Theoretical and Numerical Results and Problems in Tensors

Shmuel Friedland
Univ. Illinois at Chicago

Householder Symposium XVIII
June 14, 2011

Foreword

In the past ten years, tensors again became a hot topic of research in pure and applied mathematics. In applied mathematics it is driven by data which has a few parameters. In pure math. it is quantum information theory, and multilinear algebra. There are many interesting numerical and theoretical problems that need to be resolved. Tensors are related to matrices on one hand and on the other hand are related to polynomial maps.

To paraphrase Max Noether:
Matrices were created by God and tensors by Devil.

Overview

Ranks of 3-tensors

1. Basic facts.
2. Results and conjectures

Approximations of tensors

1. Rank one approximation.
2. Perron-Frobenius theorem
3. Rank (R_1, R_2, R_3) approximations
4. CUR approximations

Diagonal scaling of nonnegative tensors

Maxplus eigenvalue of nonnegative tensors

Characterization of tensor in $\mathbb{C}^{4\times4\times4}$ of border rank 4

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ $\dim \mathbb{U}_i = m_i, i = 1, 2, 3$, $\dim \mathbb{U} = m_1 m_2 m_3$
Tensor $\tau \in \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$

Rank one tensor $t_{i,j,k} = x_i y_j z_k, (i, j, k) = (1, 1, 1), \ldots, (m_1, m_2, m_3)$
or decomposable tensor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$

basis of $\mathbb{U}_j$: $[\mathbf{u}_{1,j}, \ldots, \mathbf{u}_{m_j,j}]$ $j = 1, 2, 3$
basis of $\mathbb{U}$: $\mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}, i_j = 1, \ldots, m_j, j = 1, 2, 3,$
$\tau = \sum_{i_1=i_2=i_3=1}^{m_1,m_2,m_3} t_{i_1,i_2,i_2} \mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}$

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \mathrm{rank}\, A_1$:
dimension of row or column subspace spanned in direction 1

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$
$\mathcal{T} = \sum_{i=1}^{m_1} T_{i,1} \mathbf{e}_{i,1}$ (convenient notation)
$R_1 := \dim \mathrm{span}(T_{1,1}, \ldots, T_{m_1,1})$.

Similarly, unfolding in directions 2, 3

$\mathrm{rank}\, \mathcal{T}$ minimal r:

$\mathcal{T} = f_r(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) := \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$,
(CANDEC, PARFAC)

Basic facts

FACT I: $\mathrm{rank}\ \mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

FACT II : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let
$T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1, m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\mathrm{rank}\ \mathcal{T} =$ minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

COR $\mathrm{rank}\ \mathcal{T} \leq \min(mn, ml, nl)$

Generic and typical ranks

$\mathcal{R}_r(m,n,l) \subset \mathbb{F}^{m\times n\times l}$: all tensors of rank $\le r$

$\mathcal{R}_r(m,n,l)$ not closed variety for $r \ge 2$

Border rank of $\mathcal{T}$ the minimum k s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, $\mathrm{rank}\ T_j = k$.

generic rank is the rank of a random tensor $\mathcal{T} \in \mathbb{C}^{m\times n\times l}$: $\mathrm{grank}(m,n,l)$

typical rank is a rank of a random tensor $\mathcal{T} \in \mathbb{R}^{m\times n\times l}$.

typical rank takes all the values $k = \mathrm{grank}(m,n,l), \ldots, \mathrm{mtrank}(m,n,l)$

In all the examples we know $\mathrm{mtrank}(m,n,l) \le \mathrm{grank}(m,n,l) + 1$

Generic rank of $\mathbb{C}^{m\times n\times l}$

THM: $\mathrm{grank}_{\mathbb{C}}(m,n,l)=\min(l,mn)$ for $(m-1)(n-1)+1\le l$.

Reason: For $l=(m-1)(n-1)+1$ a generic subspace of matrices of dimension l in $\mathbb{C}^{m\times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m\times n}$ at least at l lines which contain l linearly independent matrices

COR: $\mathrm{grank}_{\mathbb{C}}(2,n,l)=\min(l,2n)$ for $2\le n\le l$

Dimension count for $\mathbb{F}=\mathbb{C}$ and $2\le m\le n\le l\le(m-1)(n-1)+1$:

$f_r:(\mathbb{C}^m\times\mathbb{C}^n\times\mathbb{C}^l)^r\to\mathbb{C}^{m\times n\times l}$, $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}=(a\mathbf{x})\otimes(b\mathbf{y})\otimes((ab)^{-1}\mathbf{z})$

