3-Tensors

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Geometry and Representation Theory of Tensors, MSRI
July 18, 2008

Overview

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- Unfolding and (R_1, R_2, R_3) ranks

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Rank τ denoted $\mathrm{rank}\, \tau$ is the minimal k:

$\tau = \sum_{i=1}^{k} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$

(CANDEC, PARFAC)

Vector ranks of tensors

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Unfolding tensor: in direction 1:
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Similarly, unfolding in directions 2, 3

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Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

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$\exists \mathbf{U}_i \subset \mathbb{F}^{m_i}, \dim \mathbf{U}_i = R_i, i = 1, 2, 3$ s.t. $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$.

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Write $\mathbf{z}_i = \sum_{j=1}^{m_3} z_{i,j} \mathbf{e}_{j,3}$ then each $T_{k,3} \in \mathrm{span}(\mathbf{x}_1 \otimes \mathbf{y}_1, \ldots, \mathbf{x}_p \otimes \mathbf{y}_p)$.

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$\exists \mathbf{U}_i \subset \mathbb{F}^{m_i}, \dim \mathbf{U}_i = R_i, i = 1, 2, 3$ s.t. $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$.

PROP : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let
$T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\mathrm{rank}\, \mathcal{T} =$
minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

PROOF: Suppose $\tau = \sum_{i=1}^{p} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ (1)
Write $\mathbf{z}_i = \sum_{j=1}^{m_3} z_{i,j} \mathbf{e}_{j,3}$ then each $T_{k,3} \in \mathrm{span}(\mathbf{x}_1 \otimes \mathbf{y}_1, \ldots, \mathbf{x}_p \otimes \mathbf{y}_p)$.
Vise versa suppose $T_{k,3} = \sum_{i=1}^{p} a_{k,i} \mathbf{x}_i \otimes \mathbf{y}_i, k = 1, \ldots, m_3$.

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It is possible $\max_l r_l > r_1$

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Choose a new basis in $[\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{m,1}]$ in $\mathbf{U}_1 = \mathbb{C}^m$ given by A^{-1} and
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$\mathcal{T}' = I_m \mathbf{e}_{1,3} + A^{-1}B\mathbf{e}_{2,3} = \sum_{i=1}^m \mathbf{u}_{i,2} \otimes \mathbf{u}_{i,2} \otimes \mathbf{e}_{1,3} + \lambda_i \mathbf{u}_{i,2} \otimes \mathbf{u}_{i,2} \otimes \mathbf{e}_{2,3} =$
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So $\mathrm{rank}\,\mathcal{T} \leq m$. Easy $R_1 = R_2 = m$ for $\mathcal{T}'$. Hence $\mathrm{rank}\,\tau = m$.

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If $A^{-1}B$ has two distinct real eigenvalues, or $A^{-1}B = aI_2$ then $\operatorname{rank} \tau = 2$.
If $A^{-1}B$ has two distinct complex eigenvalues or it is not diagonable $\operatorname{rank} \tau = 3$.
If the subspace spanned by A, B does not contain an invertible matrix then $\operatorname{rank} \tau = 1, 2$.
(This can happen if either $\dim \operatorname{span}(A\mathbb{R}^2, B\mathbb{R}^2) = 1$ or $\dim \operatorname{span}(A^\top \mathbb{R}^2, B^\top \mathbb{R}^2) = 1$.)
For example $\tau = \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{e}_1 + \mathbf{u} \otimes \mathbf{w} \otimes \mathbf{e}_2, \mathbf{u} \neq \mathbf{0}$
If $\mathbf{v}, \mathbf{w}$ linearly independent $\operatorname{rank} \tau = 2$

Algebraic geometry & tensor rank

View tensor one rank matrices as the map

$f : \mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3} \longrightarrow \mathbb{C}^{m_1 \times m_2 \times m_3}$: $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$

