An upper bound for the minimum rank of a graph*

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Abstract

For a graph G of order n, the minimum rank of G is defined to be the smallest possible rank over all real symmetric $n \times n$ matrices A whose (i, j)th entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. We prove an upper bound for minimum rank in terms of minimum degree of a vertex is valid for many graphs, including all bipartite graphs, and conjecture this bound is true over for all graphs, and prove a related bound for all zero-nonzero patterns of (not necessarily symmetric) matrices. Most of the results are valid for matrices over any infinite field, but need not be true for matrices over finite fields.

1 Introduction

The (symmetric) minimum rank problem for a simple graph asks us to determine the minimum rank among real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph G. The solution to the minimum rank problem is equivalent to the determination of the maximum multiplicity of an eigenvalue among the same family of matrices.

This problem, and its extension to symmetric matrices over other fields, have received considerable attention recently. See [7] for a survey of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there. The AIM Minimum Rank Graph Catalog [2] is available on-line and is updated routinely.

A graph will be denoted by $G = (V(G), E(G))$. All graphs discussed in this paper are simple, meaning no loops or multiple edges, undirected, and have finite nonempty vertex sets. The *degree* of a vertex is the number of edges incident with the vertex, and the *minimum degree* over all vertices of a graph G will be denoted by $\delta(G)$. A graph is *connected* if there is a path from any vertex to any other vertex. A *component* of a graph is a maximal connected subgraph. A *cut-vertex* of a connected graph is a vertex whose deletion disconnects G.

All fields discussed are infinite except when a specific finite field is mentioned by name; F denotes an infinite field, $M_n(F)$ denotes the $n \times n$ matrices over F, and $S_n(F)$ denotes the symmetric $n \times n$ matrices over F.

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For $A \in S_n(F)$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i,j\}|\ a_{ij} \neq 0 \text{ and } i \neq j\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The *set of symmetric matrices* of graph G over field F is

$$\mathcal{S}_G^F = \{A \in S_n(F) :\ \mathcal{G}(A) = G\}.$$

We will be interested in minimum rank over a variety of fields and will also need to consider the minimum rank of a family of not-necessarily symmetric matrices. We adopt the perspective that we are finding the minimum of the ranks of the matrices in a given family $\mathcal{F}$ of matrices, and define

$$\mathrm{mr}(\mathcal{F}) = \min\{\mathrm{rank}(A) :\ A \in \mathcal{F}\}.$$

Note that what we are denoting by $\mathrm{mr}(\mathcal{S}_G^{\mathbb{R}})$ is commonly denoted by $\mathrm{mr}(G)$ in papers that study only the minimum rank of the real symmetric matrices described by a graph, and $\mathrm{mr}(\mathcal{S}_G^F)$ is sometimes denoted by $\mathrm{mr}(F, G)$ or $\mathrm{mr}^F(G)$. If G is the union of disjoint components $G_i, i = 1, \ldots, t$, then $\mathrm{mr}(\mathcal{S}_G^F) = \sum_{i=1}^{t} \mathrm{mr}(\mathcal{S}_{G_i}^F)$, so it is customary to restrict consideration to connected graphs.

The *nullity* (or *corank*) of an $n \times n$ matrix A is the dimension of the kernel of A. Let $M(\mathcal{F})$ denote the *maximum nullity* (or *maximum corank*) among matrices in $\mathcal{F}$. For $\mathcal{F} \subseteq M_n(F)$, it is immediate that

$$\mathrm{mr}(\mathcal{F}) + M(\mathcal{F}) = n. \tag{1}$$

The maximum nullity is, of course, the maximum geometric multiplicity of eigenvalue 0, but the geometric multiplicity may be less than the algebraic multiplicity (except for real symmetric matrices). If the family allows translation by an arbitrary scalar multiple of the identity matrix, then the maximum geometric multiplicity of any eigenvalue is the same. For real symmetric matrices, where geometric and algebraic multiplicity are the same, the minimum rank problem is often studied from the perspective of the problem of determining the maximum eigenvalue multiplicity of (any) eigenvalue.

