)	**78**	**7.958105292**	**0.7958105**
(6, 2)	**237**	**9.550024542**	**0.7958353**
(7, 2)	**687**	**11.14163679**	**0.7958311**
(8, 2)	**2299**	**12.73331093**	**0.7958319**
(3, 3)	**25**	**7.057039652**	**0.7841155**
(4, 3)	**158**	**9.421594940**	**0.7851329**
(5, 3)	**708**	**11.77517604**	**0.7850117**
(4, 4)	**805**	**12.57923752**	**0.7862023**

Table 2: Spectral radii for h_3

14 Lower Bounds for $\lambda_3(p)$

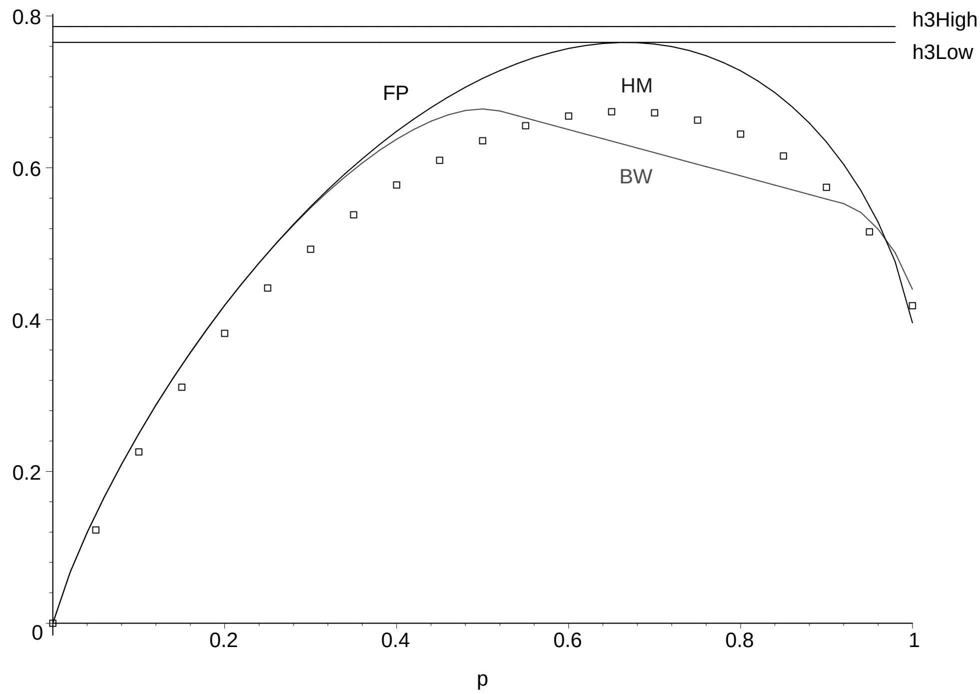


Figure 2: HM is the lower bound of Hammersley-Menon, BW is the lower bound of Bondy-Welsh, FP is the lower bound Friedland-Peled, h3Low and h3High are the best lower and upper bounds for $h_3 = \max \lambda_3(p)$.

15 Low. Bds $\widetilde{h}_3, h_3$ by spec. rad.

$$h_3 \geqslant \frac{\log \beta(p+2q, u+2s) - \log \beta(p+2q, 2s)}{up}$$

$$-\frac{\log \beta(2q, 2v)}{2vp}$$

$$\widetilde{h}_3 \geqslant \frac{\log \widetilde{\beta}(p+2q, u+2s) - \log \widetilde{\beta}(p+2q, 2s)}{up}$$

$$-\frac{\log \widetilde{\beta}(2q, 2v)}{2vp}$$

$$\beta(n,0) = \beta(0,n) = \widetilde{\beta}(n,0) = \widetilde{\beta}(0,n) = 2^n$$

$$n \in \mathbb{N}$$