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$f_k : (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k \to \mathbb{C}^{m_1 \times m_2 \times m_3}$
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$\mathrm{rank}\, J(f_k) = \dim \mathrm{span}\{\mathbf{e}_{i_1,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3}, \mathbf{x}_{l,1} \otimes \mathbf{e}_{i_2,2} \otimes \mathbf{x}_{l,3}, \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{e}_{i_3,3}\}$,
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Terracini's lemma $\sim$ 1915

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Maximal tensor rank

Lemma: $f_{k-1}((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k-1}) \subsetneqq f_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k)$ for $k = 1, \ldots, \mathrm{mrank}(m_1, m_2, m_3)$ and $f_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = \mathbb{C}^{m_1 \times m_2 \times m_3}$ for $k \geq \mathrm{mrank}(m_1, m_2, m_3)$.

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$\mathrm{mrank}(m_1, m_2, m_3)$ maximal (tensor) rank
(of $\mathcal{T} \in \mathbb{C}^{m_1 \times m_2 \times m_3}$)

Maximal tensor rank

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The computation of $\mathrm{grank}(m_1, m_2, m_3)$ difficult,
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$\mathrm{grank}(l,m,(l-1)(m-1)) \ge \lceil \frac{lm(l-1)(m-1)}{l+m+(l-1)(m-1)+2} \rceil = (l-1)(m-1)+1$

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CON 2 holds in above cases CON 1 holds

Numerical verification of Conjectures 1 &2

We verified numerically[1] the above two conjectures for $m_1 \le m_2 \le m_3 \le 10$, by finding random $k \in [2, \lceil \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rceil]$ vectors $\mathbf{x}_{l,i} \in (\mathbb{Z} \cap [-99, 99])^{m_i}$, $i = 1, 2, 3,\ l = 1, \ldots, k$ such that the rank of the Jacobian matrix at the corresponding rank k tensor

$$\mathcal{T} = \sum_{l=1}^{k} \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3} \qquad (0.1)$$

was $\min(k(m_1 + m_2 + m_3 - 2), m_1 m_2 m_3)$.

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We call (m_1, m_2, m_3) *regular* if (m_1, m_2, m_3) satisfies Conjecture 1 and $\lfloor \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rfloor$ satisfies Conjecture 2.

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COR:

for $n > m \geq 2$: $m + 1 \leq \mathrm{mrank}(2, m, m) \leq 2m - 1$

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$(2, m, n)$ - Kronecker canonical form for $(T_{1,1}, T_{2,1}) \in (\mathbb{C}^{m\times n})^2$

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(3): Assume the worst case:
$T_{1,1}, T_{2,1}, \ldots, T_{n,1}$ lin. ind. Choose new base $S_1, \ldots, S_n$ in $\mathrm{span}(T_{1,1}, \ldots, T_{n,1})$ s.t. $\mathrm{rank}\, S_1 \geq \mathrm{rank}\, S_2 \geq \ldots \geq \mathrm{rank}\, S_n$ and $\mathrm{span}(S_1, \ldots, S_i) = \mathrm{span}(T_{1,1}, \ldots, T_{i,1})$ for $i = 1, \ldots, n$.
$\mathrm{rank}\, S_1 = m, \mathrm{rank}\, S_2 = \mathrm{rank}\, S_3 = \mathrm{rank}\, S_4 = 2,$
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Proof of 2: Radon-Hurwitz numbers, i.e existence of 3- dimensional subspace of 4×4 skew symmetric nonsingular nonzero matrices

(R_1, R_2, R_3)-rank approximation of 3-tensors

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For $\mathbb{F} = \mathbb{C}, \mathbb{R}$ approximate well and fast $\mathcal{T} \in \mathbb{F}^{m_1 \times m_2 \times m_3}$ by rank (R_1, R_2, R_2) 3-tensor.

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When close to a critical point switch to Newton method on
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Similar extensions of CUR approximation to tensors

References

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