It is clear from Equation (1) that a lower bound on $M(\mathcal{F})$ gives rise to an associated upper bound on $\mathrm{mr}(\mathcal{F})$ and an upper bound on $M(\mathcal{F})$ gives rise to an associated lower bound on $\mathrm{mr}(\mathcal{F})$ (and vice versa). One strategy for computation of minimum rank, which was used extensively in [1], is to obtain equal upper and lower bounds for $\mathrm{mr}(\mathcal{S}_G^{\mathbb{R}})$ (or equivalently, $M(\mathcal{S}_G^{\mathbb{R}})$).

We are interested in the relationship between the minimum degree $\delta(G)$ and the maximum nullity of symmetric matrices, and make the following conjecture.

Conjecture 1.1. *For any graph G and infinite field F,*

$$\delta(G) \leq M(\mathcal{S}_G^F), \tag{2}$$

or equivalently,

$$\mathrm{mr}(\mathcal{S}_G^F) \leq |G| - \delta(G). \tag{3}$$

It is clear that the bounds in Conjecture 1.1 are satisfied with equality for the complete graph K_n, the path P_n, the cycle C_n, and any graph having a vertex of degree one. We will establish the conjecture for a variety of graphs, including small graphs, more than twenty families of graphs, and all bipartite graphs. We will also establish requirements on a minimal counterexample, should one exist.

Note that Conjecture 1.1 assumes the field is infinite. The bounds in Conjecture 1.1 can fail for finite fields as seen in the next example.

Example 1.2. [1, Example 3.4]

$$\mathrm{mr}(\mathcal{S}_{K_3 \square K_2}^{\mathbb{Z}_2}) = 4 > 3 = 6 - 3 = |K_3 \square K_2| - \delta(K_3 \square K_2) = \mathrm{mr}(\mathcal{S}_{K_3 \square K_2}^{\mathbb{R}}).$$

The conjecture, if established, provides an upper bound for minimum rank. Such a bound can sometimes be used in conjunction a lower for minimum rank to determine the minimum rank.

Several lower bounds for minimum rank are known. An upper bound for $M(\mathcal{S}_G^F)$, which yields an associated lower bound for $\mathrm{mr}(\mathcal{S}_G^F)$, is the parameter $Z(G)$ introduced in [1]. If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black (this is called the *color-change rule*). Given a coloring of G, the *derived coloring* is the (unique) result of applying the color-change rule until no more changes are possible. A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black. $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

Lower bounds

1. [1] $M(\mathcal{S}_G^F) \leq Z(\mathcal{S}_G^F)$ and thus $|G| - Z(G) \leq \mathrm{mr}(\mathcal{S}_G^F)$.
2. [4] The minimum rank of any induced subgraph (for which the minimum rank is known) provides a lower bound on minimum rank. In particular, if p is the length (= # of edges) of the longest induced path of G, then $p \leq \mathrm{mr}(\mathcal{S}_G^F)$.

Another application of Conjecture 1.1 is to the sum of the minimum ranks of a graph and its complement. At the American Institute of Mathematics workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns," the following question was asked:

Question 1.3. [5, Question 1.16] *How large can* $\mathrm{mr}(G) + \mathrm{mr}(\overline{G})$ *be?*

It was noted there that for the few graphs for which the minimum rank of both the graph and its complement was known,

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + 2$$

and equality in this bound is achieved by a path. It was also noted [5, Observation 1.15] that if Conjecture 1.1 is true, a consequence would be that for any regular graph G,

$$\mathrm{mr}^F(G) + \mathrm{mr}^F(\overline{G}) \leq |G| + 1.$$

2 Minimum rank of combinatorially symmetric matrices described by a graph

To establish the bounds on the minimum rank of bipartite graphs, we will need to consider not necessarily symmetric matrices. A zero-nonzero pattern is an $m \times n$ matrix with entries in $\{0, *\}$, where $*$ designates a nonzero entry. A matrix A is *combinatorially symmetric* if $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$. A combinatorially symmetric matrix has a symmetric zero-nonzero pattern. For such a matrix, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} |\ a_{ij} \neq 0 \text{ and } i \neq j\}$. The *set of matrices* of graph G over field F is

$$\mathcal{M}_G^F = \{A \in M_n(F) : A \text{ is combinatorially symmetric and } \mathcal{G}(A) = G\}.$$

Clearly $S_n(F) \subseteq M_n(F)$, $\mathcal{S}_G^F \subseteq \mathcal{M}_G^F$, and $\mathrm{mr}(\mathcal{M}_G^F) \leq \mathrm{mr}(\mathcal{S}_G^F)$. It is possible to have strict inequality, as the next example shows.

