Perron-Frobenius theorem for nonnegative multilinear forms and extensions

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Overview

1. Perron-Frobenius theorem for *irreducible* nonnegative matrices
2. Irreducibility and weak irreducibility for tensors
3. Perron-Frobenius for irreducible tensors
4. Rank one approximation of tensors
5. Nonnegative multilinear forms
6. Monotone homogeneous maps
7. Polynomial homogeneous maps
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9. Power iterations
10. Numerical counterexamples

Nonnegative irreducible and primitive matrices

$A = [a_{ij}] \in \mathbb{R}_+^{n\times n}$ induces digraph $DG(A) = DG = (V, E)$

$V = [n] := \{1, \ldots, n\}, \; E \subseteq [n] \times [n], \; (i, j) \in E \iff a_{ij} > 0$

DG strongly connected, SC,
if for each pair $i \neq j$ there exists a dipath from i to j

Claim: DG SC iff for each $\emptyset \neq I \subset [n] \; \exists j \in [n] \setminus I$ s.t. $(i, j) \in E$

A -primitive if $A^N > 0$ for some $N > 0 \iff A^N(\mathbb{R}_+^n \setminus \{\mathbf{0}\}) \subset \mathrm{int}\, \mathbb{R}_+^n$

A primitive $\iff$ A irreducible and g.c.d of all cycles in $DG(A)$ is one

Perron-Frobenius theorem

PF: $A \in \mathbb{R}_+^{n}$ irreducible. Then $0 < \rho(A) \in \mathrm{spec}\ (A)$ algebraically simple
$\mathbf{x}, \mathbf{y} > \mathbf{0}$ $A\mathbf{x} = \rho(A)\mathbf{x}, A^\top \mathbf{y} = \rho(A)\mathbf{y}$.

$A \in \mathbb{R}_+^{n\times n}$ primitive iff in addition to above $|\lambda| < \rho(A)$ for
$\lambda \in \mathrm{spec}\ (A) \setminus \{\rho(A)\}$

Collatz-Wielandt:

$$\rho(A) = \min_{\mathbf{x}>\mathbf{0}} \max_{i\in[n]} \frac{(A\mathbf{x})_i}{x_i} = \max_{\mathbf{x}>\mathbf{0}} \min_{i\in[n]} \frac{(A\mathbf{x})_i}{x_i}$$

SVD

$A \in \mathbb{R}^{m \times n}$, $\sigma_1(A) \geq \ldots \geq 0$ singular values

$A\mathbf{y}_i = \sigma_i(A)\mathbf{x}_i, \; A^\top \mathbf{x}_i = \sigma_i(A)\mathbf{y}_i$

$\pm\sigma_i(A), i = 1, \ldots$ are critical values of $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top A\mathbf{y}$
restricted to $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$

SVD of A closely related to spectral theory

$$B = \begin{bmatrix} 0_{m\times m} & A \\ A^\top & 0_{n\times n} \end{bmatrix}, \quad -\lambda(B) = \lambda(B)$$

positive singular values are the positive eigenvalues of B

$\sigma_1(A) = \max_{\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1} \mathbf{y}^\top A\mathbf{x}$

SVD of nonnegative matrices

Perron-Frobenius for $A = [a_{ij}] \in \mathbb{R}_+^{m \times n}$:

$\mathbf{u} \in \mathbb{R}_+^m, \mathbf{v} \in \mathbb{R}_+^n, \ \mathbf{u}^\top \mathbf{u} = \mathbf{v}^\top \mathbf{v} = 1 \ A\mathbf{v} = \sigma_1(A)\mathbf{u}, \ A^\top \mathbf{u} = \sigma_1(A)\mathbf{v}$

$\sigma_1(A) = \max_{\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n, \|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1} \mathbf{x}^\top A\mathbf{y} = \mathbf{u}^\top A\mathbf{v}$.

$G(A) := DG(B) = G(B) = (V_1 \cup V_2, E)$ bipartite graph on $V_1 = [m], V_2 = [n], (i, j) \in E \iff a_{ij} > 0$.

If $G(A)$ connected. Then $\mathbf{u}, \mathbf{v}$ unique, $\sigma_2(A) < \sigma_1(A)$, (as B-irreducible).

