Some open problems in matchings in graphs

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Overview

- Matchings in graphs
- Number of k-matchings in bipartite graphs and graphs as permanents and haffnians
- Upper bounds on permanents and haffnians: results and conjectures.
- Lower bounds on permanents and haffnians: results and conjectures.

Matchings

- $G = (V, E)$ undirected graph with vertices V, edges E.
- matching in G: $M \subseteq E$
  no two edges in M share a common endpoint.
- $e = (u, v) \in M$ is dimer
- v not covered by M is monomer.
- M called monomer-dimer cover of G.
- M is perfect matching $\iff$ no monomers.
- M is k-matching $\iff$ $\#M = k$.

Generating matching polynomial

- $\phi(k, G)$ number of k-matchings in G, $\phi(0, G) := 1$
- $\Phi_G(x) := \sum_k \phi(k, G)x^k$ matching generating polyn.
- roots of $\Phi_G(x)$ are real nonpositive Heilmann-Lieb 1972.
  Newton inequalities hold
- $\Phi_{G_1 \cup G_2}(x) = \Phi_{G_1}(x)\Phi_{G_2}(x)$

Examples:

$$\Phi_{K_{2r}}(x) = \sum_{k=0}^{r} \binom{2r}{2k} \frac{\prod_{j=0}^{k-1} \binom{2k-2j}{2}}{k!} x^k = \sum_{k=0}^{r} \frac{(2r)!}{(2r-2k)!2^k k!} x^k$$

$$\Phi_{K_{r,r}}(x) = \sum_{k=0}^{r} \binom{r}{k}^2 k! x^k$$

$\mathcal{G}(r, 2n) \supset \mathcal{GB}(r, 2n)$ set of r-regular and regular bipartite graphs on $2n$ vertices, respectively

$qK_{r,r} \in \mathcal{GB}(r, 2rq)$ a union of q copies of $K_{r,r}$.
$\Phi_{qK_{r,r}} = \Phi_{K_{r,r}}^q$

Formulas for k-matchings in bipartite graphs

$G = (V, E)$ bipartite $V = V_1 \cup V_2, E \subset V_1 \times V_2$,
represented by bipartite adjacency matrix
$B(G) = B = [b_{ij}]_{i,j=1}^{m\times n} \in \{0,1\}^{m\times n}$, $\#V_1 = m, V_2 = n$.

Example: Any subgraph of $\mathbb{Z}^d$ is bipartite

CLAIM: $\phi(k, G) = \mathrm{perm}_k(B(G))$.

Prf: Suppose $n = \#V_1 = \#V_2$.
Then permutation $\sigma : \langle n\rangle \to \langle n\rangle$ is a perfect match iff $\prod_{i=1}^{n} b_{i\sigma(i)} = 1$.
The number of perfect matchings in G is $\phi(n, G) = \mathrm{perm}\, B(G)$. □

Computing $\phi(n, G)$ is $\#P$-complete problem Valiant 1979

For $G = (\langle 2n\rangle, E)$ bipartite $G \in \mathcal{GB}(r, 2n) \iff \frac{1}{r}B(G) \in \Omega_n \iff$
G is a disjoint (edge) union of r perfect matchings

Matching on nonbipartite graphs

$G=(V,E), |V|=2n$,
$A(G)=[a_{ij}]\in \mathrm{S}_0(2n,\{0,1\})$ - adjacency matrix of G

$\phi(n,G)=\mathrm{haf}(A(G))=\sum_{M\in\mathcal{M}(K_{2n})}\prod_{(i,j)\in M} a_{ij}$
$\mathcal{M}(K_{2n})$ the set of perfect matchings in K_{2n}

$\phi(k,G)=\mathrm{haf}_k(A(G))=\sum_{M\in\mathcal{M}_k(K_{2n})}\prod_{(i,j)\in M} a_{ij}$
$\mathcal{M}_k(K_{2n})$ the set of k matchings in K_{2n}

Claim $\mathrm{perm}(A(G))\ge \mathrm{haf}(A(G))^2$. Equality holds if G is bipartite.

