GENERALIZATIONS OF THE ODD DEGREE THEOREM AND APPLICATIONS

Shmuel Friedland
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
email: friedlan@uic.edu

and

Anatoly S. Libgober
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
email: libgober@math.uic.edu

Abstract

Let $V \subset \mathbb{PR}^n$ be an algebraic variety, such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is irreducible of codimension $m \geq 1$. We use a sufficient condition on a linear space $L \subset \mathbb{PR}^n$ of dimension $m + 2r$ to have a nonempty intersection with V, to show that any six dimensional subspace of 5×5 real symmetric matrices contains a nonzero matrix of rank at most 3.

1 Introduction

Let $p(x) = x^k + a_1 x^{k-1} + ... + a_k \in \mathbb{R}[x]$. Then the odd degree theorem states that $p(x)$ has a real root if k is odd. Let $\mathbb{PR}^n$ and $\mathbb{P}^n := \mathbb{PC}^n$ be the real and the complex projective space of dimension n respectively. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ we view a linear space $L \subset \mathbb{PF}^n$ of dimension m as an element of the Grassmanian manifold $\mathrm{Gr}\,(m + 1, n + 1, \mathbb{F})$. Let $V \subset \mathbb{PR}^n$ be an algebraic variety, such that it complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is irreducible and has codimension $m \geq 1$. If $d = \deg V_{\mathbb{C}}$ is odd then for any linear space $L \subset \mathbb{PR}^n$ of dimension m the intersection $V \cap L \neq \emptyset$. Indeed, we have $B(V) = V$, where $B : \mathbb{P}^n \to \mathbb{P}^n$ is the involution $z \mapsto \bar{z}$. The set $V_{\mathbb{C}} \cap L_{\mathbb{C}}$ consists of exactly d points. As this set is invariant under the involution B, we deduce that there exists $z \in V_{\mathbb{C}} \cap L_{\mathbb{C}}$ such that $B(z) = z \Rightarrow z \in \mathbb{PR}^n$. The continuity argument yields that $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}\,(m+1, n+1, \mathbb{R})$.

Consider now the case when d is even. Then it is not difficult to find nontrivial examples where $V \cap L' = \emptyset$ for some $L' \in \mathrm{Gr}\,(m+1, n+1, \mathbb{R})$. We are interested in this paper in cases when V is a determinantal variety, i.e. finding nonzero real matrices of rank at most k in linear families. The examples such that for any integer $k \in [0, p)$ there exists $L' \in \mathrm{Gr}\,(m+k+1, n+1, \mathbb{R})$ satisfying $V \cap L' = \emptyset$, while $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}\,(m+p+1, n+1, \mathbb{R})$ can be found among determinantal varieties. (see §2).

Let $S_n(\mathbb{F})$ be the space of $n \times n$ symmetric matrices with entries in $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Let $V_{k,n}(\mathbb{F})$ be the variety of all matrices in $S_n(\mathbb{F})$ of rank k or less. Then the projectivization $\mathbb{P}V_{k,n}(\mathbb{F})$ is an irreducible variety of codimension $\binom{n-k+1}{2}$ in the projective space $\mathbb{P}S_n(\mathbb{F})$. Note that $V_{k-1,n}(\mathbb{F})$ is the variety of the singular points of $V_{k,n}(\mathbb{F})$ (e.g. [1, II]). Let $d(n,k,\mathbb{F})$ be the smallest integer ℓ such that every ℓ dimensional subspace of $S_n(\mathbb{F})$ contains a nonzero matrix whose rank is at most k. Then

$$d(n,k,\mathbb{C}) = \binom{n-k+1}{2} + 1, \tag{1.1}$$

and the problem is to determine $d(n,k,\mathbb{R})$. The degree of $\mathbb{P}V_{k,n}(\mathbb{C})$ was computed by Harris and Tu in [9]

$$\delta_{k,n} := \deg \mathbb{P}V_{k,n}(\mathbb{C}) = \prod_{j=0}^{n-k-1} \frac{\binom{n+j}{n-k-j}}{\binom{2j+1}{j}}. \tag{1.2}$$

It was shown in [5] that $\delta_{n-q,n}$ is odd if

$$n \equiv \pm q \ (\mathrm{mod}\ 2^{\lceil \log_2 2q \rceil}). \tag{1.3}$$

Then $d(n, n-q, \mathbb{R}) = d(n, n-q, \mathbb{C})$ for these values of n and q. It is conjectured in [5] that if $\delta_{n-q,n}$ is odd then (1.3) holds.

In this paper we show that not only the degree of complexification but also the Euler characteristic of the intersection of $\mathbb{P}V_{k,n}(\mathbb{C})$ with a generic linear space of dimension $\binom{n-k+1}{2} + 2r$ can be used to get an additional information about $d(n,k,\mathbb{R})$. Our estimate of $d(n,k,\mathbb{R})$ from above uses the following result proved in §2.

Corollary 1.1 *Let $V \subset \mathbb{PR}^n$ be an algebraic variety such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is an irreducible variety of codimension m. Assume that*

$\deg V_{\mathbb{C}}$ *is even and let* r *be a positive integer. Suppose that the codimension of the variety of the singular points of* $V_{\mathbb{C}}$ *in* $V_{\mathbb{C}}$ *is at least* $2r+1$. *Suppose furthermore that for a generic* $L \in \mathrm{Gr}\,(m+2r+1, n+1, \mathbb{C})$ *the Euler characteristic of* $V_{\mathbb{C}} \cap L$ *is odd. Then* $V \cap L \neq \emptyset$ *for any* $L \in \mathrm{Gr}\,(m+2r+1, n+1, \mathbb{R})$.

This corollary applies whenever one has an answer to the following problem:

Problem 1.1 *Assume that* $\delta_{k,n}$ *is even. Find an integer* $r \geq 1$, *preferably the smallest possible, such that*

$$2r < \binom{n-k+2}{2} - \binom{n-k+1}{2}, \tag{1.4}$$

and the Euler characteristic of $\mathbb{P}V_{k,n}(\mathbb{C}) \cap L$ *is odd for a generic* $L \in \mathrm{Gr}\,(\binom{n-k+1}{2} + 2r + 1, \binom{n+1}{2}, \mathbb{C})$.

For $k = n-1$ there is no r which satisfies the conditions of Problem 1.1, hence Corollary 1.1 is not applicable. This follows from the result that the Euler characteristic of a smooth hypersurface of an even degree is even. Let $k = n-2$. The smallest n of interest is $n = 5$ [5]. In §6 we show that the minimal solution to Problem 1.1 is $r = 1$. Hence $d(5,3,\mathbb{R}) \leq 6$. A numerical evidence supports the conjecture that $d(5,3,\mathbb{R}) = 6$ [5].