Irreducibility and weak irreducibility of nonnegative tensors

$\mathcal{F} := [f_{i_1,\ldots,i_d}] \in \otimes_{i=1}^d \mathbb{R}^{m_i} = \mathbb{R}^{m_1\times\cdots\times m_d}$ is called d-tensor, $(d \geq 3)$

$\mathcal{T} \geq 0$ if $\mathcal{T} \in \mathbb{R}_+^{m_1\times\cdots\times m_d}$

$G(\mathcal{F}) = (V, E(\mathcal{F}))$, $V = \cup_{j=1}^d V_j$, d-partite graph $V_j = [m_j], j \in [d]$.

$(i_k, i_l) \in V_k \times V_l, k \neq l$ is in $E(\mathcal{F})$ iff $f_{i_1,i_2,\ldots,i_d} > 0$ for some $d-2$ indices $\{i_1,\ldots,i_d\}\backslash\{i_k,i_l\}$.

$\mathcal{F}$ weakly irreducible if $G(\mathcal{F})$ is connected.

$\mathcal{F}$ irreducible: for each $\emptyset \neq I \subsetneqq V, J := V\backslash I$ there exists $k \in [d]$, $i_k \in I \cap V_k$ and $i_j \in J \cap V_j$ for each $j \in [d]\backslash\{k\}$ such that $f_{i_1,\ldots,i_d} > 0$.

Claim: irreducible implies weak irreducible

Perron-Frobenius theorem for nonnegative tensors I

$\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \otimes_{i=1}^d \mathbb{C}^n$ maps $\mathbb{C}^n$ to itself

$\mathcal{T}(\mathbf{x})_i = \sum_{i_2,\ldots,i_d \in [n]} t_{i,i_2,\ldots,i_d} x_{i_2} \ldots x_{i_d}, \ i \in [n]$

$\mathcal{T}$ has eigenvector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ with eigenvalue λ:

$\mathcal{T}(\mathbf{x})_i = \lambda x_i^{d-1}$ for all $i \in [n]$

Assume: $\mathcal{T} \geq 0, \mathcal{T}(\mathbb{R}_+^n \setminus \{\mathbf{0}\}) \subseteq \mathbb{R}_+^n \setminus \{\mathbf{0}\}$

$$\mathcal{T}_1 : \Pi_n \to \Pi_n, \quad \mathbf{x} \mapsto \frac{1}{\sum_{i=1}^n \mathcal{T}(\mathbf{x})_i^{\frac{1}{d-1}}} (\mathcal{T}(\mathbf{x}))^{\frac{1}{d-1}}$$

Brouwer fixed point: $\mathbf{x} \gneq \mathbf{0}$ eigenvector with $\lambda > 0$ eigenvalue

Problem When there is a unique positive eigenvector with maximal eigenvalue?

Perron-Frobenius theorem for nonnegative tensors II

Theorem Chang-Pearson-Zhang 2009 [2]
Assume $\mathcal{T} \in (\otimes_{i=1}^{d} \mathbb{R}^n)_+$ is irreducible.
Then there exists a unique nonnegative eigenvector which is positive with the corresponding maximum eigenvalue λ.
Furthermore the Collatz-Wielandt characterization holds

$$\lambda = \min_{\mathbf{x}>0} \max_{i\in[n]} \frac{(\mathcal{T}(\mathbf{x}))_i}{x_i^{d-1}} = \max_{\mathbf{x}>0} \min_{i\in[n]} \frac{(\mathcal{T}(\mathbf{x}))_i}{x_i^{d-1}}$$

Rank one approximations for 3-tensors

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\|_2 = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$

$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

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$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1, \ldots, \mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|_2^2 = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|_2^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|_2^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|_2^2$

Best rank one approximation of $\mathcal{T}$:

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$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1, \ldots, \mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|_2^2 = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|_2^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|_2^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|_2^2$

Best rank one approximation of $\mathcal{T}$:
$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{T} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|_2 = \min_{\|\mathbf{x}\|_2=\|\mathbf{y}\|_2=\|\mathbf{z}\|_2=1, a} \|\mathcal{T} - a\,\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|_2$

Equivalent: $\max_{\|\mathbf{x}\|_2=\|\mathbf{y}\|_2=\|\mathbf{z}\|_2=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T} \times \mathbf{y}\otimes\mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda \mathbf{x}$
$\mathcal{T} \times \mathbf{x}\otimes\mathbf{z} = \lambda\mathbf{y}, \ \mathcal{T} \times \mathbf{x}\otimes\mathbf{y} = \lambda\mathbf{z}$
λ singular value, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ singular vectors