Main problems

Find good estimates on

$s_n(k,r) := \min_{G\in\mathcal{G}(r,2n)} \phi(k,G) \le t_n(k,r) := \min_{G\in\mathcal{GB}(r,2n)} \phi(k,G)$

$S_n(k.r) := \max_{G\in\mathcal{G}(r,2n)}\phi(k,G) \ge T_n(k,r) := \max_{G\in\mathcal{GB}(r,2n)} \phi(k,G)$

Completely solved case $r = 2$ [8]

$S_n(k,2) = T_n(k,2)$ achieved only for $G = mK_{2,2}$ or $G = mK_{2,2} \cup C_6$.

$t_n(k,2)$ achieved only for C_{2n}

$s_n(k,2)$ achieved only for mC_3, $mC_3 \cup C_4$ or $mC_3 \cup C_5$.

The upper bound conjecture

$S_{qr}(k,r) = T_{qr}(k,r) = \phi(k, qK_{r,r})$

$k = qr$ Follows from Bregman's inequality (see also [3])
$\mathrm{perm}\, A \le \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}$

$A = [a_{ij}] \in \{0,1\}^{n \times n}$ $r_i = \sum_{j=1}^{n}, i = 1, \ldots, n$

Egorichev-Alon-Friedland for $G = (V, E), |V| = 2n$
$\phi(n, G) \le \prod_{v \in V} (\deg(v)!)^{\frac{1}{2\deg(v)}}$

Equality holds iff G a union of complete bipartite graphs

$S_n(k,r) \le \binom{2n}{2k} (r!)^{\frac{k}{r}}$
$T_n(k,r) \le \min(\binom{n}{k}^2 (r!)^{\frac{k}{r}}, \binom{n}{k} r^k)$

Friedland-Krop-Lundow-Markström [7]

The lower bounds: Bipartite case

$r^k \min_{C\in\Omega_n} \mathrm{perm}_k C \le \phi(k,G)$ for any $G \in \mathcal{GB}(r,2n)$

$J_n = B(K_{n,n}) = [1]$ the incidence matrix of the complete bipartite graph $K_{n,n}$ on $2n$ vertices

van der Waerden permanent conjecture 1926:

$$\min_{C\in\Omega_n} \mathrm{perm}\, C = \mathrm{perm}\, \frac{1}{n} J_n \ \left(= \frac{n!}{n^n} \approx \sqrt{2\pi n}\, e^{-n} \right)$$

Tverberg permanent conjecture 1963:

$$\min_{C\in\Omega_n} \mathrm{perm}_k\, C = \mathrm{perm}_k\, \frac{1}{n} J_n \ \left(= \binom{n}{k}^2 \frac{k!}{n^k} \right)$$

for all $k = 1, \ldots, n$.

History

- In 1979 Friedland showed the lower bound $\mathrm{perm}\, C \geq e^{-n}$ for any $C \in \Omega_n$ following T. Bang's announcement 1976.
  This settled the conjecture of Erdös-Rényi on the exponential growth of the number of perfect matchings in $d \geq 3$-regular bipartite graphs 1968, Voorhoeve 1979.
- van der Waerden permanent conjecture was proved by Egorichev and Falikman 1981.
- Tverberg conjecture was proved by Friedland 1982
- 79 proof is tour de force according to Bang
- 81 proofs involve directly (Egorichev) and indirectly (Falikman) use of Alexandroff mixed volume inequalities with the conditions for the extremal matrix
- 82 proof uses methods of 81 proofs with extra ingredients
- There are new simple proofs using nonnegative hyperbolic polynomials e.g. Gurvits, Friedland-Gurvits

Lower matching bounds for bipartite graphs

Voorhoeve-1979 ($r = 3$) Schrijver-1998

$$\phi(n, G) \geq \left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n \quad \textit{for} \quad G \in \mathcal{GB}(r, 2n)$$

Gurvits 2006: $A \in \Omega_n$, each column has at most r nonzero entries:

$$\operatorname{perm} A \geq \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{r-1}{r}\right)^{(r-1)n}.$$

$$\textit{Cor}: \ \phi(n, G) \geq \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n$$

$$\textit{Con FKM } 2006: \ \phi(k, G) \geq \binom{n}{k}^2 \left(\frac{nr-k}{nr}\right)^{nr-k}\left(\frac{kr}{n}\right)^k, \ G \in \mathcal{GB}(r, 2n)$$

F-G 2008 showed weaker inequalities

Positive hyperbolic polynomials

A polynomial $p = p(\mathbf{x}) = p(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ is called *positive hyperbolic* if

p is a homogeneous polynomial of degree $m \geq 0$.