Example 2.1. Let $K_{3,3,3}$ be the complete tripartite graph on three sets of three vertices each. Let $A = \begin{bmatrix} 0 & J & J \\ -J & 0 & -J \\ -J & J & 0 \end{bmatrix}$, where $J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Then (with $A \in \mathbb{R}^{9\times 9}$), $\operatorname{rank}(A) = 2$ and $A \in \mathcal{M}^{\mathbb{R}}_{K_{3,3,3}}$, so $\operatorname{mr}(\mathcal{M}^{\mathbb{R}}_{K_{3,3,3}}) \le 2 < 3 = \operatorname{mr}(\mathcal{S}^{\mathbb{R}}_{K_{3,3,3}})$, with the latter equality established in [4].

In this section we establish the bound analogous to (3) in Conjecture 1.1 for minimum rank of matrices that are not required to be symmetric, i.e. we show

$$\operatorname{mr}(\mathcal{M}^F_G) \le |G| - \delta(G).$$

In Section 3 we then use this result to establish a better bound than (3) for minimum rank of symmetric matrices described by a bipartite graph (Theorem 3.1).

Observation 2.2. *Many properties of minimum rank of symmetric matrices of a graph extend to not-necessarily symmetric matrices, even without the assumption that the field is infinite. These include:*

1. *If the connected components of G are $G_1, \dots, G_t$, then*
$$\operatorname{mr}(\mathcal{M}^F_G) = \sum_{i=1}^{t} \operatorname{mr}(\mathcal{M}^F_{G_i}).$$
2. *If G' is an induced subgraph of G then $\operatorname{mr}(\mathcal{M}^F_{G'}) \le \operatorname{mr}(\mathcal{M}^F_G)$.*
3. $\operatorname{mr}(\mathcal{M}^F_G) \le |G| - 1$.
4. *For the path on n vertices, $\operatorname{mr}(\mathcal{M}^F_{P_n}) = n - 1 = \operatorname{mr}(\mathcal{S}^F_{P_n})$.*
5. *If $\operatorname{mr}(\mathcal{M}^F_G) = |G| - 1$, then $G = P_{|G|}$.*
(If $\operatorname{mr}(\mathcal{M}^F_G) = |G| - 1$, then $|G| - 1 = \operatorname{mr}(\mathcal{S}^F_G)$ and $\operatorname{mr}(\mathcal{S}^F_G) = |G| - 1$ implies $G = P_{|G|}$.)
6. *For a connected graph G of order $n > 1$, $\operatorname{mr}(\mathcal{M}^F_G) = 1$ if and only if $G = K_n$ if and only if $\operatorname{mr}(\mathcal{S}^F_G) = 1$.*
7. *For the cycle on n vertices, $\operatorname{mr}(\mathcal{M}^F_{C_n}) = n - 2$.*
8. *If T is a tree, then $\operatorname{mr}(\mathcal{S}^F_T) = \operatorname{mr}(\mathcal{M}^F_T)$.*
(If T is a tree and $A \in \mathcal{M}^F_T$, then there exist nonsingular diagonal matrices D_1 and D_2 such that $B = D_1 A D_2$ has all off-diagonal entries equal to 0 or 1, and so $B \in \mathcal{S}^F_T$. The technique used to choose the diagonal matrices is well known - see for example [6].)
9. *If $\operatorname{mr}(\mathcal{M}^F_G) \le 2$, then G does not contain as an induced subgraph any of the graphs P_4, Dart, $\ltimes$ (shown in Figure 1 in Section 4 below).*
(Although not necessarily symmetric matrices were not discussed in [4], it is clear from the argument in that paper that $\operatorname{mr}(\mathcal{M}^F_G) = 3$ for any field and for G any of P_4, Dart, $\ltimes$.)