$\mathrm{grank}_{\mathbb{C}}(m,n,l)(m+n+l-2)\ge mnl\Rightarrow\mathrm{grank}_{\mathbb{C}}(m,n,l)\ge\lceil\frac{mnl}{(m+n+l-2)}\rceil$

Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l)=\lceil\frac{mnl}{(m+n+l-2)}\rceil$
for $2\le m\le n\le l<(m-1)(n-1)$ and $(3,n,l)\ne(3,2p+1,2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3,2p+1,2p+1)=\lceil\frac{3(2p+1)^2}{4p+3}\rceil+1$

Known cases of rank conjecture

$\mathrm{grank}(3, 2p, 2p) = \lceil \frac{12p^2}{4p+1} \rceil$ and $\mathrm{grank}(3, 2p-1, 2p-1) = \lceil \frac{3(2p-1)^2}{4p-1} \rceil + 1$
$(n, n, n+2)$ if $n \neq 2 \pmod 3$,
$(n-1, n, n)$ if $n = 0 \pmod 3$,
$(4, m, m)$ if $m \geq 4$,
(n, n, n) if $n \geq 4$
$(l, 2p, 2q)$ if $l \leq 2p \leq 2q$ and $\frac{2lp}{l+2p+2q-2}$ is integer

Easy to compute $\mathrm{grank}_{\mathbb{C}}(m, n, l)$:

Pick at random $\mathbf{w}_r := (\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r$
The minimal $r \geq \lceil \frac{mnl}{(m+n+l-2)} \rceil$ s.t. $\mathrm{rank}\, J(f_r)(\mathbf{w}_r) = mnl$
is $\mathrm{grank}_{\mathbb{C}}(m, n, l)$ (Terracini Lemma 1915)

Avoid round-off error:
$\mathbf{w}_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r$ find $\mathrm{rank}\, J(f_r)(\mathbf{w}_r)$ exact arithmetic
I checked the conjecture up to $m, n, l \leq 14$

Generic rank III - the real case

For $mn \leq l$ $\mathrm{mtrank}(m,n,l) = \mathrm{grank}(m,n,l) = mn$.

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

Closure$(\cup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}$
$\mathrm{rank}\ \mathcal{T} = \mathrm{grank}(m,n,l)$ for each $\mathcal{T} \in V_1$
$\mathrm{rank}\ \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$
$\{\rho_1, \ldots, \rho_{c(m,n,l)}\} = \{grank(m,n,l), \ldots, \mathrm{mtrank}(m,n,l)\}$

$\mathrm{mtrank}(2,n,l) = \mathrm{grank}(2,n,l) = \min(l, 2n)$ if $2 \leq n < l$ - one typical rank
$\mathrm{mtrank}(2,n,n) = \mathrm{grank}(2,n,n) + 1 = n + 1$ if $2 \leq n$ - two typical ranks

For $l = (m-1)(n-1) + 1$ $\exists m, n$:
$c(m,n,l) > 1, \mathrm{mtrank}(m,n,l) \geq \mathrm{grank}(m,n,l) + 1$

Examples [5]

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1, \ldots, \mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle^2$
$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Best rank one approximation of $\mathcal{T}$:
$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{T} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\| = \min_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1, a} \|\mathcal{T} - a\,\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|$

Equivalent: $\max_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T} \times \mathbf{y}\otimes\mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda \mathbf{x}$
$\mathcal{T} \times \mathbf{x}\otimes\mathbf{z} = \lambda\mathbf{y}, \ \mathcal{T} \times \mathbf{x}\otimes\mathbf{y} = \lambda\mathbf{z}$
λ singular value, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ singular vectors
How many distinct singular values are for a generic tensor?
(Related result of Cartwright-Sturmfels [1])

ℓ_p maximal problem and Perron-Frobenius

$\|(x_1,\ldots,x_n)^\top\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

Problem: $\max_{\|\mathbf{x}\|_p=\|\mathbf{y}\|_p=\|\mathbf{z}\|_p=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k}x_iy_jz_k$

Lagrange multipliers: $\mathcal{T}\times\mathbf{y}\otimes\mathbf{z} := \sum_{j=k=1} t_{i,j,k}y_jz_k = \lambda\mathbf{x}^{p-1}$
$\mathcal{T}\times\mathbf{x}\otimes\mathbf{z} = \lambda\mathbf{y}^{p-1},\ \mathcal{T}\times\mathbf{x}\otimes\mathbf{y} = \lambda\mathbf{z}^{p-1}$ $(p = \frac{2t}{2s-1}, t, s \in \mathbb{N})$

$p = 3$ is most natural in view of homogeneity

Assume that $\mathcal{T} \geq 0$. Then $\mathbf{x},\mathbf{y},\mathbf{z} \geq 0$

For which values of p we have an analog of Perron-Frobenius theorem?