Nonnegative multilinear forms

Associate with $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{R}_+^{m_1\times\ldots\times m_d}$
a multilinear form $f(\mathbf{x}_1,\ldots\mathbf{x}_d) : \mathbb{R}^{m_1\times\ldots\times m_d} \to \mathbb{R}$

$f(\mathbf{x}_1,\ldots,\mathbf{x}_d) = \sum_{i_j\in[m_j],j\in[d]} t_{i_1,\ldots,i_d} x_{i_1,1}\ldots x_{i_d,d}$,
$\mathbf{x}_i = (x_{1,i},\ldots,x_{m_i,i} \in \mathbb{R}^{m_i}$

For $\mathbf{u} \in \mathbb{R}^m, p \in (0,\infty]$ let $\|\mathbf{u}\|_p := (\sum_{i=1}^m |u_i|^p)^{\frac{1}{p}}$ and
$\mathrm{S}_{p,+}^{m-1} := \{\mathbf{0} \le \mathbf{u} \in \mathbb{R}^m,\ \|\mathbf{u}\|_p = 1\}$

For $p_1,\ldots,p_d \in (1,\infty)$ critical point $(\xi_1,\ldots,\xi_d) \in \mathrm{S}_{p_1,+}^{m_1-1}\times\ldots\times\mathrm{S}_{p_d,+}^{m_d-1}$
of $f|\mathrm{S}_{p_1,+}^{m_1-1}\times\ldots\times\mathrm{S}_{p_d,+}^{m_d-1}$ satisfies Lim [1]:
$\sum t_{i_1,\ldots,i_d} x_{i_1,1}\ldots x_{i_{j-1},j-1} x_{i_{j+1},j+1}\ldots x_{i_d,d} = \lambda x_{i_j,j}^{p_j-1}$,
$i_j \in [m_j], \mathbf{x}_j \in \mathrm{S}_{m_j,+}^{p_j-1}, j \in [d]$

Perron-Frobenius theorem for nonnegative multilinear forms

Theorem- Friedland-Gauber-Han [3]
$f : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_d} \to \mathbb{R}$, a nonnegative multilinear form,
$\mathcal{T}$ weakly irreducible and $p_j \geq d$ for $j \in [d]$.
Then f has unique positive critical point on $\mathrm{S}_+^{m_1-1} \times \ldots \times \mathrm{S}_+^{m_d-1}$.
If $\mathcal{F}$ is irreducible then f has a unique nonnegative critical point which is necessarily positive

Eigenvectors of homogeneous monotone maps on $\mathbb{R}^n_+$

Hilbert metric on $\mathbb{PR}^n_{>0}$: for $\mathbf{x} = (x_1, \ldots, x_n)^\top, \mathbf{y} = (y_1, \ldots, y_n)^\top > \mathbf{0}$.
Then $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \max_{i \in [n]} \log \frac{y_i}{x_i} - \min_{i \in [n]} \log \frac{y_i}{x_i}$.

$\mathbf{F} = (F_1, \ldots, F_n)^\top : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ homogeneous:
$\mathbf{F}(t\mathbf{x}) = t\mathbf{F}(\mathbf{x})$ for $t > 0, \mathbf{x} > \mathbf{0}$, and monotone $\mathbf{F}(\mathbf{y}) \geq \mathbf{F}(\mathbf{x})$ if $\mathbf{y} \geq \mathbf{x} > \mathbf{0}$.
$\mathbf{F}$ induces $\hat{\mathbf{F}} : \mathbb{PR}^n_{>0} \to \mathbb{PR}^n_{>0}$

$\mathbf{F}$ nonexpansive with respect to Hilbert metric
$\mathrm{dist}(\mathbf{F}(\mathbf{x}), \mathbf{F}(\mathbf{y})) \leq \mathrm{dist}(\mathbf{x}, \mathbf{y})$.