Lemma 2.3. *Let C be a $k \times n$ matrix over an infinite field F such that every $k \times k$ submatrix of C is nonsingular. Then $\mathbf{x}$ is the zero-nonzero pattern of a vector $\mathbf{v} \in F^n$ such that $C\mathbf{v} = 0$ if and only if there are at least $k + 1$ entries $*$ in $\mathbf{x}$.*

Lemma 2.4. *Let k and n be integers with $k < n$ and let F be an infinite field. Then there exists a $k \times n$ matrix C over F such that every $k \times k$ submatrix of C is nonsingular.*

Proposition 2.5. *Let Z be an $n \times m$ zero-nonzero pattern such that each column of Z has at least r nonzero entries. Then over an infinite field there exists realization A of Z of rank at most $n - r + 1$. Moreover, if Z is symmetric, then A can be chosen so that if $i \neq j$ and $z_{ij} = *$, then $a_{ij} + a_{ji} \neq 0$.*

Proof. Let C be a $(r-1) \times n$ matrix over F such that every $(r-1) \times (r-1)$ submatrix of F is nonsingular; its existence follows from Lemma 2.4. In particular, the rows of C are linearly independent. By Lemma 2.3, for each j there exists a vector a_j in the null space of C whose pattern is that of the jth column of Z. Hence, $A = [a_1, a_2, \ldots, a_m]$ is a matrix with zero-nonzero pattern Z, whose column space is in the nullspace of C. Hence, A has rank at most $n - r + 1$.

Moreover, for any invertible diagonal matrix D, AD has the same zero-nonzero pattern as A, and its column space is contained in the nullspace of C. Thus $\text{rank}(AD) \leq n - r + 1$. As F is infinite, there exists a D, such that if $z_{ij} = *$, then the (i,j) and (j,i)-entries of AD are not opposites. □

Since the minimum number of entries allowed to be nonzero in a column of A is $\delta(G) + 1$, we have the following corollary.

Corollary 2.6. *Let F be an infinite field. For any graph G,*

$$\text{mr}(\mathcal{M}_G^F) \leq |G| - \delta(G). \tag{4}$$

3 Minimum rank and minimum degree of bipartite graphs

Let G be a bipartite graph having bipartition $V(G) = U \cup W$. Define $\delta_W(G) = \min_{w \in W}\{\deg_G(w)\}$. Note $\delta_W(G) \leq |U|$.

Let the elements of U be $u_i, i = 1, \ldots, p$ and the elements of W be $w_j, j = 1, \ldots, q$. The $p \times q$ zero-nonzero pattern $Z = [z_{ij}]$ such that $z_{ij} = *$ if and only if u_i and w_j are adjacent in G is the *biadjacency pattern* of G for the ordered partition $(u_1, \ldots, u_p), (w_1, \ldots, w_q)$.

Theorem 3.1. *For any infinite field F and bipartite graph G having bipartition $V(G) = U \cup W$,*

$$\text{mr}(\mathcal{S}_G^F) \leq 2(|U| - \delta_W(G) + 1) \text{ and } \text{mr}(\mathcal{S}_G^F) \leq 2(|W| - \delta_U(G) + 1). \tag{5}$$

Proof. Let $\mathcal{F}$ be the family of matrices having zero-nonzero pattern described by the biadjacency pattern Z. If $A \in \mathcal{S}_G^F$ and the diagonal of A is 0, then $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ where $B \in \mathcal{F}$. The minimum number of nonzero entries of a column of Z is $\delta_W(G)$, so by Proposition 2.5, there exists $B \in \mathcal{F}$ such that $\text{rank}(B) \leq |U| - \delta_W(G) + 1$. Then $\text{rank}(A) = 2\,\text{rank}(B) \leq 2(|U| - \delta_W(G) + 1)$. □