Yes, for $p \geq 3$, No, for $p < 3$,
Friedland-Gauber-Han [2]

(R_1, R_2, R_3)-rank approximation of 3-tensors

Fundamental problem in applications:
Approximate well and fast $\mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ by rank (R_1, R_2, R_3) 3-tensor.

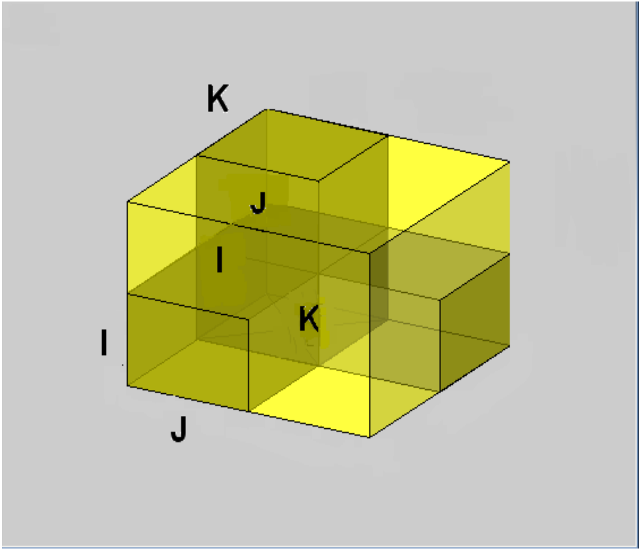
Best (R_1, R_2, R_3) approximation problem:
Find $\mathbb{U}_i \subset \mathbb{F}^{m_i}$ of dimension R_i for $i = 1, 2, 3$ with maximal $||P_{\mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3}(\mathcal{T})||$.

Relaxation method:
Optimize on $\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3$ by fixing all variables except one at a time
This amounts to SVD (Singular Value Decomposition) of matrices:
Fix $\mathbb{U}_2, \mathbb{U}_3$. Then $\mathbb{V} = \mathbb{U}_1 \otimes (\mathbb{U}_2 \otimes \mathbb{U}_3) \subset \mathbb{R}^{m_1 \times (m_2 \cdot m_3)}$

$\max_{\mathbb{U}_1} \|P_{\mathbb{V}}(\mathcal{T})\|$ is an approximation in 2-tensors=matrices

Use Newton method on Grassmannians - Eldén-Savas 2009 [2]

Fast low rank approximation I:



Fast low rank approximations II:

Approximate $A \in \mathbb{R}^{m\times n}$ by CUR where $C \in \mathbb{R}^{m\times p}$, $R \in \mathbb{R}^{q\times n}$ for some submatrices of A.

$\min_{U\in\mathbb{C}^{p\times q}} \|A - CUR\|_F$ achieved for $U = C^\dagger A R^\dagger$

Faster choice: $U = A[I, J]^\dagger$
(corresponds to best CUR approximation on the entries read)
Problem: How to choose good I, J?

For given $\mathcal{A} \in \mathbb{R}^{m\times n\times l}$, $F \in \mathbb{R}^{m\times p}$, $E \in \mathbb{R}^{n\times q}$, $G \in \mathbb{R}^{l\times r}$,
where $\langle p\rangle \subset \langle n\rangle \times \langle l\rangle, \langle q\rangle \subset \langle m\rangle \times \langle l\rangle, \langle r\rangle \subset \langle m\rangle \times \langle l\rangle$

$\min_{\mathcal{U}\in\mathbb{C}^{p\times q\times r}} \|\mathcal{A} - \mathcal{U} \times F \times E \times G\|_F$ achieved for $\mathcal{U} = \mathcal{A} \times E^\dagger \times F^\dagger \times G^\dagger$

CUR approximation of $\mathcal{A}$ obtained by choosing E, F, G submatrices of unfolded $\mathcal{A}$ in the mode $1, 2, 3$.