The contents of the paper are as follows. In §2 we give a generalization of the odd degree theorem. It is a straightforward consequence of the Lefschetz fixed point theorem, the Hodge decomposition and the Poincaré duality. We also recall the exact value of the gap $d(n, n-1, \mathbb{R}) - d(n, n-1, \mathbb{C})$. In §3 we recall some known results about the projectivized complex bundles and the corresponding Chern classes of their tangent bundles. Next we discuss a resolution of the singularities of $V_{k,n}(\mathbb{C})$ and $\mathbb{P}V_{k,n}(\mathbb{C})$. Let $\tau, \kappa \to \mathrm{Gr}\,(k, n, \mathbb{C})$ be the tautological k-bundle and its quotient bundle respectively. Then $\mathrm{Sym}^2\,\tau$, $\mathrm{Sym}^2\,\kappa$ are resolutions of $V_{k,n}(\mathbb{C})$, $V_{n-k,n}(\mathbb{C})$ respectively. The projectivized bundle $\mathbb{P}(\mathrm{Sym}^2\,\tau)$, $\mathbb{P}(\mathrm{Sym}^2\,\kappa)$ are resolutions of $\mathbb{P}V_{k,n}(\mathbb{C})$, $\mathbb{P}V_{n-k,n}(\mathbb{C})$ respectively. In §4 we discuss $\mathbb{P}(\mathrm{Sym}^2\,\tau)$ for $k = 1$. In §5 we discuss $\mathbb{P}(\mathrm{Sym}^2\,\tau)$ for $k = 2$ and mostly for $n = 4$. In §6 we discuss $\mathbb{P}(\mathrm{Sym}^2\,\kappa)$ for $k = 2$, $n = 5$ modulo 2.

2 Generalizations of the odd degree theorem

Lemma 2.1 *Let* $W \subset \mathbb{P}\mathbb{R}^n$ *be an algebraic variety such that its complexification* $W_{\mathbb{C}} \subset \mathbb{P}^n$ *is a smooth irreducible variety of (complex) dimension*

$m \geq 1$. *Then for any nonnegative integer* r

$$
\begin{array}{l}
\operatorname{trace}(B^*|H^{2r+1}(W_{\mathbb{C}},\mathbb{R})) = 0, \\
\operatorname{trace}(B^*|H^{2r}(W_{\mathbb{C}},\mathbb{R})) = \operatorname{trace}(B^*|H^{r,r}(W_{\mathbb{C}})) = \\
(-1)^m \operatorname{trace}(B^*|H^{m-r,m-r}(W_{\mathbb{C}}))
\end{array}
\tag{2.1}
$$

where B *is conjugation in* $\mathbb{P}^n$.

Proof. Since $B^*(H^{p,q}(W_{\mathbb{C}})) = H^{q,p}(W_{\mathbb{C}})$ we have for $p \neq q$

$$\operatorname{trace}(B^*|(H^{p,q}(W_{\mathbb{C}}) \oplus H^{q,p}(W_{\mathbb{C}})) = 0.$$

The Hodge decomposition of $H^k(W_{\mathbb{C}},\mathbb{R})$ yields the claim since B^* reverses the orientation of $W_{\mathbb{C}}$ if m is odd and preserves the orientation of $W_{\mathbb{C}}$ if m is even. □

Corollary 2.1 *Let the assumptions of Lemma 2.1 hold. Then the Lefschetz number* $\lambda(W_{\mathbb{C}})$ *of* $B|W_{\mathbb{C}}$ *is given by*

$$
\begin{array}{l}
\lambda(W_{\mathbb{C}}) = 0, \quad \text{if } m \text{ is odd}, \\
\lambda(W_{\mathbb{C}}) = \operatorname{trace}(B^*|H^m(W_{\mathbb{C}})) + 2\displaystyle\sum_{r=0}^{\frac{m-2}{2}} \operatorname{trace}(B^*|H^{2r}(W_{\mathbb{C}})) \in \mathbb{Z}, \\
\text{if } m \text{ is even}.
\end{array}
\tag{2.2}
$$

If $\lambda(W_{\mathbb{C}}) \neq 0$ *then* $W \cap \mathbb{PR}^n \neq \emptyset$.

Proof. This is a consequence of the last lemma and the Lefschetz fixed point theorem. □

Corollary 2.2 *Let* W *be as in Lemma 2.1. Suppose that* m *is even and* $b_m(W_{\mathbb{C}})$ *(equivalently the Euler characteristic* $\chi(W_{\mathbb{C}})$*) is odd. Then* $W \cap \mathbb{PR}^n \neq \emptyset$.

Proof. Since the eigenvalues of $B^*|H^m(W_{\mathbb{C}})$ are ± 1 we have that $b_m(W_{\mathbb{C}}) = \lambda(W_{\mathbb{C}}) \bmod 2$. □

Theorem 2.1 *Let $V \subset \mathbb{PR}^n$ be an algebraic variety such that its complexification $V_{\mathbb{C}} \subset \mathbb{P}^n$ is an irreducible variety of codimension m. Suppose that the codimension of the variety of the singular points of $V_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is at least k. Then for a generic $L \in \mathrm{Gr}\ (m+k, n+1, \mathbb{R})$ $\lambda(V_{\mathbb{C}} \cap L_{\mathbb{C}})$ is equal to zero if k is even and is equal to $b_{k-1}(V_{\mathbb{C}} \cap L_{\mathbb{C}}) \bmod 2$ if k is odd. In particular, if $k = 2r+1$ and $b_{2r}(V_{\mathbb{C}} \cap L_{\mathbb{C}})$ is odd, or more generally $\lambda(V_{\mathbb{C}} \cap L_{\mathbb{C}}) \neq 0$, then $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}\ (m+2r+1, n+1, \mathbb{R})$.*

Proof. For $k=1$ $V_{\mathbb{C}} \cap L_{\mathbb{C}}$ consists of $\deg V_{\mathbb{C}}$ distinct points for a generic L and the theorem follows. Assume that $k>1$. Let $W = V \cap L$, $W_{\mathbb{C}} = V_{\mathbb{C}} \cap L_{\mathbb{C}}$. The assumptions of the theorem yield that for a generic L $W_{\mathbb{C}}$ is a smooth irreducible variety. Hence $\lambda(B|W_{\mathbb{C}})$ is given by Corollary 2.1. Other claims of the theorem follow from Corollaries 2.1 and 2.2. $\square$

Clearly Corollary 1.1 follows from Theorem 2.1. The values of $d(n, n-1, \mathbb{R})$ were computed by Adams, Lax and Phillips in [3] using the work of Adams [2] on the maximal number of linearly independent vector fields on the $n-1$ dimensional sphere S^{n-1}. Write $n = (2a+1)2^{c+4d}$, where a and d are nonnegative integers, and $c \in \{0,1,2,3\}$. Then $\rho(n) = 2^c + 8d$ is the Radon-Hurwitz number. Let $\rho(x) = 0$ if x is not a positive integer.

Then
$$d(n, n-1, \mathbb{R}) = \rho(\frac{n}{2}) + 2.$$
Let
$$p := d(n, n-1, \mathbb{R}) - d(n, n-1, \mathbb{C}) = \rho(\frac{n}{2}). \qquad (2.3)$$
Note that either p is even or $p=1$. Assume that n is even. Let $V = \mathbb{P}V_{n-1,n}(\mathbb{R})$. Then $V_{\mathbb{C}} = \mathbb{P}V_{n-1,n}(\mathbb{C})$. The codimension of the variety of singular points of $V_{\mathbb{C}}$ in $V_{\mathbb{C}}$ is 2. Then for any $k < p$ there exists a linear space $L' \in \mathrm{Gr}\ (2+k, \binom{n+1}{2}, \mathbb{R})$ such that $V \cap L' = \emptyset$. It is shown in [3] that $V \cap L \neq \emptyset$ for any $L \in \mathrm{Gr}\ (2+p, \binom{n+1}{2}, \mathbb{R})$.