Corollary 3.2. *For any infinite field F and bipartite graph G having bipartition $V(G) = U \cup W$, $\text{mr}(\mathcal{S}_G^F) \leq |G| - \delta(G)$.*

Proof. If $\delta(G) \leq 1$ then the bound is valid, so we assume $\delta(G) \geq 2$, and without loss of generality assume $|W| - \delta_U(G) + 1 \leq |U| - \delta_W(G) + 1$. Then

$$\begin{aligned} \operatorname{mr}(\mathcal{S}_G^F) &\leq 2(|W| - \delta_U(G) + 1) \\ &\leq |W| - \delta_U(G) + 1 + |U| - \delta_W(G) + 1 \\ &= |G| - \delta_U(G) - \delta_W(G) + 2 \\ &\leq |G| - \delta(G) - \delta(G) + 2 \\ &\leq |G| - \delta(G) \quad \square \end{aligned}$$

Note that the bound in Theorem 3.1 is at least as good as that in the preceding corollary provided $\delta(G) > 1$, and often is much better, as in the case of the complete bipartite graph $K_{p,q}$, for which $2(|U| - \delta_W(G) + 1) = 2(|W| - \delta_U(G) + 1) = 2 = \operatorname{mr}(\mathcal{S}_{K_{p.q}}^F)$.

4 Minimum rank and minimum degree

We can establish a relationship between $\delta(G)$ and $Z(G)$ and use this to show many families of graphs satisfy Conjecture 1.1.

Proposition 4.1. *For any graph G, $\delta(G) \leq Z(G)$.*

Proof. Let $Z \subset V(G)$ be a minimal zero forcing set (necessarily Z is a proper subset of $V(G)$). Then it is necessary that the color-change rule be applied at least once. To apply the color-change rule it is necessary to have a black vertex with all but one neighbor black. Let the set consisting of this vertex and its neighbors be denoted by W. Then $\delta(G) \leq |W| - 1 \leq |Z| = Z(G)$. □

In [1] it was shown that $M(\mathcal{S}_G^{\mathbb{R}}) = Z(G)$ for numerous families of graphs (cf. [1, Table 1]), and thus by Proposition 4.1, Conjecture 1.1 is true for all these graphs. In fact, Conjecture 1.1 is true for all graphs listed in the AIM Minimum Rank Graph Catalogs [2].

Next we show that Conjecture 1.1 is true for graphs having extreme minimal degree. We need an easy lemma.

Lemma 4.2. *Let H be an induced subgraph of a graph G. Then $\delta(G) \leq |G| - (|H| - \delta(H))$.*

Proof. Since at least $|H| - 1 - \delta(H)$ edges must be missing (in both G and H) from some vertex of H,

$$\delta(G) \leq |G| - 1 - (|H| - 1 - \delta(H)) = |G| - (|H| - \delta(H)). \quad \square$$

A 2-*connected partial linear* 2-*tree*, also called a linear singly edge-articulated cycle graph in [8], is a "path" of cycles built up one cycle at a time by identifying an edge of a new cycle with an edge (that has a vertex of degree 2) of the most recently added cycle.

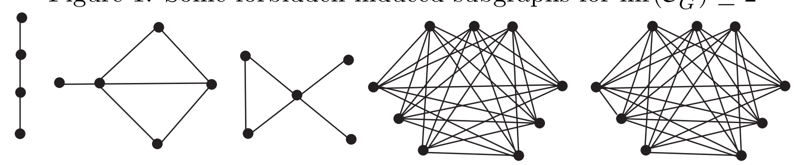
Proposition 4.3. *Let G be a graph of order n and let F be an infinite field. If $\delta(G) \leq 3$ or $\delta(G) \geq |G| - 2$ then $\operatorname{mr}(\mathcal{S}_G^F) \leq |G| - \delta(G)$.*

Proof.