Scaling of nonnegative tensors to tensors with given row, column and depth sums

$0 \le \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m\times n\times l}$ has given row, column and depth sums:
$\mathbf{r} = (r_1,\dots,r_m)^\top, \mathbf{c} = (c_1,\dots,c_n)^\top, \mathbf{d} = (d_1,\dots,d_l)^\top > \mathbf{0}$:
$\sum_{j,k} t_{i,j,k} = r_i > 0,\ \sum_{i,k} t_{i,j,k} = c_j > 0, \sum_{i,j} t_{i,j,k} = d_k > 0$
$\sum_{i=1}^m r_i = \sum_{j=1}^n c_j = \sum_{k=1}^l d_k$

Find nec. and suf. conditions for scaling:
$\mathcal{T}' = [t_{i,j,k}e^{x_i+y_j+z_k}], \mathbf{x},\mathbf{y},\mathbf{z}$ such that $\mathcal{T}'$ has given row, column and depth sum
Solution: Convert to the minimal problem:
$\min_{\mathbf{r}^\top\mathbf{x}=\mathbf{c}^\top\mathbf{y}=\mathbf{d}^\top\mathbf{z}=0} f_{\mathcal{T}}(\mathbf{x},\mathbf{y},\mathbf{z}), \quad f_{\mathcal{T}}(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i,j,k} t_{i,j,k}e^{x_i+y_j+z_k}$

Any critical point of $f_{\mathcal{T}}$ on $\mathcal{S} := \{\mathbf{r}^\top\mathbf{x} = \mathbf{c}^\top\mathbf{y} = \mathbf{d}^\top\mathbf{z} = 0\}$ gives rise to a solution of the scaling problem (Lagrange multipliers)
$f_{\mathcal{T}}$ is convex
$f_{\mathcal{T}}$ is strictly convex implies $\mathcal{T}$ is not decomposable: $\mathcal{T} \ne \mathcal{T}_1 \oplus \mathcal{T}_2$.

Scaling of nonnegative tensors II

if $f_{\mathcal{T}}$ is strictly convex and is ∞ on $\partial\mathcal{S}$, $f_{\mathcal{T}}$ achieves its unique minimum

Equivalent to: the inequalities $x_i + y_j + z_k \le 0$ if $t_{i,j,k} > 0$ and equalities $\mathbf{r}^\top \mathbf{x} = \mathbf{c}^\top \mathbf{y} = \mathbf{d}^\top \mathbf{z} = 0$ imply $\mathbf{x} = \mathbf{0}_m, \mathbf{y} = \mathbf{0}_n, \mathbf{z} = \mathbf{0}_l$.

Fact: For $\mathbf{r} = \mathbf{1}_m, \mathbf{c} = \mathbf{1}_n, \mathbf{d} = \mathbf{1}_l$ Sinkhorn scaling algorithm works.

Newton method works, since the scaling problem is equivalent finding the unique minimum of strict convex function

Hence Newton method has a quadratic convergence versus linear convergence of Sinkhorn algorithm
True for matrices too

$\rho_{\mathrm{trop}}(\mathcal{F})$ - maxplus eigenvalue

if $\mathcal{F}$ weakly irreducible then $\mathcal{F}$ has positive tropical eigenvector
$\max_{i_2,\ldots,i_d} f_{i,i_2,\ldots,i_d} x_{i_2} \ldots x_{i_d} = \lambda x_i^{d-1}, \quad i \in [n], \mathbf{x} > \mathbf{0}$

generalization of Engel-Schneider [3], (Collatz-Wielandt)
$\rho_{\mathrm{trop}}(\mathcal{F}) = \inf_{(t_1,\ldots,t_n)^\top \in \mathbb{R}^n} \max_{i_1,\ldots,i_d} e^{-dt_{i_1} + \sum_{j=1}^{d} t_{i_j}} f_{i_1,\ldots,i_d}$

Friedland 1986: $\rho_{\mathrm{trop}}(A)$ (maximal tropical eigenvalue)
is the maximum geometric average of cycle products of $A \in \mathbb{R}_+^{n\times n}$.

$d-1$ cycle on $[m]$ vertices is $d-1$ regular strongly connected digraph
$D = ([m], A)$,
i.e. indegree and outdegree of each vertex is $d-1$,
i.e. the digraph adjacency matrix $A(D) \in \mathbb{Z}_+^{m\times m}$ has row and column sum $d-1$.