$p(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$.

$\phi(t) := p(\mathbf{x} + t\mathbf{u})$, for $t \in \mathbb{R}$, has m-real t-roots for each $\mathbf{u} > \mathbf{0}$ and each $\mathbf{x}$.

Ex. 1: $A = (a_{ij})_{i=j=1}^{m,n} \in \mathbb{R}_+^{m \times n}$

$p_{k,A}(\mathbf{x}) := \sum_{1 \leq i_1 < \ldots i_k \leq m} \prod_{j=1}^{k} (A\mathbf{x})_{i_j}, \mathbf{x} \in \mathbb{R}^n$

Ex. 2: $A_1, \ldots, A_n \in \mathbb{C}^{m \times m}$ hermitian, nonnegative definite matrices such that $A_1 + \ldots + A_n$ is a positive definite matrix. Let $p(\mathbf{x}) = \det \sum_{i=1}^{n} x_i A_i$. Then $p(\mathbf{x})$ is positive hyperbolic.

Ex. 3: $B \in \mathbb{R}_+^{m \times m}$ symmetric. Then $\mathbf{x}^\top B\mathbf{x}$ positive hyperbolic iff B has exactly on positive eigenvalue.

Capacity

$p(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ positive hyperbolic polynomial of degree $m \geq 1$.

Gurvits $\mathrm{Cap}\, p := \inf_{\mathbf{x}>0, x_1 \ldots x_n = 1} p(\mathbf{x})$

$A \in \mathbb{R}_+^{n \times n}$ doubly stochastic. Then $\mathrm{Cap}\, p_{k,A} = \binom{n}{k}$.

Let $B = D_1 A D_2$, D_1, D_2 positive diagonal, A doubly stochastic matrix. Let $p_{n,B}$ be defined as above. Then $\mathrm{Cap}\, p_{n,B} = \frac{1}{\det D_1 D_2}$.

Lemma: $p : \mathbb{R}^n \to \mathbb{R}$ positive hyperbolic of degree $m \geq 1$. Assume that $\mathrm{Cap}\, p > 0$. Then $\deg_i p \geq 1$ for $i = 1, \ldots, n$. For $m = n \geq 2$ $\mathrm{Cap}\, \frac{\partial p}{\partial x_i}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geq (\frac{\deg_i p - 1}{\deg_i p})^{\deg_i p - 1} \mathrm{Cap}\, p$ for $i = 1, \ldots, n$, where $0^0 = 1$.

Friedland-Gurvits inequality

Let $p : R^n \to \mathbb{R}$ be positive hyperbolic of degree $m \in [1, n]$. Assume that $\deg_i p \le r_i \in [1, m]$ for $i = 1, \ldots, n$. Rearrange the sequence $r_1, \ldots, r_n$ in an increasing order $1 \le r_1^* \le r_2^* \le \ldots \le r_n^*$. Let $k \in [1, n]$ be the smallest integer such that $r_k^* > m - k$. Then

$$\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) \ge$$

$$\frac{n^{n-m}}{(n-m)!} \frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1} \left(\frac{r_j^* + n - m - 1}{r_j^* + n - m}\right)^{r_j^* + n - m - 1} \mathrm{Cap}\, p. \quad (0.1)$$

(Here $0^0 = 1$, and the empty product for $k = 1$ is assumed to be 1.) If $\mathrm{Cap} > 0$ and $r_i = m$ for $i = 1, \ldots, m$ equality holds if and only if $p = C(\frac{x_1 + \ldots + x_n}{n})^m$ for each $C > 0$.