Let us consider $d(n,k,\mathbb{R})$ for $k=1$. We have $\mathbb{P}V_{1,n}(\mathbb{C}) \subset \mathbb{P}S_n(\mathbb{C}) \sim \mathbb{P}^{\binom{n+1}{2}-1}$. The variety $\mathbb{P}V_{1,n}(\mathbb{C})$ is biholomorphic to $\mathbb{P}^{n-1}$. Indeed, identify $\mathbb{P}^{n-1}$ with the lines in $\mathbb{C}^n$ spanned by the nonzero column vectors $x \in \mathbb{C}^n$. Then
$$q : \mathbb{P}^{n-1} \to \mathbb{P}V_{1,n}(\mathbb{C}), \quad q(x) = xx^T \qquad (2.4)$$
is a biholomorphism.

In [6] the linear subspace $L_0 \subset \mathbb{P}S_n(\mathbb{C})$ (of codimension 1) of matrices of trace 0 was considered. Clearly $\mathbb{P}V_{1,n}(\mathbb{R}) \cap L_0 = \emptyset$. Hence [6]

$$d(n,1,\mathbb{R}) = \binom{n+1}{2}.$$

Corollary 1.1 yields that for any generic complex linear subspace $L \subset \mathbb{P}S_n(\mathbb{C})$ of codimension $m,\ 1 \le m \le n-1$ the middle Betti number of $L \cap \mathbb{P}V_{1,n}(\mathbb{C})$ is even. (Since $L \cap \mathbb{P}V_{1,n}(\mathbb{C})$ is biholomorphic to a nonsingular quadric this Betti number is either 0 or 2 depending on parity of n.) Similarly for $n > 1$ $\mathbb{P}V_{1,n}(\mathbb{R}) \cap L_0 = \emptyset$ yields that deg $\mathbb{P}V_{1,n}(\mathbb{C})$ is even. (This fact follows also from the formula (1.2).)

Since for an odd n the middle Betti number of $\mathbb{P}V_{1,n}(\mathbb{C})$ is 1, we see that the parity of the Euler characteristic of smooth variety in $\mathbb{P}^n$ is independent of the parity of its degree. Though a complete intersection of even degree has an even Euler characteristic.

3 Chern classes for desingularizations of determinantal varieties.

In this section we shall collect the formulas for the Chern classes of projectivizations of certain bundles. The main reference is [7]. We also specify how such projectivizations come up as desingularizations of determinantal varieties.

Let E be an ℓ-bundle over smooth complex manifold M with the Chern classes $c_1(E), ..., c_\ell(E)$. Let $u_i,\ i = 1, ..., \ell$ be the roots of the Chern polynomial

$$c(E,t) = \sum_{j=0}^{\ell} c_j(E) t^j$$

of E, i.e.

$$c(E,t) = \prod_{i=1}^{\ell} (1 + u_i t)$$

We have (cf. [4, §4.20])

$$c(\mathrm{Sym}^2\, E, t) = \prod_{1 \le i \le j \le \ell} (1 + (u_i + u_j)t). \tag{3.1}$$

Let $\mathbb{P}(E)$ be the projectivization of E. (As a set it consists of the pairs $(x, [v])$, where $x \in M$ and $[v]$ is a line in E over x spanned by a nonzero point $v \in E$ over x.) Let $\tilde{E}$ be the tautological line bundle over $\mathbb{P}(E)$ (given by the line $[v]$ over the point $(x, [v])$). Let E^* be the pull back of E to $\mathbb{P}(E)$ induced by the projection $\pi_1 : \mathbb{P}(E) \to M$. $\tilde{E}$ is a subbundle of E^* (cf. [7, B.5.5]).

Lemma 3.1 *Let M be a complex manifold of dimension n. Let $E \to M$ be a complex vector bundle vector of rank $\ell \geq 1$ and $\pi : \mathbb{P}(E) \to M$ be its projectivization. Let $\tilde{E}$ be the tautological line bundle over $\mathbb{P}(E)$, and $q = c_1(\tilde{E})$ be its first Chern class (resp $h = -q$ is the first Chern class of $\tilde{E}'$, which is the dual to $\tilde{E}$). Then the cohomology ring $\mathrm{H}^*(\mathbb{P}(E), \mathbb{C})$ is $\mathrm{H}^*(M, \mathbb{C})[q]$ together with the relation*

$$q^\ell + \sum_{i=1}^{\ell} (-1)^i c_i(E) q^{\ell - i} = 0. \tag{3.2}$$

Let

$$c(T_M, t) = \sum_{i=0}^{n} c_i(T_M) t^i, \quad c_0(T_M) = 1$$

be the Chern polynomial of the tangent bundle of M. Then the Chern polynomial of the tangent bundle of $\mathbb{P}(E)$ is given by

$$c(T_{\mathbb{P}(E)}, t) = c(T_M, t)(\sum_{j=0}^{\ell} c_j(E) t^j (1 - qt)^{\ell - j}). \tag{3.3}$$

Proof. For the proof of (3.2) see [10], [8, §4.6, pp. 606] or [4, §4.20]. On the other hand for the relative tangent bundle $T_{\mathbb{P}(E)/M}$, which fits into exact sequence:

$$0 \to \pi^*(T_M) \to T_{\mathbb{P}(E)} \to T_{\mathbb{P}(E)/M} \to 0,$$

we have

$$T_{\mathbb{P}(E)/M} = \tilde{E} \otimes Q, \tag{3.4}$$

where Q is the universal quotient bundle: $E^*/\tilde{E}$. (cf. [7, B.5.8]). This yileds (3.3). $\square$

For example if E is trivial and has rank m then $\mathbb{P}(E) = M \times \mathbb{P}^{m-1}$ and (3.3) becomes:

$$c(T_{\mathbb{P}(E)}) = c(T_M)(1 - qt)^m, \quad q^m = 0. \tag{3.5}$$

In the next sections the following situation will arise:

Lemma 3.2 *Let M be a complex manifold of dimension n and $E \to M$ be a trivial complex vector bundle vector of rank $m \geq 2$. Denote by $\tilde{E}'$ the dual to the tautological bundle $\tilde{E}$. Let $U \subset \mathbb{P}(E)$ be a connected complex submanifold of dimension d. Consider hypersurfaces $\tilde{H}_i$ $i = 1, ..., k$ in $\mathbb{P}(E)$ each being the zero set of a generic section of $\tilde{E}'$. Let $W = U \cap_{i=1}^{i=k} H_i$ and ι be the embedding W in U. Then*

$$c(T_W, t) = \iota^* c(T_U|W, t)(1 - tq)^{-k}, \tag{3.6}$$

and

$$\chi(W) = h^d c(T_U)(1 - tq)^{-k}[U], \tag{3.7}$$

where $[U]$ is the fundamental class of U and h is the restriction on U of the first Chern class $c_1(\tilde{E}')$.