$\alpha_{max}\mathbf{x} \leq \mathbf{y} \leq \beta_{\min}\mathbf{x} \Rightarrow$
$\alpha_{\max}\mathbf{F}(\mathbf{x}) = \mathbf{F}(\alpha_{\max}\mathbf{x}) \leq \mathbf{F}(\mathbf{y}) \leq \mathbf{F}(\beta_{\min}\mathbf{x}) = \beta_{\min}\mathbf{F}(\mathbf{x})$
$\Rightarrow \mathrm{dist}(\mathbf{F}(\mathbf{x}), \mathbf{F}(\mathbf{y})) \leq \log \frac{\beta_{\min}}{\alpha_{\max}} = \mathrm{dist}(\mathbf{x}, \mathbf{y})$

$\mathbf{x} > \mathbf{0}$ eigenvector of $\mathbf{F}$ if $\mathbf{F}(\mathbf{x}) = \lambda\mathbf{F}(\mathbf{x})$.
So $\mathbf{x} \in \mathbb{PR}^n_+$ fixed point of $\mathbf{F}|\mathbb{PR}^n_+$.

Existence of positive eigenvectors of **F**

1. If **F** contraction: $\mathrm{dist}(\mathbf{F}(\mathbf{x}), \mathbf{F}(\mathbf{y})) \le K\mathrm{dist}(\mathbf{x}, \mathbf{y})$ for $K < 1$, then **F** has unique fixed point in $\mathbb{PR}^n_+$ and power iterations converge to the fixed point

2. Use Brouwer fixed and irreducibility to deduce existence of positive eigenvector

3. [4, Theorem 2]: for $u \in (0, \infty), J \subseteq [n]$ let $\mathbf{u}_J = (u_1, \ldots, u_n)^\top > \mathbf{0}$: $u_i = u$ if $i \in J$ and $u_i = 1$ if $i \notin J$. $F_i(\mathbf{u}_J)$ nondecreasing in u. di-graph $\mathcal{G}(\mathbf{F}) \subset [n] \times [n]$: $(i, j) \in \mathcal{G}(\mathbf{F})$ iff $\lim_{u \to \infty} F_i(\mathbf{u}_{\{j\}}) = \infty$.

Thm: $\mathbf{F} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ homogeneous and monotone. If $\mathcal{G}(\mathbf{F})$ strongly connected then **F** has positive eigenvector

Uniqueness and convergence of power method for F

Thm 2.5, Nussbaum 88: $\mathbf{F}:\mathbb{R}^n_{>0}\to\mathbb{R}^n_{>0}$ homogeneous and monotone. Assume; $\mathbf{u}>\mathbf{0}$ eigenvector $\mathbf{F}$ with the eigenvalue $\lambda>0$, $\mathbf{F}$ is C^1 in some open neighborhood of $\mathbf{u}$, $A=\mathrm{D}\mathbf{F}(\mathbf{u})\in\mathbb{R}_+^{n\times n}$ $\rho(A)(=\lambda)$ a simple root of $\det(xI-A)$. Then $\mathbf{u}$ is a unique eigenvector of $\mathbf{F}$ in $\mathbb{R}^n_{>0}$.

Cor 2.5, Nus88: In the above theorem assume $A=\mathrm{D}\mathbf{F}(\mathbf{u})$ is primitive.
Let $\psi\gneq\mathbf{0},\psi^\top\mathbf{u}=1$.
Define $\mathbf{G}:\mathbb{R}^n_{>0}\to\mathbb{R}^n_{>0}$ $\mathbf{G}(\mathbf{x})=\frac{1}{\psi^\top\mathbf{F}(\mathbf{x})}\mathbf{F}(\mathbf{x})$
Then $\lim_{m\to\infty}\mathbf{G}^{\circ m}(\mathbf{x})=\mathbf{u}$ for each $\mathbf{x}\in\mathbb{R}^n_{>0}$.

Outline of the uniqueness of pos. crit. point of f

Define: $F : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+ \to \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$:

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{i,1} = \left(\|\mathbf{x}\|_p^{p-3} \sum_{j=k=1}^{n,l} t_{i,j,k} y_j z_k\right)^{\frac{1}{p-1}}, i = 1,\ldots,m$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} t_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\ldots,n$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{k,3} = \left(\|\mathbf{z}\|_p^{p-3} \sum_{i=j=1}^{m,n} t_{i,j,k} x_i y_j\right)^{\frac{1}{p-1}}, k = 1,\ldots,l$$

Assume $\sum_{j=k=1}^{n,l} t_{i,j,k} > 0, i = 1,\ldots,m$,
$\sum_{i=k=1}^{m,l} t_{i,j,k} > 0, j = 1,\ldots,n$, $\sum_{i=j=1}^{m,n} t_{i,j,k} > 0, k = 1,\ldots,l$

F 1-homogeneous monotone, maps open positive cone $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ to itself.