p-matching and total matching entropies

$G = (V, E)$ infinite, degree of each vertex bounded by N,

$p \in [0,1]$-matching entropy, (p-dimer entropy) of G

$$h_G(p) = \sup_{\text{on all sequences}} \limsup_{l\to\infty} \frac{\log \phi(k_l, G_l)}{\#V_l}$$

and total matching entropy, (monomer-dimer entropy)

$$h_G = \sup_{\text{on all sequences}} \limsup_{l\to\infty} \frac{\log \sum_{k=0}^{0.5(\#V_l)} \phi(k, G_l)}{\#V_l},$$

$G_l = (E_l, V_l), l \in \mathbb{N}$ a sequence of finite graphs converging to G, and

$$\lim_{l\to\infty} \frac{2k_l}{\#V_l} = p$$

$h_G = \max_{p\in[0,1]} h_G(p)$

Asymptotic versions

$$Sa(p,r) = \limsup_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log S_{n_j}(k_j,r)}{2n_j}$$

$$Ta(p,r) = \limsup_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log T_{n_j}(k_j,r)}{2n_j}$$

$$sa(p,r) = \liminf_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log s_{n_j}(k_j,r)}{2n_j}$$

$$ta(p,r) = \liminf_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log t_{n_j}(k_j,r)}{2n_j}$$

Next slide gives the graphs of AUMC and the upper bounds for $Ta(p,4)$.

Expected values of k-matchings for bipartite graphs

- Permutation $\sigma : \langle nr\rangle \to \langle nr\rangle$ induces $G(\sigma) \in \mathcal{GB}_{\mathrm{mult}}(r, 2n)$ and vice versa
  $G(\sigma) = \{(i, \lceil \frac{\sigma((i-1)r+j)}{r} \rceil), j = 1, \ldots, r, \ i = 1, \ldots, n\} \subset \langle n\rangle \times \langle n\rangle$
  number of different σ inducing the same simple G is $(r!)^n$
- μ probability measure on $\mathcal{GB}_{\mathrm{mult}}(r, 2n)$:
  $\mu(G(\sigma)) = ((nr)!)^{-1}$
- FKM 06:
  $E(k, n, r) := \mathrm{E}(\phi(k, G)) = \binom{n}{k}^2 r^{2k} k!(nr-k)!)(nr)!)^{-1}$,
  $k = 1, \ldots, n$
- $1 \le k_l \le n_l, l = 1, \ldots,$ increasing sequences of integers s.t. $\lim_{l\to\infty} \frac{k_l}{n_l} = p \in [0, 1]$. Then

$$\lim_{l\to\infty} \frac{\log E(k_l, n_l, r)}{2n_k} = f(p, r)$$

$$f(p, r) := \frac{1}{2}(p \log r - p \log p - 2(1-p)\log(1-p) + (r-p)\log(1-\frac{p}{r}))$$

Asymptotic Lower and Upper Matching conjectures

FKLM JOSS 08 :

$G_l = (E_l, V_l) \in \mathcal{G}(r, \#V_l), l = 1, 2, \ldots,$ and $\lim_{l\to\infty} \frac{2k_l}{\#V_l} = p$.

$$\mathrm{low}_r(p) := \inf_{\text{all allowable sequences}} \liminf_{l\to\infty} \frac{\log \phi(k_l, G_l)}{\#V_l}$$

ALMC: $\mathrm{low}_r(p) = f(p, r)$ (For most of the sequences $\liminf = f(p, r)$)

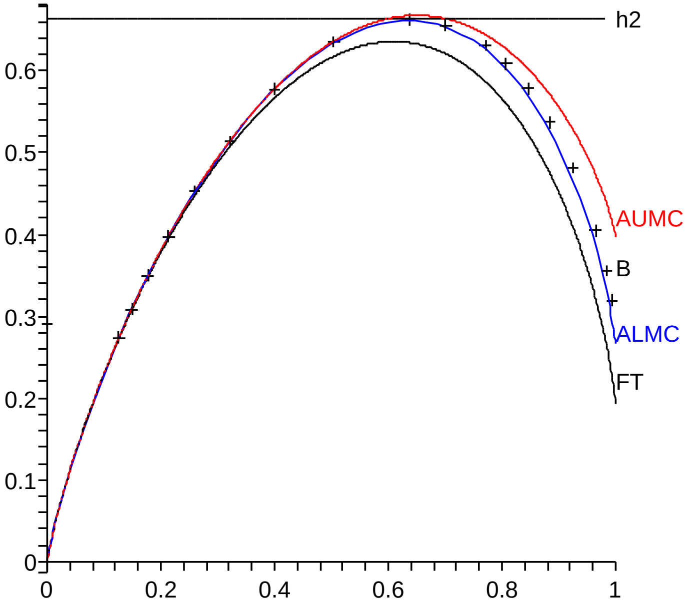
$$\mathrm{upp}_r(p) := \sup_{\text{all allowable sequences}} \limsup_{l\to\infty} \frac{\log \phi(k_l, G_l)}{\#V_l}$$