Proof. (3.6) is a consequence of the exact sequence:

$$0 \to T_W \to T_U|W \to \sum_{i=1}^{k} \oplus N_{\tilde{H}_i}|W \to 0.$$

(3.7) is similar to [10, 9.3]. □

Let $E \to M$ a trivial m-bundle, and $F \to M$ is an ℓ-subbundle of E. As above q_E (resp. q_F) be the first Chern class of the tautological bundle $\tilde{E}$ (resp. $\tilde{F}$) on $\mathbb{P}(E)$ (resp. $\mathbb{P}(F)$). Then $\mathbb{P}(F) \subset \mathbb{P}(E)$ and if ι is the embedding then:

$$q_F = \iota^* q_E. \tag{3.8}$$

We describe now a smooth resolutions of $V_{k,n}(\mathbb{C})$ and $\mathbb{P}V_{k,n}(\mathbb{C})$ for $1 \leq k \leq n-1$. This construction is similar to the one described in [1, II]. We have the following exact sequence of three bundles over Gr $(k, n, \mathbb{C})$:

$$0 \to \tau \to \mathbb{C}^n \to \kappa \to 0. \tag{3.9}$$

Here τ is the tautological k-bundle, $\mathbb{C}^n$ is the n-trivial bundle and $\kappa := \mathbb{C}^n/\tau$ the $n-k$ quotient bundle.

Lemma 3.3 *Let $1 \le k < n$. Then the bundles* $\mathrm{Sym}^2\ \tau$ *and* $\mathrm{Sym}^2\ \kappa$ *are smooth resolutions of* $V_{k,n}(\mathbb{C})$ *and* $V_{n-k,n}(\mathbb{C})$ *respectively. Furthermore the projectivized bundles* $\mathbb{P}(\mathrm{Sym}^2\ \tau)$ *and* $\mathbb{P}(\mathrm{Sym}^2\ \kappa)$ *are smooth resolutions of* $\mathbb{P}V_{k,n}(\mathbb{C})$ *and* $\mathbb{P}V_{n-k,n}(\mathbb{C})$ *respectively.*

Proof. Viewing A as a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ yields the two linear subspaces: Range A and Ker A of $\mathbb{C}^n$, which are the range and kernel of the operator A respectively. Note that if $a \in \mathbb{C}^*$ then Range A =Range aA and Ker A =Ker aA. Let

$$\begin{aligned}
&X := S_n(\mathbb{C}) \times \mathrm{Gr}\ (k,n,\mathbb{C}), \quad \tilde{X} := \mathbb{P}S_n(\mathbb{C}) \times \mathrm{Gr}\ (k,n,\mathbb{C}),\\
&Y := \{(A,V) \in X : \quad \mathrm{Range}\ A \subset V\},\\
&\tilde{Y} := \{(A,V) \in \tilde{X} : \quad \mathrm{Range}\ A \subset V\},\\
&Z := \{(B,V) \in X : \quad \mathrm{Kernel}\ B \supset V\},\\
&\tilde{Z} := \{(B,V) \in \tilde{X} : \quad \mathrm{Kernel}\ B \supset V\}.
\end{aligned} \tag{3.10}$$

Let $\pi_1 : X \to S_n(\mathbb{C})$, $\pi_2 : X \to \mathrm{Gr}\ (k,n,\mathbb{C})$ be the projections on the first and second coordinates respectively. Clearly

$$\begin{aligned}
&\pi_1(Y) = V_{k,n}(\mathbb{C}), \quad \pi_2(Y) = \mathrm{Gr}\ (k,n,\mathbb{C}),\\
&\pi_1(Z) = V_{n-k,n}(\mathbb{C}), \quad \pi_2(Z) = \mathrm{Gr}\ (k,n,\mathbb{C}).
\end{aligned}$$

The map π_1 is a resolution. Indeed it is birational of degree one since it is $1-1$ on

$$\pi_1^{-1}(V_{k,n}(\mathbb{C}) \backslash V_{k-1,n}(\mathbb{C})) \subset Y \quad \text{and} \quad \pi_1^{-1}(V_{n-k,n}(\mathbb{C}) \backslash V_{n-k-1,n}(\mathbb{C})) \subset Z.$$

Similar situation takes place for π_2

Finally the fiber of the projection of Y on Gr $(k,n,\mathbb{C})$ over V can be identified with the space of symmetric transformations of V which yields the identification of Y with $Sym^2\tau$. Similarly Z can be identified with $\mathrm{Sym}^2\ \kappa$. Hence $\mathbb{P}(\mathrm{Sym}^2\ \tau)$ and $\mathbb{P}(\mathrm{Sym}^2\ \kappa)$ are smooth resolutions of $\mathbb{P}V_{k,n}(\mathbb{C})$ and $\mathbb{P}V_{n-k,n}(\mathbb{C})$ respectively. □

We review now some known facts about the cohomology of Grassmanians used in the rest of the paper. Let $c_1, ..., c_k$ and $s_1, ..., s_{n-k}$ be the Chern classes of τ and κ respecively. Denote by $c(\tau, t)$, $c(\kappa, t)$ the Chern polynomials

$$c(\tau,t)=1+\sum_{i=1}^{\infty}c_it^i,\quad c(\kappa,t)=1+\sum_{j=1}^{\infty}s_jt^j,$$

where $c_i=s_j=0$ for $i>k,\ j>n-k$. Recall that

$$c(\tau,t)c(\kappa,t)=1. \tag{3.11}$$

Then the cohomology ring of Gr $(k,n,\mathbb{C})$ has the following representation [7, Ex. 14.6.6] or [4, §4.23]

$$\mathrm{H}^*(\mathrm{Gr}\ (k,n,\mathbb{C}),\mathbb{C})=\mathbb{C}[(c_1,...,c_k)]/(s_{n-k+1},...,s_n). \tag{3.12}$$

Here we use the formula

$$c(\kappa,t)=\frac{1}{1+c_1t+...+c_kt^k}. \tag{3.13}$$

With the help of these formulas we can compute the Chern classes of

$$\mathrm{Sym}^2\ \tau,\ \mathrm{Sym}^2\ \kappa\subset E$$

as polynomials in $c_1,...,c_k$ and $s_1,...,s_{n-k}$ respectively. Here

$$E\to\mathrm{Gr}\ (k,n,\mathbb{C})\text{ is a trivial bundle with the fiber }S_n(\mathbb{C})=\mathrm{Sym}^2\ \mathbb{C}^n. \tag{3.14}$$

Then $\mathbb{P}(E)$ is identified with $\mathbb{P}S_n(\mathbb{C})\times$ Gr $(k,n,\mathbb{C})$. Furthermore, $q=-h$ is the tautological line bundle over $\mathbb{P}(E)$. Thus

$$\mathrm{H}^*(\mathbb{P}S_n(\mathbb{C})\times\mathrm{Gr}\ (k,n,\mathbb{C}),\mathbb{C})=\mathrm{H}^*(\mathrm{Gr}\ (k,n,\mathbb{C}),\mathbb{C})[q],\quad q^{\binom{n+1}{2}}=0. \tag{3.15}$$