- If $\delta(G) \leq 1$, then obviously $\operatorname{mr}(\mathcal{S}_G^F) \leq |G| - \delta(G)$.
- If $\delta(G) = 2$ then $G \neq P_n$ so $\operatorname{mr}(\mathcal{S}_G^F) \leq n - 2 = n - \delta(G)$.
- If $\delta(G) = 3$ then G is not a 2-connected partial linear 2-tree (which has minimal degree equal to 2), so by [8] $\operatorname{mr}(\mathcal{S}_G^F) \leq n - 3$.

- If $\delta(G) = n-1$ then $G = K_n$ so $\mathrm{mr}(\mathcal{S}_G^F) = 1 = n - \delta(G)$.
- If $\delta(G) = n-2$, then Lemma 4.2 assures that G cannot contain as an induced subgraph H any of the graphs P_4, Dart, $\ltimes$, $K_{3,3,3}$, $\overline{P_3 \cup 2K_3}$ (see Figure 1), since for these graphs $|H| - \delta(H) = 3, 4, 4, 3, 3$, respectively. By [4], any graph having $\mathrm{mr}(\mathcal{S}_G^F) > 2$ over an infinite field F must have one of these subgraphs as an induced subgraph, so $\mathrm{mr}(\mathcal{S}_G^F) \leq 2$. □

Figure 1: Some forbidden induced subgraphs for $\mathrm{mr}(\mathcal{S}_G^F) \leq 2$



We now turn our attention to determining what properties are required for a minimal counterexample to the conjecture (minimal in the sense that there is no proper induced subgraph that is a counterexample). The proof of the following result is straightforward.

Proposition 4.4. *If $\delta(\mathcal{S}_G^{\mathbb{R}}) \leq M(\mathcal{S}_G^{\mathbb{R}})$ and $\delta(\mathcal{S}_H^{\mathbb{R}}) \leq M(\mathcal{S}_H^{\mathbb{R}})$, then $\delta(\mathcal{S}_{G\Box H}^{\mathbb{R}}) \leq M(\mathcal{S}_{G\Box H}^{\mathbb{R}})$.*

Corollary 4.5. *Suppose G is a minimal counterexample to Conjecture 1.1 over the real numbers. Then G is not a non-trivial Cartesian product.*

The next theorem shows that a minimal counterexample cannot have a cut-vertex. In [3] the *rank-spread* of G at vertex v was defined as $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G-v)$, and it was shown that

$$\mathrm{mr}(G) = \sum_1^h \mathrm{mr}(G_i - v) + \min\left\{\sum_1^h r_v(G_i),\ 2\right\}. \tag{6}$$

where the ith component of $G - v$ is H_i and G_i is the subgraph induced by $\{v\} \cup V(H_i)$.

Theorem 4.6. *Let G be a connected graph with cut-vertex v and let $H_i, i = 1, \ldots, h$ be the connected components of $G - v$. If $\mathrm{mr}(\mathcal{S}_{H_i}^F) \leq |H_i| - \delta(H_i)$ for all $i = 1, \ldots, h$, then $\mathrm{mr}(\mathcal{S}_G^F) \leq |G| - \delta(G)$.*

Proof. Note that for all i, $\delta(H_i) \geq \delta(G) - 1$, since

$$\delta(G) \leq \min_{u \in V(H_i)} \{\deg_G(u)\} \leq \min_{u \in V(H_i)} \{\deg_{H_i}(u) + 1\} = \delta(H_i) + 1.$$

If $\delta(G) = 1$, then $\mathrm{mr}(\mathcal{S}_G^F) \leq |G| - \delta(G)$, so without loss of generality we assume $\delta(G) \geq 2$, and hence $\delta(H_i) \geq 1$ for all i.