AUMC: $\mathrm{upp}_r(p) = h_{K(r)}(p)$, $K(r)$ countable union of $K_{r,r}$

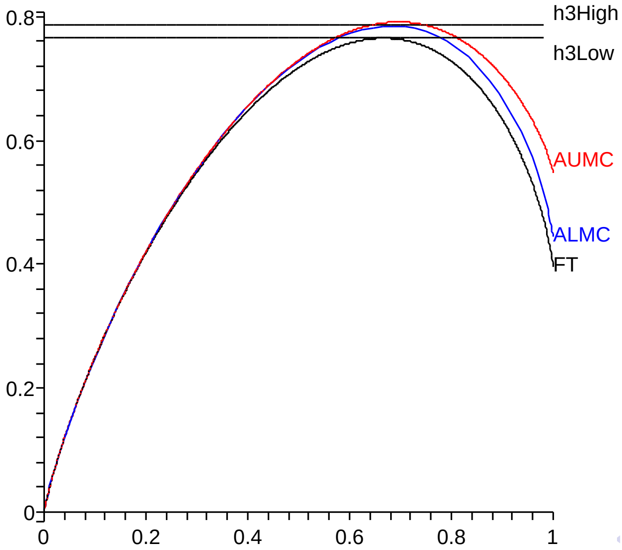
$$P_r(t) := \frac{\log \sum_{k=0}^{r} \binom{r}{k}^2 k!\, e^{2kt}}{2r}, \; t \in \mathbb{R},$$

$$p(t) := P_r'(t) \in (0, 1), \quad h_{K(r)}(p(t)) := P_r(t) - tp(t)$$

$r = 4$



$r = 6$



$r = 4$ upper bounds

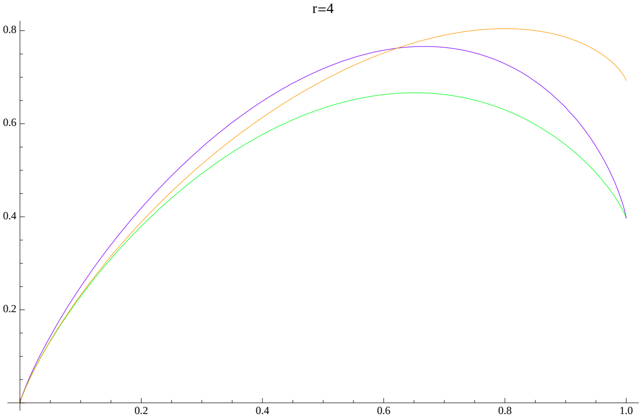


Figure: $h_{K(4)}$-green, $\mathrm{upp}_{4,1}$-blue, $\mathrm{upp}_{4,2}$-orange

Lower asymptotic bounds Friedland-Gurvits 2008

Thm: $r \ge 3, s \ge 1$ integers,
$B_n \in \Omega_n, n = 1, 2, \ldots$ each column of B_n has at most r-nonzero entries.
$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$ then

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{k_n} B_n}{2n} \geqslant \frac{1}{2}(-p\log p - 2(1-p)\log(1-p)) + \frac{1}{2}(r+s-1)\log(1-\frac{1}{r+s}) - \frac{1}{2}(s-1+p)\log(1-\frac{1-p}{s})$$

Prf combines properties positive hyperbolic polynomials, capacity and the measure on $\mathcal{G}(r, 2n)$

- Cor: r-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$
- Con: under Thm assumptions

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{k_n} B_n}{2n} \geqslant f(r, p) - \frac{p}{2}\log r$$

- For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$ conjecture holds

Lower bounds for matchings in regular non-bipartite graphs

Petersen's THM: A bridgeless cubic graph has a perfect match

Problem: Find the minimum of the biggest match in $\mathcal{G}(r, 2n)$ for $r > 2$.