From the proof of Lemma 3.3 it follows that $\mathbb{P}(\mathrm{Sym}^2\ \tau),\ \mathbb{P}(\mathrm{Sym}^2\ \kappa)$ are subvarieties of $\mathbb{P}(E)$, which can be identified with the smooth subvarieties $\tilde{Y},\tilde{Z}\subset\mathbb{P}S_n(\mathbb{C})\times\mathrm{Gr}\ (k,n)$. Then on $\tilde{Y},\ \tilde{Z}$ the generator q satisfies the corresponding relation

$$q^{\binom{k+1}{2}}+\sum_{i=1}^{\binom{k+1}{2}}(-1)^ic_i(\mathrm{Sym}^2\ \tau)q^{\binom{k+1}{2}-i}=0, \tag{3.16}$$

$$q^{\binom{n-k+1}{2}}+\sum_{j=1}^{\binom{n-k+1}{2}}(-1)^jc_j(\mathrm{Sym}^2\ \kappa)q^{\binom{n-k+1}{2}-j}=0.$$

To find the Chern classes of the tangent bundles of $T_{\tilde{Y}}$, $T_{\tilde{Z}}$ we use Lemma 3.1. To find the Chern class of the tangent bundle of Gr $(k, n, \mathbb{C})$ recall the following identity (cf. [7, §B.6]) :

$$T_{Gr(k,n,\mathbb{C})} \sim \kappa \otimes \tau' \tag{3.17}$$

Then

$$\begin{aligned} c(\tau', t) &= 1 + \sum_{i=1}^{k} (-1)^i c_i(\tau) t^i = \prod_{i=1}^{k} (1 + \alpha_i t), \\ c(\kappa, t) &= 1 + \sum_{j=1}^{n-k} s_j t^j = \prod_{j=1}^{n-k} (1 + \beta_j t), \\ c(\kappa \otimes \tau', t) &= \prod_{i,j=1}^{k,n-k} (1 + (\alpha_i + \beta_j) t) = 1 + \sum_{\ell=1}^{k(n-k)} v_\ell t^\ell. \end{aligned} \tag{3.18}$$

4 Gr $(1, n, \mathbb{C})$

As an illustration of the above formulas let us consider the case Gr $(1, n, \mathbb{C}) = \mathbb{P}^{n-1}$. The Chern class of the tautological line bundle τ of Gr $(1, n, \mathbb{C})$ is c_1. The basic relation is $c_1^n = 0$. Note that $-c_1$ is the dual action of the hyperplane section. So $c(\tau, t) = 1 + c_1 t$. The Chern polynomial of $T_{\mathbb{P}^{n-1}}$ is $(1 - c_1 t)^n$, e.g. [8, §3.3]. Recall that $\mathrm{Sym}^2\, \tau = \mathbb{P}V_{1,n}(\mathbb{C})$. Hence

$$c(\mathbb{P}V_{1,n}(\mathbb{C}), t) = c(\mathrm{Sym}^2\, \tau, t) = 1 + w_1 t, \quad w_1 = 2c_1.$$

Let $q = -h$ be the tautological line bundle of $\mathbb{P}(E)$. Then $-h = q|\mathbb{P}(\mathrm{Sym}^2\, \tau)$ satisfies the equation (3.2) which is $-h = q = w_1 = 2c_1$. Hence $H^*(\mathbb{P}V_{1,n}(\mathbb{C}), \mathbb{C})$ is generated by c_1. The equality (3.3) yields the obvious equality

$$c(T_{\mathbb{P}V_{1,n}(\mathbb{C})}) = (1 - c_1 t)^n ((1 - tq) + w_1 t) = (1 - c_1 t)^n,$$

as $\mathbb{P}V_{1,n}(\mathbb{C}) \sim \mathbb{P}^{n-1}$. We now compute the degree of $\mathbb{P}V_{1,n}(\mathbb{C})$. It is equal to the Chern number of the hyperplane section

$$h^{n-1} = (-q)^{n-1} = (-2c_1)^{n-1} = 2^{n-1}(-c_1)^{n-1}.$$

Since $-c_1$ is the class of the hyperplane section in $\mathbb{P}^{n-1}$ it follows that deg $\mathbb{P}V_{1,n}(\mathbb{C}) = 2^{n-1}$, which agrees with the formula (1.2). We now compute the Euler characteristic of the intersection of $\mathbb{P}V_{1,n}(\mathbb{C})$ with a generic linear subspace of codimension $k \geq 1$. Let $U = \mathbb{P}(\mathrm{Sym}^2\,\tau)$. Then by (3.6)

$$c(T_W, t) = (1 - c_1 t)^n (1 - 2c_1 t)^{-k}.$$

Hence

$$c_{n-1-k}(T_W) = (-c_1)^{n-k-1} \sum_{j=0}^{n-1-k} \binom{n}{j} \binom{-k}{n-1-k-j} 2^{n-1-k-j}.$$

(3.6) yields

$$\chi(W) = 2^k \sum_{j=0}^{n-1-k} \binom{n}{j} \binom{-k}{n-1-k-j} 2^{n-1-k-j}.$$

For $k = n-2$ W is a smooth curve with the Euler characteristic

$$\chi(W) = 2^{n-2}(4-n).$$

5 Gr $(2, 4, \mathbb{C})$

We now consider Gr $(2, n, \mathbb{C})$ for $n \geq 3$. Then

$$\begin{aligned}
&c(\tau, t) = 1 + c_1 t + c_2 t^2, \\
&c(\tau', t) = 1 - c_1 t + c_2 t^2 = (1 + \alpha_1 t)(1 + \alpha_2 t), \\
&\alpha_1 + \alpha_2 = -c_1, \quad \alpha_1\alpha_2 = c_2, \\
&c(\kappa, t) = 1 + \sum_{j=1}^{\infty} s_j t^j = \prod_{j=1}^{n-2}(1 + \beta_j t) = \\
&\frac{1}{1 + c_1 t + c_2 t^2} = \frac{1}{(1 - \alpha_1 t)(1 - \alpha_2 t)}, \\
&s_p = \sum_{i=0}^{p} \alpha_1^i \alpha_2^{p-i}, \quad p = 1, \ldots
\end{aligned} \tag{5.1}$$

A straightforward calculation shows

$$\begin{aligned}
&s_1 = -c_1,\ s_2 = c_1^2 - c_2,\ s_3 = -c_1^3 + 2c_1c_2, \\
&s_4 = c_1^4 - 3c_1^2 c_2 + c_2^2,\ s_5 = -c_1^5 + 4c_1^3 c_2 - 3c_1 c_2^2.
\end{aligned} \tag{5.2}$$

Thus

$$H^*(\text{Gr }(2,4,\mathbb{C}),\mathbb{C}) = \mathbb{C}[c_1,c_2]/(-c_1^3+2c_1c_2, c_1^4-3c_1^2c_2+c_2^2), \tag{5.3}$$
$$H^*(\text{Gr }(2,5,\mathbb{C}),\mathbb{C}) = \mathbb{C}[c_1,c_2]/(c_1^4-3c_1^2c_2+c_2^2, -c_1^5+4c_1^3c_2-3c_1c_2^2).$$