$\mathcal{T} = [t_{i,j,k}]$ induces tri-partite graph on $\langle m\rangle, \langle n\rangle, \langle l\rangle$:
$i \in \langle m\rangle$ connected to $j \in \langle n\rangle$ and $k \in \langle l\rangle$ iff $t_{i,j,k} > 0$, sim. for j, k

If tri-partite graph is connected then F has unique positive eigenvector

If F completely irreducible, i.e. F^N maps nonzero nonnegative vectors to positive, nonnegative eigenvector is unique and positive

Perron-Frobenius theorem for nonnegative polynomial maps

$\mathbf{P} = (P_1, \ldots, P_n)^\top : \mathbb{R}^n \to \mathbb{R}^n$ polynomial map $\deg P_i = d_i \geq 1$ with nonnegative coefficients. Let $\delta_i \geq d_i, i \in [n]$.
Consider the system $P_i(\mathbf{x}) = \lambda x_i^{\delta_i}, i \in [n], \mathbf{x} \geq \mathbf{0}$
Assume $\mathbf{P}$ weakly irreducible. Then for each $a, p > 0$ $\exists! \mathbf{x} > \mathbf{0}$, depending on a, p, satisfying above equation and $\|\mathbf{x}\|_p = a$.
If $\mathbf{P}$ irreducible then above system has a unique solution, depending on a, p satisfying $\|\mathbf{x}\|_p = a$, with all coordinates positive

Collatz-Wielandt

Collatz-Wielandt [1, Lemma 2.8]:
$\mathbf{P} = (P_1, \ldots, P_n)^\top : \mathbb{R}^n \to \mathbb{R}^n$, P_i homogeneous polynomial of degree $d \geq 1$ with nonnegative coefficients.
Assume $\mathbf{P}$ weakly irreducible. Then there exists unique scalar λ, $\mathbf{u}$ with $P_i(\mathbf{u}) = \lambda u_i^d, i \in [n]$ which satisfies
$\lambda = \inf_{\mathbf{x} \in \text{interior } \mathbb{R}_+^n} \max_{i \in [n]} \frac{P_i(\mathbf{x})}{x_i^d} =$
$\sup_{\mathbf{x} \in \mathbb{R}_+^n \setminus \{0\}} \min_{\substack{i \in [n] \\ x_i \neq 0}} \frac{P_i(\mathbf{x})}{x_i^d}$

Geometric convergence of power algorithm

P weakly primitive if the di-graph $G(\mathbf{P})$ is strongly connected and if the gcd of the lengths of its circuits is equal to one.

Cor. 2.5, Nussbaum 88 yields

Thm: Let **P** and d be above and assume that **P** is weakly primitive. Then

$x_i^{(k+1)} = (\psi^\top \mathbf{F}(\mathbf{x}^{(k)}))^{-1} F_i(\mathbf{x}^{(k)}), k = 1, \ldots,$

converges to the unique vector $\mathbf{u} \in \text{interior } \mathbb{R}_+^n$ satisfying

$P_i(\mathbf{u}) = \lambda u_i^d, i \in [n]$, where $\psi^\top \mathbf{u} = 1$

Numerical counterexamples

Numerical counterexamples

$\mathcal{F} := [f_{i,j,k}] \in \mathbb{R}_+^{2\times 2\times 2}$: $f_{1,1,1} = f_{2,2,2} = a > 0$ otherwise, $f_{i,j,k} = b > 0$.

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = b(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) + (a - b)(x_1 y_1 z_1 + x_2 y_2 z_2).$$

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For $p_1 = p_2 = p_3 = p > 1$ positive singular vectors:
$\mathbf{x} = \mathbf{y} = \mathbf{z} = (0.5^{1/p}, 0.5^{1/p})^\top$.

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For $a = 1.2, b = 0.2$ and $p = 2$ additional positive singular vectors:
$\mathbf{x} = \mathbf{y} = \mathbf{z} \approx (0.9342, 0.3568)^\top$,
$\mathbf{x} = \mathbf{y} = \mathbf{z} \approx (0.3568, 0.9342)^\top$.

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For $a = 1.001, b = 0.001$ and $p = 2.99$ additional positive singular vectors:
$\mathbf{x} = \mathbf{y} = \mathbf{z} \approx (0.9667, 0.4570)^\top$,
$\mathbf{x} = \mathbf{y} = \mathbf{z} \approx (0.4570, 0.9667)^\top$

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