Friedland-Gaubert: $\rho_{\mathrm{trop}}(\mathcal{F})$ is the maximum geometric average of cycle products of $\mathcal{F} \in ((\mathbb{R}^n)^{\otimes d})_+$

Characterization of tensor in $\mathbb{C}^{4\times 4\times 4}$ of border rank 4

Major problem in algebraic statistics:
phylogenic trees and their invariants [1]:
Characterize tensors of border rank at most 4 in $\mathbb{C}^{4\times 4\times 4}$

$\mathbf{W} \subset \mathbb{C}^{4\times 4}$ subspace spanned by four sections of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$
If $\mathbf{W}$ contains identity matrix then $\mathbf{W}$ space of commuting matrices

If $\mathbf{W}$ contains an invertible matrix Z then any other $X, Y \in \mathbf{W}$ satisfy
$X(\mathrm{adj}Z)Y = Y(\mathrm{adj}Z)X$ - equations of degree 5
Landsberg-Manivel showed that there is an additional set of degree 6 equations

Degree 9 symmetrization conditions for $3 \times 3 \times 4$ subtensor of $\mathcal{T}$

Friedland [1] one needs a equations of degree 16
Friedland-Gross [3]: $5, 6, 9$ degrees suffice

References I

- E.S. Allman and J.A. Rhodes, Phylogenic ideals and varieties for general Markov model, *Advances in Appl. Math.*, 40 (2008) 127-148.
- R.B. Bapat D_1AD_2 theorems for multidimensional matrices, *Linear Algebra Appl.* 48 (1982), 437-442.
- R.B. Bapat and T.E.S. Raghavan, An extension of a theorem of Darroch and Ratcliff in loglinear models and its application to scaling multidimensional matrices, *Linear Algebra Appl.* 114/115 (1989), 705-715.
- R.A. Brualdi, Convex sets of nonnegative matrices, *Canad. J. Math* 20 (1968), 144-157.
- R.A. Brualdi, S.V. Parter and H. Schneider, The diagonal equivalence of a nonnegative matrix to a stochastic matrix, *J. Math. Anal. Appl.* 16 (1966), 31–50.

References II

- D. Cartwright and B. Sturmfels, The Number of Eigenvalues of a Tensor, arXiv:1004.4953, to appear in *Linear Algebra Appl.* .
- Lar. Eldén and B. Savas, A Newton Grassmann method for computing the Best Multi-Linear Rank-(r1; r2; r3) Approximation of a Tensor, *SIAM J. Matrix Anal. Appl.* 31 (2009), 248–271.
- G.M. Engel and H. Schneider, Diagonal similarity and equivalence for matrices over groups with 0, *Czechoslovak Mathematical Journal* 25 (1975), 389–403.
- J. Franklin and J. Lorenz, On the scaling of multidimensional matrices, *Linear Algebra Appl.* 114/115 (1989), 717-735.
- S. Friedland, On the generic rank of 3-tensors, arXiv: 0805.3777, to appear in *Linear Algebra Appl.* .
- S. Friedland, Positive diagonal scaling of a nonnegative tensor to one with prescribed slice sums, *Linear Algebra Appl.*, 434 (2011), 1615–1619.

References III

- S. Friedland,On tensors of border rank l in $\mathbb{C}^{m\times n\times l}$, arXiv:1003.1968, to appear in *Linear Algebra Appl.* .
- S. Friedland, S. Gauber and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms, *arXiv:0905.1626*, to appear in *Linear Algebra Appl.* .
- S. Friedland and E. Gross, A proof of the set-theoretic version of the salmon conjecture, arXiv:1104.1776.
- S. Friedland, V. Mehrmann, A. Miedlar, and M. Nkengla, Fast low rank approximations of matrices and tensors, to appear in *ELA*, www.matheon.de/preprints/4903.
- S. Friedland and V. Mehrmann, Best subspace tensor approximations, arXiv:0805.4220,

References IV

- S.A. Goreinov, E.E. Tyrtyshnikov, N.L. Zmarashkin, A theory of pseudo-skeleton approximations of matrices, *Linear Algebra Appl.* 261 (1997), 1-21.
- L.H. Lim, Singular values and eigenvalues of tensors: a variational approach, *CAMSAP* 05, 1 (2005), 129-132.
- M.W. Mahoney and P. Drineas, CUR matrix decompositions for improved data analysis, *PNAS* 106, (2009), 697-702.
- M.V. Menon, Matrix links, an extremisation problem and the reduction of a nonnegative matrix to one with with prescribed row and column sums, *Canad. J. Math* 20 (1968), 225-232.
- V. Strassen, Rank and optimal computations of generic tensors, *Linear Algebra Appl.* 52/53: 645-685, 1983.