We now compute the four Chern classes v_1, v_2, v_3, v_4 of the tangent bundle of Gr $(2,4,\mathbb{C})$ interms of c_1, c_2 using (3.18). Note that the power series corresponding to terms contributed by only α and β respectively correspond to the polynomials

$$(1-c_1t+c_2t^2)^2 = 1-2c_1t+(c_1^2+2c_2)t^2-2c_1c_2t^3+c_2^2t^4,$$
$$(1+s_1t+s_2t^2)^2 = 1+2s_1t+(s_1^2+2s_2)t^2+2s_1s_2t^3+s_2^2t^4.$$

Hence

$$\begin{aligned}
&v_1 = 2(-c_1+s_1) = -4c_1,\\
&v_2 = c_1^2+2c_2+s_1^2+2s_2+3(\alpha_1+\alpha_2)(\beta_1+\beta_2) = 7c_1^2,\\
&v_3 = -2c_1c_2+2s_1s_2+\\
&(\alpha_1^2+\alpha_2^2+4\alpha_1\alpha_2)(\beta_1+\beta_2)+(\alpha_1+\alpha_2)(\beta_1^2+\beta_2^2+4\beta_1\beta_3) = -6c_1^3\\
&v_4 = c_2^2+s_2^2+\alpha_1\alpha_2(\alpha_1+\alpha_2)(\beta_1+\beta_2)+(\alpha_1+\alpha_2)(\beta_1+\beta_2)\beta_1\beta_2+\\
&(\alpha_1^2+\alpha_2^2)\beta_1\beta_2+\alpha_1\alpha_2(\beta_1^2+\beta_2^2)+2\alpha_1\alpha_2\beta_1\beta_2 = c_1^4+4c_2^2 = 3c_1^4.
\end{aligned} \tag{5.4}$$

Here we used the two identities in $H^*(\text{Gr }(2,4,\mathbb{C}),\mathbb{C})$ given in (5.3). This agrees with the folowing well known computation of the tangent bundle of Gr $(2,4,\mathbb{C})$. Recall the classical result that Gr $(2,4,\mathbb{C})$ imbeds as a smooth quadric in $\mathbb{P}^5$. The tangent bundle of $\mathbb{P}^5$ is $(1+h)^6$, while the normal bundle of the quadric is $(1+2h)$. Hence the tangent bundle of the quadric is given by $\frac{(1+h)^6}{(1+2h)}$. So $c_1 = -h$.

We now consider the 3-bundle $\text{Sym}^2\ \tau$. Let w_1, w_2, w_3 be its Chern classes. Then

$$\begin{aligned}
&c(\text{Sym}^2\ \tau, t) = 1+\sum_{i=1}^{3} w_it^i = (1-2\alpha_1t)(1-2\alpha_2t)(1-(\alpha_1+\alpha_2)t) =\\
&(1+2c_1t+4c_2t^2)(1+c_1t) = (1+3c_1t+(2c_1^2+4c_2)t^2+4c_1c_2t^3.
\end{aligned} \tag{5.5}$$

Then the cohomology ring of $\mathbb{P}(\text{Sym}^2\ \tau)$ is H*(Gr $(2,4,\mathbb{C})[h]$ $(h=-q)$ with the relation

$$h^3+3c_1h^2+(2c_1^2+4c_2)h+4c_1c_2 = h^3+3c_1h^2+(2c_1^2+4c_2)h+2c_1^3 = 0. \tag{5.6}$$

Use (3.4) to deduce that

$$c(T_{\mathbb{P}(\mathrm{Sym}^2\,\tau)/\mathrm{Gr}\,(2,4,\mathbb{C})},t) = 1 + (3h+3c_1)t + (3h^2+3c_1h+2c_1^2+4c_2)t^2.$$

Hence

$$c(T_{\mathbb{P}(\mathrm{Sym}^2\,\tau)},t) = (1 + (3h+3c_1)t + (3h^2+3c_1h+2c_1^2+4c_2)t^2) \times \\ (1-4c_1t+7c_1^2t^2-6c_1^3t^3+3c_1^4t^4). \tag{5.7}$$

Observe also that any monomial in c_1, c_2 of total degree greater than 4 is zero, since the dimension of Gr $(2,4,\mathbb{C})$ is 4. Consider the intersection of $\mathbb{P}V_{2,4}(\mathbb{C})$ with a linear subspace of codimension 6. This is equivalent to the class of h^6 in $\mathbb{P}(\mathrm{Sym}^2\ \tau)$. We want to find out the generator of the top cohomology of $\mathbb{P}(\mathrm{Sym}^2\ \tau)$ and the class of h^6 in terms of this generator. Using the equation (5.6) we can express h^6 in as a quadratic polynomial in q with polynomial coefficients in c_1, c_2:

$$\begin{aligned}
&h^3 = -3c_1h^2 - (2c_1^2+4c_2)h - 2c_1^3,\\
&h^4 = -3c_1(-3c_1h^2 - (2c_1^2+4c_2)h - 2c_1^3) - (2c_1^2+4c_2)h^2 - 2c_1^3h = \\
&(7c_1^2-4c_2)h^2 + 10c_1^3h + 6c_1^4,\\
&h^5 = (7c_1^2-4c_2)(-3c_1h^2 - (2c_1^2+4c_2)h - 2c_1^3) + 10c_1^3h^2 + 6c_1^4h = \\
&-5c_1^3h^2 - 10c_1^4h - 10c_1^5 = -5c_1^3h^2 - 10c_1^4h,\\
&h^6 = -5c_1^3(-3c_1h^2 - (2c_1^2+4c_2)h - 2c_1^3) - 10c_1^4h^2 = \\
&5c_1^4h^2 = 10c_1^2c_2h^2 = 10c_2^2h^2.
\end{aligned}$$

Mulitiply h^5 by c_1, h^4 be c_1^2 and h^3 by c_1^3 respectively to conclude the following relations:

$$h^6 = -c_1h^5 = c_1^2h^4 = 5c_1^4h^2 = 10c_1^2c_2h^2 = 10c_2^2h^2, \quad c_1^3h^3 = -3c_1^4h^2. \tag{5.8}$$

Recall the result of Harris and Tu [9] that the degree of $\mathbb{P}V_{2,4}(\mathbb{C})$ is 10. Hence $c_2^2h^2$ is the generator in the top cohomology of $\mathbb{P}(\mathrm{Sym}^2\ \tau)$. (This can be concluded directly.)

We now compute the Euler characteristic of the smooth curve W, obtained by a generic plane section of codimension 5 with $\mathbb{P}V_{2,4}(\mathbb{C})$. It is the class of h^5 times the first Chern class a (the coefficient of t) in the product

$$c(T_{\mathbb{P}(\mathrm{Sym}^2\,\tau)},t)(1+ht)^{-5}.$$

A straightforward calculation shows $a = -2h - c_1$. Hence

$$h^5 a = -2h^6 - h^5 c_1 = -h^6,$$

and $\chi(W) = -10$.