Does every $G \in \mathcal{G}(r, 2n)$ has a match of size $\lfloor \frac{2n}{3} \rfloor$? (True for $r = 2$.)

Esperet-Kardos-King-Král-Norine:
Every cubic bridgeless graph has at least $2^{\frac{|V|}{3656}}$ perfect matchings

Cygan-Pilipczuk-Skrekovski:
$\exists$ inf-family of cubic 3-colored connected graphs $G = (V, E)$ s.t.
$\mathrm{haf}(A(G)) \approx c_F |V| (\frac{1+\sqrt{5}}{2})^{\frac{|V|}{12}}$, $|V| = 12k + 4$, $k = 1, 2, \ldots$.

An analog the van der Waerden conjecture

THM Edmonds 1965: A symmetric doubly stochastic matrix with zero diagonal of even order $A = [a_{ij}]_{i,j=1}^{2n}$ is a convex combination of symmetric permutation matrices with zero diagonal if and only if $\sum_{i,j\in S} a_{ij} \le |S| - 1$ for any odd subset $S \subset \{1, \ldots, 2n\}$ (*)

Denote by Ψ_{2n} the subset of all symmetric doubly stochastic matrices of the above form

Problem: Find $\mu_{n,n} := \min \mathrm{haf}(A), A \in \Psi_{2n}$

FALSE CONJECTURE: The minimum is achieved only for the matrix $\frac{1}{2n-1}A(K_{2n})$

$$\mathrm{haf}(\tfrac{1}{2n-1}A(K_{2n})) \approx e^{-n}\sqrt{2e} < \mathrm{haf}(\tfrac{1}{n}A(K_{n,n})) \approx e^{-n}\sqrt{2\pi n}$$

CONJECTURE: $\mu := \lim_{n\to\infty} \frac{\log \mu_{n,n}}{n} > -\infty$

C-P-S $\mu \le \frac{\log \frac{1+\sqrt{5}}{2}}{6} - \log 3$

Hyperbolic polynomials

THM: Good lower bounds hold for $\mathrm{haf}_k(A)$ if $A \in \Psi_{2n}$ n-1 $n-1$ eigenvalues of A are nonpositive

Outline of proof: Fact $\mathbf{x}^\top A\mathbf{x}$ is a hyperbolic polynomial for a nonnegative symmetric matrix iff A has all but one nonpositive eigenvalues [5]

$$\mathrm{haf}_k A = (2^k k!)^{-1} \sum_{1\le i_1<\ldots<i_{2k}\le 2n} \frac{\partial^{2k}}{\partial x_{i_1}\ldots\partial x_{i_{2k}}}(\mathbf{x}^\top A\mathbf{x})^k$$

Use the arguments of [2] to show

$$\mathrm{haf}_n(B) \ge \left(\frac{n-1}{n}\right)^{(n-1)n} \approx e^{-n}\sqrt{e}$$

$$\mathrm{haf}_k(B) \ge \frac{(2n)^{2n-2k}(2n-k)!(2n)^k}{(2n-2k)!(2n-k)^{2n-k}2^k k!}\left(\frac{(2n-k-1)}{2n-k}\right)^{(2n-k-1)k}$$