We first consider the special case in which $h = 2, \delta(H_i) = 1$ for $i = 1, 2$, and $\delta(G) = 2$. If both H_1 and H_2 are paths, then $r_v(G_i) = 0$ for $i = 1, 2$ (where G_i is the graph induced by $V(H_i) \cup \{v\}$). Thus

$$\begin{aligned} \mathrm{mr}(\mathcal{S}_G^F) &= \mathrm{mr}(\mathcal{S}_{H_1}^F) + \mathrm{mr}(\mathcal{S}_{H_1}^F) \\ &= (|H_1| - 1) + (|H_2| - 1) \\ &= |H_1| + |H_1| + 1 - 3 \\ &< |G| - \delta(G). \end{aligned}$$

If (for example) H_1 is not a path, then $\mathrm{mr}(\mathcal{S}^F_{H_1}) \le |H_1| - 2$, so

$$\begin{aligned}
\mathrm{mr}(\mathcal{S}^F_G) &\le \mathrm{mr}(\mathcal{S}^F_{H_1}) + \mathrm{mr}(\mathcal{S}^F_{H_1}) + 2 \\
&\le (|H_1| - 2) + (|H_2| - 1) + 2 \\
&= |H_1| + |H_1| + 1 - 2 \\
&= |G| - \delta(G).
\end{aligned}$$

Next suppose there exists an i such that $\delta(H_i) \ge \delta(G)$. By renumbering if necessary, assume $\delta(H_1) \ge \delta(G)$. Then

$$\begin{aligned}
\mathrm{mr}(\mathcal{S}^F_G) &\le \sum_{i=1}^{h} \mathrm{mr}(\mathcal{S}^F_{H_1}) + 2 \\
&\le \sum_{i=1}^{h} |H_i| + 1 - \delta(H_1) - \left(\sum_{i=2}^{h} \delta(H_i) - 1\right) \\
&\le |G| - \delta(G),
\end{aligned}$$

since $h \ge 2$ and $\delta(H_2) \ge 1$.

So the only remaining case is $\delta(G) \ge 3$ or $h \ge 3$ and $\delta(H_i) = \delta(G) - 1$ for all i. Then

$$\begin{aligned}
\mathrm{mr}(\mathcal{S}^F_G) &\le \sum_{i=1}^{h} \mathrm{mr}(\mathcal{S}^F_{H_1}) + 2 \\
&\le \sum_{i=1}^{h} (|H_i| - \delta(H_i)) + 2 \\
&= \sum_{i=1}^{h} |H_i| + 1 - \sum_{i=1}^{h} \delta(H_i) + 1 \\
&= |G| - \sum_{i=1}^{h} (\delta(G) - 1) + 1 \\
&= |G| - h\delta(G) + h + 1 \\
&= |G| - \delta(G) - (h-1)\delta(G) + h + 1.
\end{aligned}$$

To establish $\mathrm{mr}(\mathcal{S}^F_G) \le |G| - \delta(G)$, we show $-(h-1)\delta(G) + h + 1 \le 0$ if $h \ge 3$ or $\delta(G) \ge 3$ (note that $h \ge 2$ and $\delta(G) \ge 2$). If $\delta(G) \ge 2, h \ge 3$ then $-(h-1)\delta(G) + h + 1 \le -(h-1)2 + h + 1 = -h + 3 \le 0$. If $\delta(G) \ge 3, h \ge 2$ then $-(h-1)\delta(G) + h + 1 \le -(h-1)3 + h + 1 = -2h + 4 \le 0$. □

Corollary 4.7. *Suppose G is a minimal counterexample to Conjecture 1.1. Then G cannot have a cut-vertex.*

It should be noted that for a graph having a cut-vertex, it is usually preferable to use the cut-vertex reduction formula (6) for minimum rank first as that will provide a better bound than use of the bound (3) on the whole graph.

5 Conclusion

We have verified the upper bound on minimum rank given in Conjecture 1.1 (for infinite fields) for all graphs and families of graphs listed in the on-line AIM Graph Minimum Rank Catalog [2]. We have also established that this bound is valid for bipartite graphs, and shown that a counterexample of minimal order cannot have a cut-vertex and cannot be a Cartesian product.

This bound is of course useless for a graph with pendent vertices. However, such a graph has a cut-vertex, and thus the computation of its minimum rank can be reduced to the computation of minimum ranks of smaller induced subgraphs by the cut-vertex reduction formula (6). So if the conjecture is true and could be established, it could provide a useful upper bound for minimum rank that is easy to compute. For a bipartite graph, the bound (5) in Theorem 3.1 is better and can be useful in the absence of cut-vertices.

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