We now compute the Euler characteristic of the smooth surface W obtained by a generic plane section of codimension 4 with $\mathbb{P}V_{2,4}(\mathbb{C})$. It is the class of h^4 times the second Chern class b (the coefficient of t) in the product

$$c(T_{\mathbb{P}(\mathrm{Sym}^2\,\tau)}, t)(1 + ht)^{-4}.$$

A straightforward calculation shows $b = h^2 - 5c_1 h - 3c_1^2$. Hence

$$h^4 b = h^6 - 5c_1 h^5 - c_1^2 h^4 = 7h^6,$$

and $\chi(W) = 70$.

Corollary 5.1 *A generic linear space of codimension* 5 *in* $\mathbb{P}S_4(\mathbb{C})$ *intersects* $\mathbb{P}V_{2,4}(\mathbb{C})$ *at a smooth curve of degree* 10 *and Euler characteristic* -10. *A generic linear space of codimension* 4 *in* $\mathbb{P}S_4(\mathbb{C})$ *intersects* $\mathbb{P}V_{2,4}(\mathbb{C})$ *at a smooth surface of degree* 10 *and Euler characteristic* 70.

Hence we can not conclude from these resutls that any linear subspace $L \subset S_4(\mathbb{R})$ of dimension 6 contains a nonzero matrix of rank 2 at most. In [6] we show (using different topological methods) the sharp result that any linear subspace $L \subset S_4(\mathbb{R})$ of dimension 5 contains a nonzero matrix of rank 2 at most. It is of interest to check if the conjugation map $z \to \bar{z}$, described in the beginning of this paper, for $L \cap \mathbb{P}V_{2,4}(\mathbb{C})$, where $L \subset \mathbb{P}S_4(\mathbb{C})$ is a generic linear space of dimension 5, has a nonzero Lefschetz number.

6 Gr $(2, 5, \mathbb{C})$ modulo 2

Theorem 6.1 *Let* $L \subset PS_5(\mathbb{C})$ *be a generic linear space of dimension* 5. *Then* $L \cap \mathbb{P}V_{3,5}(\mathbb{C})$ *is a smooth surface with an odd Euler characteristic.*

Proof. Let τ, κ be the tautological and the quotient bundles of Gr $(2, 5, \mathbb{C})$. Then $\mathrm{Sym}^2\,\kappa \to$ Gr $(2, 5, \mathbb{C})$ is the subbundle of the trivial bundle $E \to$ Gr $(2, 5, \mathbb{C})$ given in (3.14). By Lemma 3.3

$$\tilde{Z} = \mathbb{P}(\mathrm{Sym}^2\,\kappa) \subset \mathbb{P}(E) = \mathbb{P}S_5(\mathbb{C}) \times \mathrm{Gr}\,(2, 5, \mathbb{C})$$

is a resolution of $\mathbb{P}V_{3,5}(\mathbb{C})$. Then $\mathrm{H}^*(\tilde{Z},\mathbb{C}) = \mathrm{H}^*(\mathrm{Gr}\ (2,5,\mathbb{C}),\mathbb{C})[q])$, where q satisfied the second indentity of (3.16). Recall that the tangent bundle of Gr $(2,5,\mathbb{C})$ is isomorphic to $\kappa \otimes \tau'$. The tangent bundle of $\mathbb{P}(\mathrm{Sym}^2\ \kappa)$ is given by the formulas (3.3). As the singular points of $\mathbb{P}V_{3,5}(\mathbb{C})$ is the variety $\mathbb{P}V_{2,5}(\mathbb{C})$ of codimension $\binom{4}{2} = 6$ it follows that $L \cap \mathbb{P}V_{2,5}(\mathbb{C}) = \emptyset$. Hence $L \cap \mathbb{P}V_{3,5}(\mathbb{C})$ is a smooth surface. It then follows that

$$L \cap \mathbb{P}V_{3,5}(\mathbb{C}) = \tilde{Z} \cap_{k=1}^{9} \tilde{H}_i,$$

where $\tilde{H}_i,\ i = 1, ..., 9$ are 9 linearly independent fiber hyperplanes in general position, as in Lemma 3.2.

Let b be the coefficient of t^2 in the product

$$c(\kappa \otimes \tau', t)c(T_{\mathbb{P}(\mathrm{Sym}^2\ \kappa)/\mathrm{Gr}\ (2,5,\mathbb{C})}, t)(1 + ht)^{-9} \tag{6.1}$$

Then Lemma 3.2 yields that

$$\chi(L \cap \mathbb{P}V_{3,5}(\mathbb{C})) = h^9 b[\tilde{Z}]. \tag{6.2}$$

Since we are interested in the parity of $\chi(L \cap \mathbb{P}V_{3,5}(\mathbb{C}))$ we will do all the computations modulo 2. (That is our computations are in $\mathrm{H}^*(\tilde{Z},\mathbb{Z}_2)$.) This will simplify our computations significantly.

We first consider $\mathrm{H}^*(\mathrm{Gr}\ (2,5,\mathbb{C}),\mathbb{Z}_2)$. It is generated by c_1, c_2 with the two simpler relations induced by the second part of (5.3)

$$c_1^4 + c_1^2 c_2 + c_2^2 = 0, \quad c_1^5 = c_1 c_2^2. \tag{6.3}$$

Muliply the first equality by c_1 and use the second identity to deduce

$$c_1^3 c_2 = 0 \Rightarrow c_1^4 c_2 = 0. \tag{6.4}$$

Multiply the first equality in (6.3) by c_2 and use (6.4). Multilply the second equality of (6.3) by c_1. Then

$$c_1^6 = c_1^2 c_2^2 = c_2^3. \tag{6.5}$$

Hence the generator of the top cohomology in $\mathrm{H}^*(\mathrm{Gr}\ (2,5,\mathbb{C}),\mathbb{Z}_2)$ is any 6-form in the (6.5).

Recall (5.1) for $n = 5$. The equalites (5.2) modulo 2 yield

$$s_1 = c_1,\ s_2 = c_1^2 + c_2,\ s_3 = c_1^3.$$

We now compute the first two Chern classes of $\kappa \otimes \tau'$, which gives the first two Chern classes v_1, v_2 of the tangent bundle of $\mathrm{Gr}\,(2,5,\mathbb{C})$. Observe that the terms in v_1, v_2, expressed either in terms of α or β are coming from either $c(\tau', t)^3$ or $c(\kappa, t)^2$:

$$\begin{aligned}
&c(\tau', t)^3 = (1 - c_1 t + c_2 t^2)^3 = 1 - 3c_1 t + 3(c_2 + c_1^2)t^2 + \text{higher order terms},\\
&c(\kappa, t)^2 = (1 + s_1 t + s_2 t^2 + s_3 t^3)^2 =\\
&1 + 2s_1 t + (2s_2 + s_1^2)t^2 + \text{higher order terms.}
\end{aligned}$$

Using the equalities in (5.2) we obtain.

$$\begin{aligned}
&v_1 = -3c_1 + 2s_1 = -5c_1,\\
&v_2 = 3(c_2 + c_1^2) + (2s_2 + s_1^2) + 5(\alpha_1 + \alpha_2)(\beta_1 + \beta_2 + \beta_3) = c_2 + c_1^2.
\end{aligned}$$

The coefficient 5 in the product of α's and β's is obtaines as follows. Consider the product $\alpha_1\beta_1$. It comes twice from the terms $(\alpha_1 + \beta_1)(\alpha_1 + \beta_i),\ i = 2, 3$ and three times from the terms $(\alpha_1 + \beta_i)(\alpha_2 + \beta_1),\ i = 1, 2, 3$.