References

- R.J. Baxter, Dimers on a rectangular lattice, *J. Math. Phys.* 9 (1968), 650–654.
- L.M. Bregman, Some properties of nonnegative matrices and their permanents, *Soviet Math. Dokl.* 14 (1973), 945-949.
- M. Cygan, M. Pilipczuk and R. Skrekovski, A bound on the number of perfect matchings in Klee-graphs, University of Ljubljana, Preprint series, vol. 47 (2009), 1105, http://www.imfm.si/preprinti/PDF/01105.pdf.
- G.P. Egorichev, Proof of the van der Waerden conjecture for permanents, *Siberian Math. J.* 22 (1981), 854–859.
- P. Erdös and A. Rényi, On random matrices, II, *Studia Math. Hungar.* 3 (1968), 459-464.
- L. Esperet, F. Kardos, A. King, D. Kral and S. Norine, Exponentially many perfect matchings in cubic graphs, arXiv:1012.2878.
- D.I. Falikman, Proof of the van der Waerden conjecture regarding the permanent of doubly stochastic matrix, *Math. Notes Acad. Sci. USSR* 29 (1981), 475–479.
- M.E. Fisher, Statistical mechanics of dimers on a plane lattice, *Phys. Rev.* 124 (1961), 1664–1672.
- R.H. Fowler and G.S. Rushbrooke, Statistical theory of perfect solutions, *Trans. Faraday Soc.* 33 (1937), 1272–1294.

References

- S. Friedland, A lower bound for the permanent of doubly stochastic matrices, *Ann. of Math.* 110 (1979), 167-176.
- S. Friedland, A proof of a generalized van der Waerden conjecture on permanents, *Lin. Multilin. Algebra* 11 (1982), 107–120.
- S. Friedland, FPRAS for computing a lower bound for weighted matching polynomial of graphs, arXiv:cs/0703029.
- S. Friedland, Analogs of the van der Waerden and Tverberg conjectures for haffnians, arXiv:1102.2542.
- S. Friedland and L. Gurvits, Generalized Friedland-Tverberg inequality: applications and extensions, arXiv:math/0603410v2.
- S. Friedland and L. Gurvits, Lower bounds for partial matchings in regular bipartite graphs and applications to the monomer-dimer entropy, *Combinatorics, Probability and Computing*, 2008, 15pp.
- S. Friedland, E. Krop, P.H. Lundow and K. Markström, Validations of the Asymptotic Matching Conjectures, *Journal of Statistical Physics*, 133 (2008), 513-533, arXiv:math/0603001v3.
- S. Friedland, E. Krop and K. Markström, On the Number of Matchings in Regular Graphs, *The Electronic Journal of Combinatorics*, 15 (2008), #R110, 1-28, arXiv:0801.2256v1 [math.Co] 15 Jan 2008.

References

- S. Friedland and U.N. Peled, Theory of Computation of Multidimensional Entropy with an Application to the Monomer-Dimer Problem, *Advances of Applied Math.* 34(2005), 486-522.
- L. Gurvits, Hyperbolic polynomials approach to van der Waerden/Schrijver-Valiant like conjectures, STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, 417–426, ACM, New York, 2006.
- J. Hammersley and V. Menon, A lower bound for the monomer-dimer problem, *J. Inst. Math. Applic.* 6 (1970), 341–364.
- O.J. Heilmann and E.H. Lieb, Theory of monomer-dimer systems., *Comm. Math. Phys.* 25 (1972), 190–232.
- P.W. Kasteleyn, The statistics of dimers on a lattice, *Physica* 27 (1961), 1209–1225.
- L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland Mathematical Studies, vol. 121, North-Holland, Amsterdam, 1986.
- P.H. Lundow, Compression of transfer matrices, *Discrete Math.* 231 (2001), 321–329.

References

- C. Niculescu, A new look and Newton' inequalties, *J. Inequal. Pure Appl. Math.* 1 (2000), Article 17.
- L. Pauling, *J. Amer. Chem. Soc.* 57 (1935), 2680–.
- J. Radhakrishnan, An Entropy Proof of Bregman's Theorem, *J. Combin. Theory Ser. A* **77** (1997), 161-164.
- A. Schrijver, Counting 1-factors in regular bipartite graphs, *J. Comb. Theory* B 72 (1998), 122–135.
- H. Tverberg, On the permanent of bistochastic matrix, *Math. Scand.* 12 (1963), 25-35.
- L.G. Valiant, The complexity of computing the permanent, *Theoretical Computer Science* 8 (1979), 189-201.
- M. Voorhoeve, A lower bound for the permanents of certain $(0, 1)$-matrices, *Neder. Akad. Wetensch. Indag. Math.* 41 (1979), 83-86.
- B.L. van der Waerden, Aufgabe 45, *Jber Deutsch. Math.-Vrein.* 35 (1926), 117.