Modulo 2 we get

$$v_1 = c_1, \quad v_2 = c_2 + c_1^2. \tag{6.6}$$

We next compute the Chern polynomial of $\mathrm{Sym}^2\,\kappa$ modulo 2. Then

$$c(\mathrm{Sym}^2\,\kappa, t) = \prod_{1 \le i \le j \le 3} (1 + (\beta_i + \beta_j)t) = c(\kappa, 2t) \prod_{1 \le i < j \le 3} (1 + (\beta_i + \beta_j)t).$$

Hence modulo 2

$$c(\mathrm{Sym}^2\,\kappa, t) = \prod_{1 \le i < j \le 3} (1 + (\beta_i + \beta_j)t) = 1 + w_1 t + w_2 t^2 + w_3 t^3.$$

Then modulo 2

$$\begin{aligned}
&w_1 = 2\sum_{i=1}^{3} \beta_i = 0,\\
&w_2 = (\beta_1 + \beta_2)(\beta_1 + \beta_3 + \beta_2 + \beta_3) + (\beta_1 + \beta_3)(\beta_2 + \beta_3) =\\
&\beta_1^2 + \beta_2^2 + \beta_3^2 + s_2 = s_1^2 - s_2 = c_2,\\
&w_3 = (s_1 - \alpha_1)(s_1 - \alpha_2)(s - \alpha_3) = s_1^3 - s_1 s_1^2 + s_2 s_1 - s_3 = c_2 c_1.
\end{aligned}$$

Use (3.4) modulo 2 to get

$$\begin{aligned}
&c(T_{\mathbb{P}(\mathrm{Sym}^2\,\kappa)/\mathrm{Gr}\,(2,5)}, t) = (1 + ht)^6 + c_2 t^2 (1 + ht)^4 + c_2 c_1 t^3 (1 + ht)^3 =\\
&1 + (c_2 + h^2)t^2 + \text{higher order terms.}
\end{aligned}$$

Then the coefficient b of t^2 in (6.1) is equal modulo 2 to the coefficient of t^2 in the product

$$(1 + c_1t + (c_2 + c_1^2)t^2 + ...)(1 + (c_2 + h^2)t^2 + ...)(1 + ht + h^2t^2 + ...) =$$
$$1 + (c_1 + h)t + (c_1^2 + c_1h)t^2 + ...$$

Hence modulo 2

$$b = c_1^2 + c_1h. \tag{6.7}$$

We now consider the second identity of (3.16) for $q = -h$ modulo 2.

$$h^6 = c_2h^4 + c_2c_1h^3. \tag{6.8}$$

Multiply by h, h^2, h^3, h^4, h^5 the above equality, use (6.3-6.5) and the fact that any form in c_1, c_2 of degree greater than 6 equals to 0 to obtain

$$\begin{aligned}
&h^7 = c_2h^5 + c_2c_1h^4,\\
&h^8 = c_2h^6 + c_2c_1h^5 = c_2(c_2h^4 + c_2c_1h^3) + c_2c_1h^5 = c_2c_1h^5 + c_2^2h^4 + c_2^2c_1h^3,\\
&h^9 = c_2c_1h^6 + c_2^2h^5 + c_2^2c_1h^4 = c_2c_1(c_2h^4 + c_2c_1h^3) + c_2^2h^5 + c_2^2c_1h^4 =\\
&c_2^2h^5 + c_2^2c_1^2h^3,\\
&h^{10} = c_2^2h^6 + c_2^2c_1^2h^4 = c_2^2(c_2h^4 + c_2c_1h^3) + c_2^2c_1^2h^4 = 0,\\
&h^{11} = 0.
\end{aligned} \tag{6.9}$$

The equality $h^{11} = 0$ (mod 2) means that $h^{11}[\tilde{Z}]$ is an even number. By Harris-Tu this number, the degree of $\mathbb{P}V_{3,5}(\mathbb{C})$, is equal to 20. We claim that the generator of the top cohomology of $\mathrm{H}^*(\tilde{Z}, \mathbb{Z}_2)$ is

$$c_1^6h^5 = c_1^2c_2^2h^5 = c_2^3h^5. \tag{6.10}$$

First consider all the monomials in h, c_1, c_2 of degre 6 in h and total degree 11

$$c_1^5h^6,\ c_1^3c_2h^6 = 0,\ c_1c_2^2h^6.$$

We used here (6.4). Multiply (6.8) by c_1^5 and $c_1c_2^2$ respectively to deduce

$$c_1^5h^6 = c_1^3c_2h^6 = c_1c_2^2h^6 = 0.$$

Second consider all the monomial in h, c_1, c_2 of degre 7 in h and total degree 11

$$c_1^4h^7,\ c_1^2c_2h^7,\ c_2^2h^7.$$

Multiply h^7 in (6.9) by an appropriate monomial of c_1, c_2 to get

$$c_1^4 h^7 = 0,\ c_1^2 c_2 h^7 = c_1^2 c_2^2 h^5,\ c_2^2 h^7 = c_2^3 h^5.$$

Hence all the nonzero terms are equal to the terms in (6.10). Third consider all the monomial in h, c_1, c_2 of degre 8 in h and total degree 11

$$c_1^3 h^8,\ c_1 c_2 h^8.$$

Multiply h^8 in (6.9) by an appropriate monomial of c_1, c_2 to get

$$c_1^3 h^8 = 0,\ c_1 c_2 h^8 = c_1^2 c_2^2 h^5.$$

Thus the nonzero term is equal to the terms in (6.10). Fourth consider all the monomial in h, c_1, c_2 of degre 9 in h and total degree 11

$$c_1^2 h^9,\ c_2 h^9.$$

Multiply h^8 in (6.9) by an appropriate monomial of c_1^2 and c_2 to get

$$c_1^2 h^9 = c_1^2 c_2^2 h^5,\ c_2 h^9 = c_2^3 h^5.$$

Hence all the terms are equal to the terms in (6.10). As $h^{10} = 0$ we deduce that $c_1^2 h^9$ is the generator of the top cohomology in $\mathrm{H}^*(\tilde{Z}, \mathbb{Z}_2)$. Clearly, mod 2

$$h^9 b = c_1^2 h^9 + c_1 h^{10} = c_1^2 h^9.$$

Hence $\chi(L \cap \mathbb{P}V_{3,5}(\mathbb{C}))$ is odd. $\square$

Corollary 6.1 $d(5, 3, \mathbb{R}) \leq 6$. *That is, every six dimensional real subspace* $L' \subset S_5(\mathbb{R})$ *contains a nonzero matrix of rank* 3 *or less.*

In [5] the authors give an example of five dimenisonal subspace $L_1 \subset S_5(\mathbb{R})$, for which a numerical evidence suggests that every nonzero matrix is of rank 4 at least. Hence the above Corollary suggests that $d(5, 3, \mathbb{R}) = 6$.

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